Chapter 1

Strain gauge measurements

1.1 Resistance strain gauges

Let us consider a wire conductor, consisting of a material of resistivity ρ, of lenght ℓ and of section A; its resistance is given by the relationship:

$$R = \frac{\rho\ell}{A} \tag{1.1}$$

if the conductor is subjected to a deformation, e.g. to a traction which causes an increase in its length and a contraction in its section, a resistance variation is obtained according to:

$$dR \;=\; \left(\frac{\partial R}{\partial \rho}\right) d\rho \;+\; \left(\frac{\partial R}{\partial \ell}\right) d\ell \;+\; \left(\frac{\partial R}{\partial A}\right) dA \tag{1.2}$$

that becomes:

$$\frac{dR}{R} \;=\; \frac{d\rho}{\rho} \;+\; \frac{d\ell}{\ell} \;-\; \frac{dA}{A} \tag{1.3}$$

If we devote by r a characteristic dimension of the wire cross-section, e.g. the radius, and by ν the Poisson's ratio of the conductor material, we have:

$$\frac{dA}{A} \;=\; \frac{2dr}{r} \;=\; -\;\frac{2\nu d\ell}{\ell} \tag{1.4}$$

in the case of a circular cross-section wire we have:

$$\begin{aligned} A &= \pi r^2 && (1.5) \\ dA &= 2\pi r \; dr && (1.6) \\ & && (1.7) \end{aligned}$$

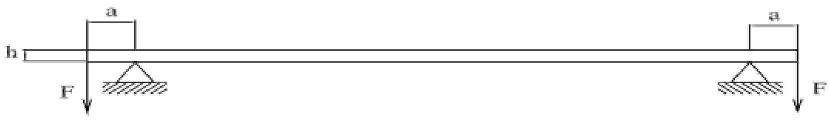

Figure 1.1: Strain gauge calibration test.

and so:

$$\frac{dA}{A} = \frac{2dr}{r} \tag{1.8}$$

then:

$$\frac{dR}{R} = \frac{d\rho}{\rho} + \frac{d\ell}{\ell} + \frac{2\nu d\ell}{\ell} \tag{1.9}$$

and placing:

$$\frac{d\ell}{\ell} = \varepsilon_x \tag{1.10}$$

we get:

$$\frac{dR}{R} = \frac{d\rho}{\rho} + \varepsilon_x(1+2\nu) \tag{1.11}$$

By indicating with K the extensometer "calibration factor" which represents its sensitivity, i.e. the variation in resistance as a function of strain, Eq. 1.11 can be written as:

$$\frac{dR}{R} = K\,\varepsilon_x \tag{1.12}$$

where:

$$K = (1+2\nu) + \left(\frac{1}{\varepsilon_x}\right)\frac{d\rho}{\rho} \tag{1.13}$$

in case there is no variation of resistivity ρ as a function of deformation, from Eq. 1.13 we have:

$$K = 1+2\nu \tag{1.14}$$

Eq. 1.14 identifies K values ranging between 1.5 and 2, typically of around 1.6. Actually, its value for a resistance strain gauge is higher: generally K is very close to 2. This increment with respect to that proposed by Eq. 1.14 is due to a resistivity variation. This variation, however, is practically constant with strain, at least for a sufficiently high range of strain values (up to a few per cent with a limit value of up to 4 %).
The K value to be used in the measurement is determined experimentally and provided by the manufacturer. Table 1 shows the characteristics of some alloys commonly used for resistance strain gauges.

Material	**Composition**	**K factor**
Constantan	45 Ni, 55 Cu	2.1
Nickel chrome	80 Ni, 20 Cr	2.1
Karma	74 Ni,20 Cr,3 Al,3 Fe	2.0
Platinum tungsten	92 Pt, 8 W	4.0

