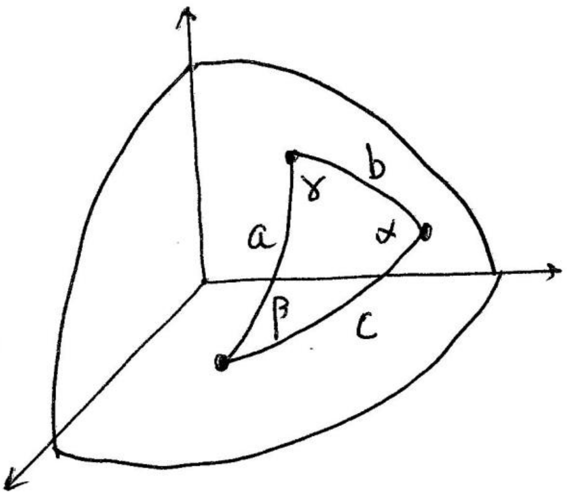
MATH FUNDAMENTALS

• Spherical trigonometry



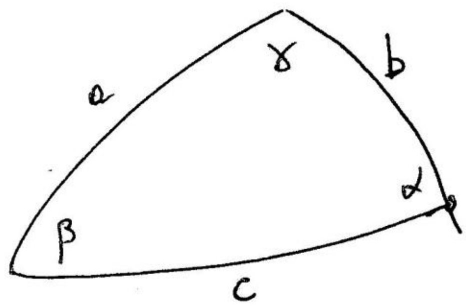
Object:

Relations between SIDES and ANGLES over a unit sphere

SIDES : arcs belonging to GREAT CIRCLES

ANGLES are between two SIDES

OBLIQUE TRIANGLE

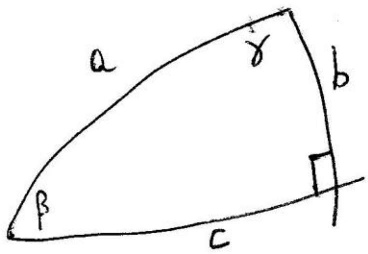


$$\frac{s_a}{s_\alpha} = \frac{s_b}{s_\beta} = \frac{s_c}{s_\gamma} \quad \text{(SINE LAW)}$$

$$\begin{cases} c_a = c_b c_c + s_b s_c c_\alpha \\ c_b = c_a c_c + s_a s_c c_\beta \\ c_c = c_a c_b + s_a s_b c_\gamma \end{cases} \quad \text{(COSINE LAW)}$$

$$\text{def} \begin{cases} c[\ ] = \cos[\ ] \\ s[\ ] = \sin[\ ] \end{cases}$$

RIGHT TRIANGLE



$\alpha = \frac{\pi}{2}$

$$\frac{s_b}{s_\beta} = s_a = \frac{s_c}{s_\gamma} \quad , \quad c_a = c_c c_b$$

$$\frac{\tan b}{\tan \beta} = s_c \ ; \ \frac{\tan c}{\tan \gamma} = s_b$$

$$\tan c = \tan a \cos\beta \ ; \ \tan b = \tan a \, c_\gamma$$

• Vectors and matrices

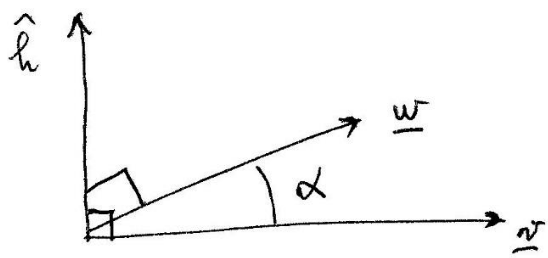
A vector is usually written with respect to a reference system, associated with the orthonormal sequence $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ (right-handed)

$$\underline{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} = v_1 \hat{e}_1 + v_2 \hat{e}_2 + v_3 \hat{e}_3$$