Table 1: characteristics of alloys for extensometry

As seen from Eq. 1.12, the resistance variation due to deformation is very small; for example, with:

$$\begin{aligned} R &= 120\ ohm \\ \varepsilon_x &= 100\ \mu s \\ K &= 2 \end{aligned} \tag{1.15}$$

is obtained from Eq. 1.12:

$$dR = K\ \varepsilon_x\ R = 2.4\ 10^{-2}\ ohm \tag{1.16}$$

For a manufacturing point of view, the strain gauges can be either wire or photo-etched.
Wire strains gauges have quite large areas, of the order of mm^2, and wire diameter varies between 0.01 and 0.025 mm. They have some advantages, such as robustness, but various limitations due to the heating related to the Joule effect.
Nowadays, photo-etched strain gauges are much more common, which make it possible to "draw" grids with a wide variety of configurations. The minimum grid size can be as small as 0.05 mm, compared to 2 mm for wire strain gauges, and the non-circular cross-section of the conductor allows a larger radiating surface and thus greater heat dissipation.
The experimental determination of the calibration factor K is carried out by subjecting a test specimen, of known geometry and elastic characteristics, to a strain state which is also known. Let us consider a beam subjected to a force F, Fig. 1.1; between the supports it is stressed by a constant bending moment given by:

$$M_f = F\ a \tag{1.17}$$

and therefore the deformation on the surface of the beam where the extensometer is glued is constant. If the beam geometry and the material elastic characteristics are known, it is possible to evaluate the K factor from the experimental measurement of dR/R using the following relation:

$$K = \left(\frac{1}{\varepsilon}\right)\frac{dR}{R} \tag{1.18}$$

Due to the bonding and the extensometer thickness, the measuring grid is actually at a distance from the beam neutral which is different from the geometric reference distance h. However, the additional thickness h^* is a few hundreads of a mm, which is much less than the beam geometric dimension $h/2$, and therefore its effect is negligible.
Of course, K can be evaluated by other tests than the one presented, e.g. by simple tensile tests for for which the position does not matter; in general, the uncertainty with which the K factor is 1%.
The strain gauge provides correct values as long as it is used in the case of monoaxial stress on a material with a Poisson's coefficient, ν, equal to that of the material used for extensometer calibration (generally $\nu_{tar} = 0.285$). Two calibration factors must be considered, denoted by K_a and K_t; the subscripts a and t stand for axial and transverse, which correspond respectively to the case in which the strain gauge axis is parallel to the uniaxial strain direction and to the case in which it is orthogonal. Thus we have:

$$\frac{dR}{R} = K_a\ \varepsilon_a + K_t\ \varepsilon_t \tag{1.19}$$

where ε_a, ε_t are respectively the deformations in the parallel and orthogonal directions to the strain gauge axis. We have:

$$\frac{\varepsilon_t}{\varepsilon_a} \; = \; -\,\nu_{tar} \tag{1.20}$$

where ν_{tar} is the Poisson's coefficient of the material for which the extensometer has been calibrated. Therefore:

$$\frac{dR}{R} \; = \; K_a\;\varepsilon_a \; - \; K_t\nu_{tar}\varepsilon_a \; = \; K_a\left(1-\nu_{tar}\frac{K_t}{K_a}\right)\varepsilon_a \tag{1.21}$$

Therefore, the calibration factor K in Eq. 1.12 is:

$$K \; = \; K_a\;\left(1-\nu_{tar}\frac{K_t}{K_a}\right) \tag{1.22}$$

The ratio between the transversal calibration coefficient and the axial one is known as "transverse sensitivity":

$$S_t \; = \; \frac{K_t}{K_a} \tag{1.23}$$

Then 1.22 can be written as:

$$K \; = \; K_a\;\left(1-\nu_{tar}S_t\right) \tag{1.24}$$

An error occurs if we use Eq. 1.12 with $\nu \neq \nu_{tar}$ unless $S_t \; = \; 0$ or the deformation field is uniaxial. If refer is made to a strain gauge in a biaxial field with axial and transverse deformations, ε_a and ε_t, the magnitude of the error can be assessed with the following procedure. Starting from:

$$\frac{dR}{R} \; = \; K_a\left(1+S_t\frac{\varepsilon_t}{\varepsilon_a}\right)\varepsilon_a \; = \; \left[\frac{K}{(1-\nu_{tar}S_t)}\right]\left(1+S_t\frac{\varepsilon_t}{\varepsilon_a}\right)\;\varepsilon_a \tag{1.25}$$

we get:

$$\varepsilon_a \; = \; \left[\frac{(dR/R)}{K}\right]\frac{(1-\nu_{tar}S_t)}{\left(1+S_t\dfrac{\varepsilon_t}{\varepsilon_a}\right)} \tag{1.26}$$

The deformation $\varepsilon_a^{'}$, obtained by considering only the extensometer factor, can be found:

$$\varepsilon_a^{'} \; = \frac{(dR/R)}{K} \tag{1.27}$$

therefore:

$$\varepsilon_a \; = \; \frac{\left[\varepsilon_a^{'}\,(1-\nu_{tar}S_t)\right]}{\left[1+S_t\left(\dfrac{\varepsilon_t}{\varepsilon_a}\right)\right]} \tag{1.28}$$

From the definition:

$$\eta_a = \frac{\left(\varepsilon_a^{'} - \varepsilon_a\right)}{\varepsilon_a} \tag{1.29}$$

the error due to transverse deformation is:

$$\eta_a = \frac{\left[S_t\left(\frac{\varepsilon_t}{\varepsilon_a} + \nu_{tar}\right)\right]}{(1 - \nu_{tar} S_t)} \simeq S_t\left(\frac{\varepsilon_t}{\varepsilon_a} + \nu_{tar}\right) \tag{1.30}$$

For example, in the case of a strain gauge with $S_t = 0.03$, bounded to a stressed specimen with biaxial strains $\varepsilon_t/\varepsilon_a = -0.4$, an error of approx. 0.3% occurs.
The effect of transverse sensitivity is important from a general measurement point of view. However, if the transverse sensitivity is of the order of one per thousand, the consequent error can be considered included in the uncertainty with which the calibration factor K is provided (generally around one per cent). For example if $S_t = 0.001$, under the same conditions as above, we have $\eta_a = 0.0001$ which is negligible as it is included in the uncertainty of the K factor.
Nevertheless, by using two strain gauges simultaneously, it is possible to obtain the actual deformations with the following relations

$$\varepsilon_x = (1 - \nu_{tar} S_t)\left(\varepsilon_x^* - S_t \varepsilon_y^*\right) \tag{1.31}$$

$$\varepsilon_y = (1 - \nu_{tar} S_t)\left(\varepsilon_y^* - S_t \varepsilon_x^*\right) \tag{1.32}$$

where ε_x^*, ε_y^* are the measured deformations and ε_x, ε_y are the actual ones.
Materials used to manufacture strain gauges are likely to have:

- linear variation of resistance with deformation, both in tension and in compression;
- high calibration factor K;
- high elastic limit;
- great fatigue resistance.

Table 2 shows some fundamental characteristics for two materials normally used in extensometry: Costantana and Karma.

characteristic	**constantan**	**karma**
calibration factor K	2.1	2.1
resistivity $(\Omega m \times 10^{-4})$	48	125
coeff. of variation of resistance γ $(\Delta R/R/\ ^oC \times 10^{-4})$	0.3	0.2
yield stress (MPa)	460	1000

Table 2: characteristics of materials for strain gauges

Temperature variations have both a direct effect on material behaviour and an indirect effect on connection characteristics leading to changes in the calibration factor K.
Fig. 1.2 shows the trend of $\Delta K/K$ depending on the temperature; it can be seen that in a range