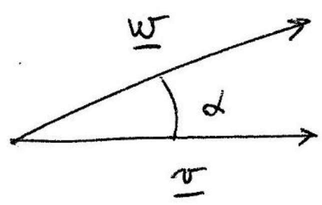
$$v = |\underline{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} \qquad \text{(magnitude)}$$

$$\hat{v} = \frac{\underline{v}}{v} \qquad \text{(unit vector associated with } \underline{v}\text{)}$$

Cross product :
$$\underline{v} \times \underline{w} = v\, w\, s_\alpha\, \hat{h} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$



Dot product:
$$\underline{v} \cdot \underline{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 = v\, w\, c_\alpha$$



- Composite products:

$$\underline{a} \times (\underline{b} \times \underline{c}) = \underline{b}\,(\underline{a} \cdot \underline{c}) - \underline{c}\,(\underline{a} \cdot \underline{b})$$

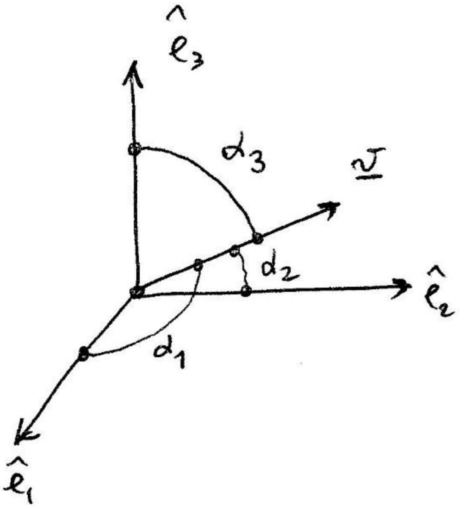
$$\underline{a} \cdot (\underline{b} \times \underline{c}) = \underline{c} \cdot (\underline{a} \times \underline{b}) = \underline{b} \cdot (\underline{c} \times \underline{a})$$

- Direction cosines:

$$\hat{v} \cdot \hat{e}_1 = c_{\alpha_1}$$

$$\hat{v} \cdot \hat{e}_2 = c_{\alpha_2}$$

$$\hat{v} \cdot \hat{e}_3 = c_{\alpha_3}$$



- Matrices $(n \times m)$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}$$

If $n = m \rightarrow$ square matrix

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$A^{-1} = \begin{bmatrix} A_{11} & \cdots & A_{13} \\ \vdots & & \vdots \\ A_{31} & \cdots & A_{33} \end{bmatrix} \quad \text{where} \quad A_{ji} = \frac{\mathrm{Compl}(ij)}{\det A}$$

and $\mathrm{compl}(ij) = (-)^{i+j} \det \bar{A}_{ij}$

$\bar{A}_{ij} = A$ without i-th row and j-th column

- Symmetric matrix: $A^T = A$
- Skew symmetric matrix: $A^T = -A$
- Orthogonal matrix is made up of

  mutually orthogonally row or column unit vectors

  and satisfies:

  (a) $A^T = A^{-1}$

  (b) $\det A = \pm 1$

  (c) if A_1 and A_2 are orthogonal matrices, then

  $A_1 A_2$ is orthogonal as well

• Eigenvalues and eigenvectors of a square matrix

- Eigenvalues and eigenvectors of a square matrix A $(n \times n)$

  $\det(A - \lambda I) = 0$ is the eigenval's equation

  $\{\lambda_K^{(E)}\}$ are the eigenvalues

  solving $\left[A - \lambda_K^{(E)}\right] \underline{v}_k^{(E)} = \underline{0}$ $\quad k = 1, .., n$

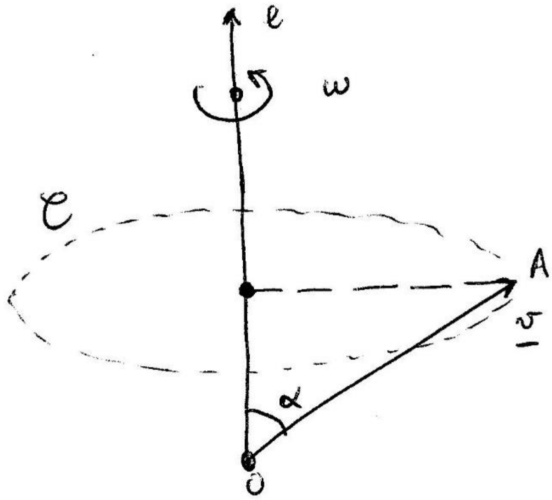
  yields the eigenvectors (either distinct or not) $\underline{v}_k^{(E)}$