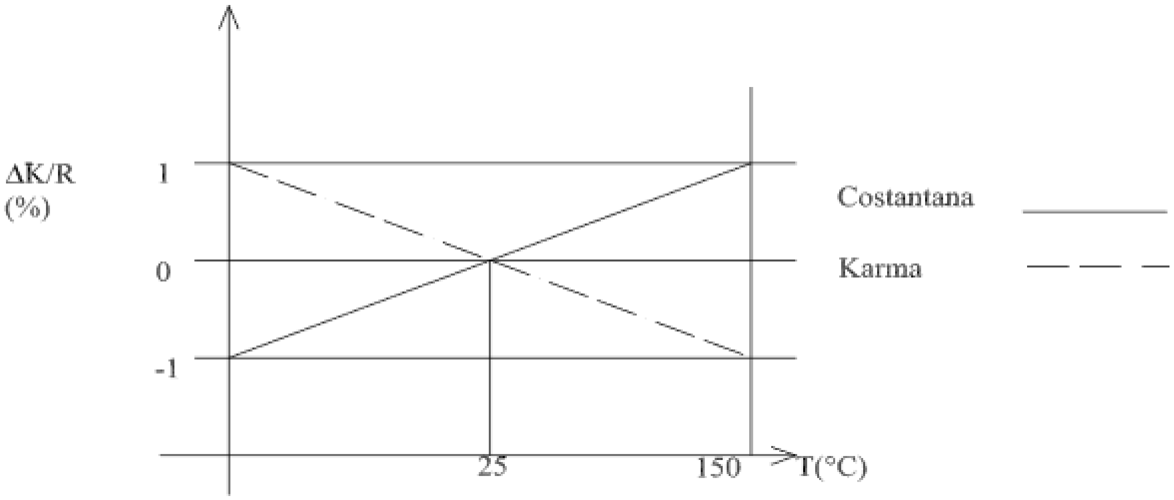

Figure 1.2: Typical curves of the strain gauge calibration factor as a function of temperature.

of plus or minus 50 ^oC around room temperature the K variation is limited to one percent. In the case of linear trend we have:

$$K(T) \; = \; K\,(1 + \beta^*(T - T_0)) \tag{1.33}$$

where the coefficient β^* is estimated from the trends shown in Fig. 1.2.

1.2 Temperature effect

The resistance variation due to temperature is given by:

$$\Delta R \; = \; \gamma \; R \; \Delta T \tag{1.34}$$

where ΔT is the temperature change with respect to a reference value, generally the reference T_0 is the room temperature, and γ is the coefficient of variation of resistance with temperature, expressed in $\Delta R/R/\ ^oC$ with typical values of 40×10^{-4} for platinum and of 3×10^{-5} for costantana.
Therefore, in the case of a platinum extensometer, there is a resistance variation per degree equal to:

$$\frac{\Delta R}{R} \; = \; 40 \times 10^{-4} \; 1/\ ^oC \tag{1.35}$$

to which corresponds an apparent deformation:

$$\varepsilon_{app} \; = \; (1/K)\Delta R/R \; = \; 830 \; \mu s/\ ^oC \tag{1.36}$$

Strain gauges made of platinum have higher K values than those made of costantana, generally $K_{pla} = 4.8$.
In the case of costantana strain gauge we have:

$$\varepsilon_{app} \; = \; 15 \; \mu s/\ ^oC \tag{1.37}$$

The difference is great (about two orders of magnitude) and leads to major difficulties in using platinum extensometer. However, also the relatively small value of the apparent deformation for the constantana strain gauge causes a non-negligible effect, even for temperature variations of only a few degrees.

The effect due to the temperature can be compensated for in different ways. For example, two strain gauges can be used on two active branches of the measuring bridge: they are brought to the same temperature but only one is stressed, so that the apparent deformation due solely to temperature variation can be eliminated.

As an example of self-compensation, let us consider a constantana strain gauge, with a thermal expansion coefficient of $\alpha_{ext} = 15 \times 10^{-6}\ ^oC^{-1}$ placed on a steel test specimen, with $\alpha_{spec} = 11 \times 10^{-6}\ ^oC^{-1}$ which is heated to a temperature T. Since the thermal expansion coefficient of constantan is greater than that of steel, the stran gauge would expand more than the specimen. Assuming that the strain gauge completely follows the deformation of the structure, the strain gauge contracts resulting in a decrease in resistance given by: :

$$\frac{\Delta R}{R} = K\ \varepsilon_{therm} \tag{1.38}$$

but:

$$\varepsilon_{therm} = (\alpha_{spec} - \alpha_{ext})\,\Delta T \tag{1.39}$$

therefore the resistance variation is:

$$(\Delta R/R)_{therm} = K\ (\alpha_{spec} - \alpha_{ext})\,\Delta T \tag{1.40}$$

The total resistance variation, due to the temperature variation ΔT, is:

$$(\Delta R/R)_{tot} = (\gamma + K\,(\alpha_{spec} - \alpha_{ext}))\,\Delta T \tag{1.41}$$

hence the total apparent deformation due to temperature:

$$\varepsilon_{app} = (\gamma/K + (\alpha_{spec} - \alpha_{ext}))\,\Delta T \tag{1.42}$$

ε_{app} varies linearly with the temperature in the case where γ, α_{spec}, α_{ext}, K are constant with temperature (which actually is not the case). Tab 2.1 shows the thermal expansion coefficients of some materials.

material	$\alpha\ \times\ 10^{-6}\ ^oC^{-1}$
Quartz	.5
Glass	9.0
Titanium	9.3
Steel	11.8
Copper	17.6
Aluminum	22.5
Magnesium	25.9
Epoxy resin	90.0
Acrylic resin	180.0

Table 2.1: thermal expansion coefficients.

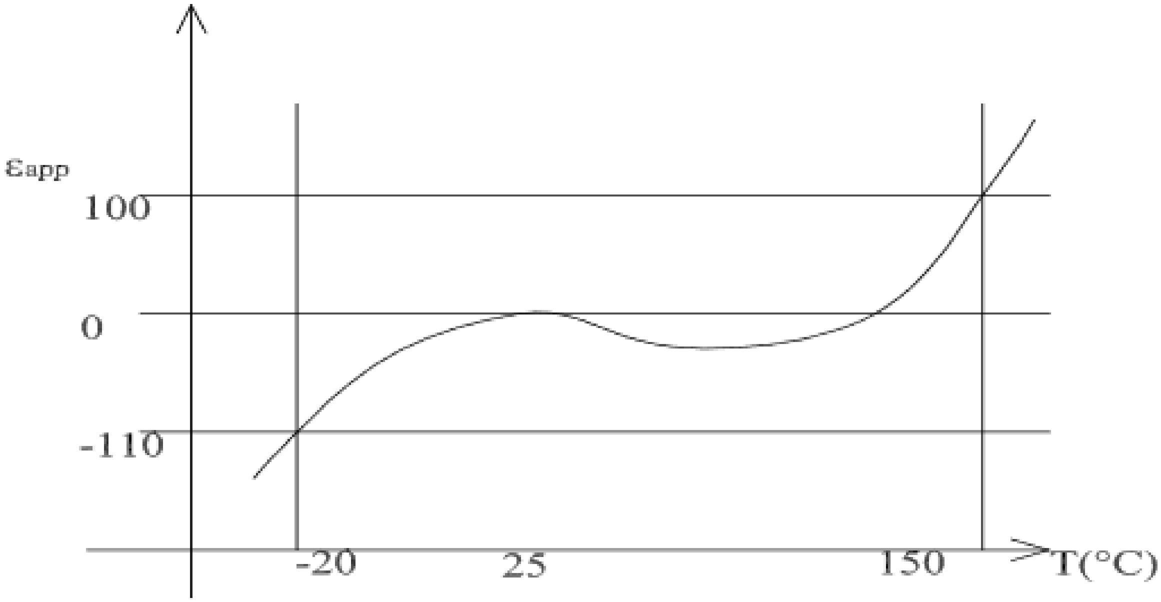

Figure 1.3: Trend of ε_{app} as a function of temperature.

It is observed that, at least for a certain temperature range, a self-compensation effect can be obtained if:

$$\gamma/K + (\alpha_{prov} - \alpha_{est}) = 0 \tag{1.43}$$

Therefore, for each strain gauge and depending on the material on which it is to be applied, a self-compensation curve of the type shown in Fig. 1.3 is provided. The self-compensation curve provides ε_{app} as a function of temperature allowing its value to be assessed under working condition:

$$\varepsilon^* \;=\; \varepsilon_{mis} - \varepsilon_{app} \tag{1.44}$$

where ε^* is the actual strain value and ε_{mis} is the measured one.
Of course the curve provides a partial compensation not including, e.g, the variation of the strain gauge K factor with temperature. To take this effect into account, one can apply the formula:

$$\varepsilon^* \;=\; (\varepsilon_{mis} - \varepsilon_{app})\, K/K_T \tag{1.45}$$

Environmental effects are also important. In particular, the need for protection from atmospheric humidity can become very complex in the case of prolonged exposure to saline conditions.
Furthermore, pressure acting on the strain gauge may lead to a deformation:

$$\varepsilon_p \;=\; -\,(1-2\nu)p/E \;=\; K_p p \tag{1.46}$$

where K_p is the compressibility factor; several experiments have shown that the correction effect proposed by Eq. 1.46 is not sufficient.