→ A symmetric matrix has

  (1) REAL eigenvalues

  (2) ORTHOGONAL eigenvectors

• Vector differentiation

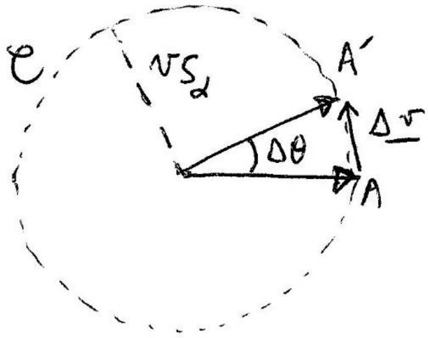


Objective: time derivative of $\underline{v}$ due only to rotation

$\underline{v}$ rotates in counterclockwise sense around $\hat{e}$

↓

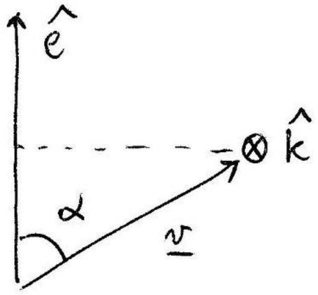
$\underline{v}$ describe a cone with angle α at the vertex



$\underline{v}$ points ~~in a~~ from O to A

A goes to A′ and describes a circle $\mathcal{C}$

$$\Delta v = v s_\alpha \Delta\theta$$



$$\frac{\Delta v}{\Delta t} \xrightarrow[\Delta t \to 0]{} \frac{dv}{dt} = v s_\alpha \left.\frac{\Delta\theta}{\Delta t}\right|_{\Delta t \to 0}$$

i.e. $$\frac{dv}{dt} = v s_\alpha \dot{\theta} = v s_\alpha \omega$$

$\Delta\underline{v} \uparrow\uparrow \otimes\hat{k}$; Moreover $\hat{e} \times \underline{v} = v s_\alpha \hat{k}$

This means that $$\frac{d\underline{v}}{dt} = v s_\alpha \omega \frac{\hat{e} \times \underline{v}}{|\hat{e} \times \underline{v}|} = \omega \hat{e} \times \underline{v}$$

letting $\underline{\omega} := \omega\hat{e}$ one has the derivative of $\underline{v}$ when $\underline{v}$ rotates (in time)

$$\left[\frac{d\underline{v}}{dt} = \underline{\omega} \times \underline{e}\right]$$ Poisson formula

• Transport theorem

This theorem relates the time derivatives of vectors as seen from different reference frames.

Letting $\frac{{}^B d}{dt}\underline{x}$ = derivative of $\underline{x}$ as seen from B

and $\underline{\omega}_{B/N}$ = angular velocity of frame B as seen from N (relative to N)

For a generic vector $\underline{v}$ in the B frame ($[\hat{e}_1, \hat{e}_2, \hat{e}_3]$)

$$\underline{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} = v_1 \hat{b}_1 + v_2 \hat{b}_2 + v_3 \hat{b}_3$$

$$\frac{{}^B d\underline{v}}{dt} = \dot{v}_1 \hat{b}_1 + \dot{v}_2 \hat{b}_2 + \dot{v}_3 \hat{b}_3 \quad \text{because} \quad \frac{{}^B d\hat{b}_i}{dt} = 0 \quad (i=1,2,3)$$

Then

$$\frac{{}^N d\underline{v}}{dt} = \dot{v}_1 \hat{b}_1 + \dot{v}_2 \hat{b}_2 + \dot{v}_3 \hat{b}_3 + \underline{\omega}_{B/N} \times \underline{v} =$$