Worthnoting is the case of cyclic stress conditions, i.e. fatigue tests, as the experiment can last millions of cycles. A first effect is the zero "drift": during the measurement the reference value shifts and an apparent fatigue deformation follows; the manufacturer provides a graph which indicates the zero drift as a function of the number of cycles performed. A second effect is the possible triggering of "cracks", which can cause the extensometer to break.
Typical resistance values of commercial strain gauges are: 120, 350, 600 and 1000 *ohms*. On the one hand, a high resistance value favors measurement sensitivity but it is accompanied by practical problems (e.g. the higher the resistance, the higher the electical insulation must be). On the other hand, small resistance values, in addition to lower sensitivity, present greater problems with regard to the disturbing effect of connecting cables.
The effect related to heat dissipation is also very important; the dissipated power, when the strain gauge is connected to the Wheatstone bridge, is given by:

$$P = V^2/R = i^2R \tag{1.47}$$

where V is the bridge supply voltage and R is the strain gauge resistance;
Important factor in assessing dissipation characteristics are:

- extensometer size;
- grid configuration;
- adhesive type;
- specimen material;
- protective treatment;
- ventilation.

The power density, P_D is the ratio between the power to be dissipated, P, and the surface of the extensometer, A_e.

$$P_D = P/A_e \tag{1.48}$$

Table 2.2 shows the admissible P_D for different materials to which the extensometer could be attached.

material	P_D (W/mm^2)
Aluminum	0.008—0.016
Steel	0.003 —0.008
Glass, ceramics	0.0003 —0.0008
Plastics	0.00003 —0.00008

Tabella 2.2: dissipable power density

With a Wheatstone bridge configuration, using four equal active branches, the bridge voltage V_B is related to the allowable power density:

$$V_B^2 \; = \; 4 \; A_e \; P_D \; R \tag{1.49}$$

Since the supply voltages used in the bridge are usually of the order of a few Volts, high resistance strain gauges should be used when connecting to materials with low heat dissipation.
In addition, it is important to consider the effect of very high and very low temperatures on strain gauge measurements. At high temperatures the resistance is a function of deformation, temperature and time $R \; = \; f(\varepsilon, T, t)$. We have therefore:

$$\Delta R/R \; = \; (1/R)(\partial f \varepsilon)\Delta\varepsilon + (1/R)(\partial f T)\Delta T + (1/R)(\partial f t)\Delta t \tag{1.50}$$

If we denote by:

$$\begin{aligned} K \; &= \; (1/R)\partial f/\partial\varepsilon \\ K_T \; &= \; (1/R)\partial f/\partial T \\ K_T \; &= \; (1/R)\partial f/\partial t \end{aligned} \tag{1.51}$$

respectively the extensometer calibration factor, the temperature sensitivity and the time duration sensitivity, we have:

$$\Delta R/R \; = \; K\Delta\varepsilon \; + \; K_T\Delta T \; + \; K_t\Delta t \tag{1.52}$$

As seen above, sensitivity to temperature variations is minimal in the temperature range of $-20\ ^oC$ to $70\ ^oC$. For much higher temperature values, compensation is not sufficient and corrective factors must be used to account for apparent deformation.
In general, karma is more suitable than constantana for use at high temperature up to $260\ ^oC$.
In the case of very low temperatures, such as a $-196\ ^oC$, there are two effects. The first is related to the variation of the factor K with temperature, as shown in Fig. 1.4. These variations are limited, around -2% for costantana and about 4% for karma at a temperature of $-200\ ^o$ C. The second effect is related to high apparent deformation as a function of small temperature variations. If temperature sensors are used together with the extensometer, then the measured temperatures can be used to evaluate apparent deformations.
Typically, cryogenic temperatures are achieved with liquid nitrogen, liquid hydrogen or liquid helium. Since these are insulating materials, they do not require special protection between the strain gauge and the liquid.
Furthermore, at very low temperature there are significant changes in mechanical properties, e.g. the modulus of elasticity for some materials varies significantly (at temperatures as low as $-200\ ^oC$ there wuold be an increase of between 5% and 20% compared to the corresponding value at room temperature).

1.3 Semiconductor strain gauges

This is a sensor consisting of a crystal, e.g. silicon, which has a very high sensitivity to deformation: calibration factor K of the order of 200 can be achieved, depending on the type and extent

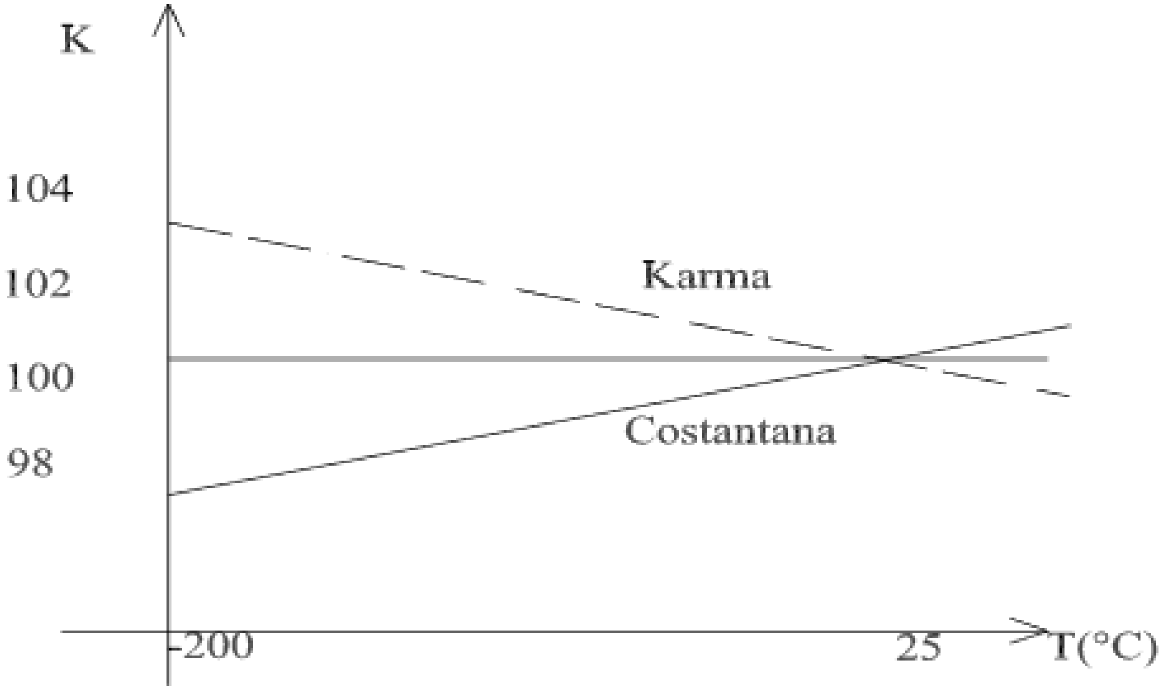

Figure 1.4: Strain gauge factors as a function of temperature.

of "doping" [1] in the crystal. The commercial development of this type of transducer dates back to the 1960s.

The extensometer factor, as seen previously in the case of the resistance extensometer (in the introductory example for an ideal cylindrical conductor), is given by:

$$K \;=\; (1+2\nu)+(1/\varepsilon)d\rho/\rho \tag{1.54}$$

the term $(1+2\nu)$ is close to 1.6 while the term related to the variation of resistivity (which in the case of resistance strain gauges is between 0.4 and 2.5) can reach much higher values, of the order of 100 or 200. In practice, K values ranging from negative values of -150 to positive values of +175 [2] can be obtained. This is a much higher sensitivity, about two orders of magnitude, than that of resistance strain gauges; furthermore, strain gauges with negative K values can be used with appropriate connections in the Wheatstone bridge.