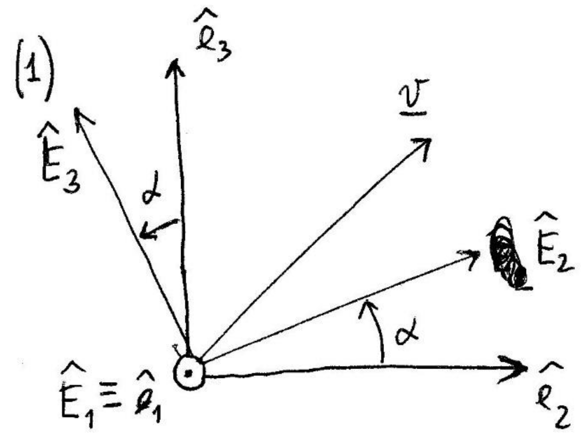
$$= \frac{{}^B d\underline{v}}{dt} + \underline{\omega}_{B/N} \times \underline{v}$$

No assumption is made on N ~~and~~ or B, which can be either inertial or rotating

Notation: if N is inertial, then the superscript is omitted: $\frac{{}^N d}{dt} = \frac{d}{dt}$

• Elementary rotations

(1)



Counterclockwise rotation about axis 1

$$\underline{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}$$

$$= \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \end{bmatrix}$$

From inspection of the above figure

$$\hat{E}_1 = \hat{e}_1$$

$$\hat{E}_2 = \hat{e}_2 \, c_\alpha + \hat{e}_3 \, s_\alpha \quad \rightarrow \quad \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\alpha & s_\alpha \\ 0 & -s_\alpha & c_\alpha \end{bmatrix}}_{R_1(\alpha)} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}$$

$$\hat{E}_3 = \hat{e}_3 \, c_\alpha - \hat{e}_2 \, s_\alpha$$

The coordinates (v_1, v_2, v_3) have a similar relation with (V_1, V_2, V_3). In fact

$$\underline{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} = \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} R_1(\alpha) \cdot \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}$$

Therefore one can easily recognize that

$$[v_1 \;\; v_2 \;\; v_3] = [V_1 \;\; V_2 \;\; V_3]\, R_1(\alpha) \qquad \text{or}$$

$$[V_1 \;\; V_2 \;\; V_3] = [v_1 \;\; v_2 \;\; v_3]\, R_1^{-1}(\alpha) \underset{\substack{\uparrow \\ R_1 \text{ is orthogonal}}}{=} [v_1 \;\; v_2 \;\; v_3]\, R_1^{T}(\alpha)$$

After transposition,

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = R_1(\alpha) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \qquad \text{because } (AB)^T = B^T A^T \text{ (in general)}$$

In a similar way one obtains the remaining elementary rotations:

(2) Counterclockwise about axis 2: $R_2(\alpha) = \begin{bmatrix} C_\alpha & 0 & -S_\alpha \\ 0 & 1 & 0 \\ S_\alpha & 0 & C_\alpha \end{bmatrix}$

(3) Counterclockwise about axis 3: $R_3(\alpha) = \begin{bmatrix} C_\alpha & S_\alpha & 0 \\ -S_\alpha & C_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

It is easy to check that

$$R_k(-\alpha) = R_k^{-1}(\alpha) = R_k^{T}(\alpha)$$

i.e. the clockwise rotation is the inverse of the counterclockwise rotation

- Composite rotations

Let the reference $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ be the result of two consecutive rotations : $R_k(\alpha_1)$ followed by $R_j(\alpha_2)$ starting from $(\hat{E}_1, \hat{E}_2, \hat{E}_3)$

$$\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} \xleftarrow[R_j(\alpha_2)]{} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} \xleftarrow[R_k(\alpha_1)]{} \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \end{bmatrix}$$

$$\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = R_k(\alpha_1) \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} = R_j(\alpha_2) \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$

Therefore

$$\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} = \underbrace{R_j(\alpha_2)\, R_k(\alpha_1)} \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \end{bmatrix}$$

composite rotation matrix $R_{IB} = \underset{B \leftarrow I}{R}$ (notation)

(from frame I to frame B)

I $(\hat{E}_1, \hat{E}_2, \hat{E}_3)$ B $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$