Generally, silicon crystals are used. Boron is used for P-type impurities, which give a positive K value, and arsenic for N-type impurities, which give a negative K value.

The great value of the material resistivity[3] (about a thousand times higher than that of costan-

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[1] The semiconductor, due to impurities with concentrations of $10^{16}-10^{20}$ $atoms/cm^3$, can be expressed by:

$$\rho \;=\; 1/(e\;N\;\mu) \tag{1.53}$$

where e is the electron charge, which depends on the type of doping, N and μ are the number and mobility of the particles and depend on the amount of doping and on the magnitude and direction of deformation.

[2] Positive K factors are obtained with P-type doping (e.g. with barium) and negative K factors are obtained with N-type doping (e.g. with arsenic)

[3]

tana) allows not to build the "strain gauge grid" as only one element is sufficient to make a highly sensitive strain gauge. Therefore, these sensors can detect very small and can be used in miniature transducers.
A semiconductor crystal is electrically anisotropic. The relationship between the electric field and the current density is established in relation to the crystal axes, denoted by the indices 1, 2, 3, accordingly to the following matrix relation:

$$\begin{Bmatrix} E_{c_1} \\ E_{c_2} \\ E_{c_3} \end{Bmatrix} = \begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{bmatrix} \begin{Bmatrix} j_1 \\ j_2 \\ j_3 \end{Bmatrix} \tag{1.55}$$

the first subscript in the term ρ_{ij} of the *resistivity matrix* indicates the field component while the second one denotes the current component.
Isotropic conduction occurs only if:

$$\begin{Bmatrix} E_{c_1} \\ E_{c_2} \\ E_{c_3} \end{Bmatrix} = \begin{bmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho \end{bmatrix} \begin{Bmatrix} j_1 \\ j_2 \\ j_3 \end{Bmatrix} \tag{1.56}$$

In the case of an unstressed crystal 1.56 is verified and 1.55 reduces to:

$$E_{c_1} = \rho\, j_1 \qquad E_{c_2} = \rho\, j_2 \qquad E_{c_3} = \rho\, j_3 \tag{1.57}$$

When the crystal is stressed there is a piezoresistive effect that can be described by the relation:

$$\rho_{ij} = \delta_{ij}\rho + \pi_{ijkl}\sigma_{kl} \tag{1.58}$$

where the subscripts $ijkl$ range from 1 to 3 and the tensor π is a function of the crystal and of the type and extent of impurities.
Considering a silicon cubic crystal and referring to its intrinsic reference system, the piezoresistive coefficients are reduced to three independent ones and the π matrix has the structure:

$$\pi = \rho \begin{bmatrix} \pi_{11} & \pi_{12} & \pi_{12} & 0 & 0 & 0 \\ \pi_{12} & \pi_{11} & \pi_{12} & 0 & 0 & 0 \\ \pi_{12} & \pi_{12} & \pi_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \pi_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \pi_{44} \end{bmatrix} \tag{1.59}$$

From 1.59 we have the relationships:

$$\pi_{1111} = \pi_{2222} = \pi_{3333} = \rho\pi_{11} \tag{1.60}$$

and similar ones, thus introducing three independent coefficients. Then we have:

$$\begin{aligned} \rho_{11} &= \rho\left[1 + \pi_{11}\sigma_{11} + \pi_{12}\left(\sigma_{22} + \sigma_{33}\right)\right] \\ \rho_{22} &= \rho\left[1 + \pi_{11}\sigma_{22} + \pi_{12}\left(\sigma_{33} + \sigma_{11}\right)\right] \\ \rho_{33} &= \rho\left[1 + \pi_{11}\sigma_{33} + \pi_{12}\left(\sigma_{11} + \sigma_{22}\right)\right] \\ \rho_{12} &= \rho\pi_{44}\sigma_{12} \\ \rho_{23} &= \rho\pi_{44}\sigma_{23} \\ \rho_{31} &= \rho\pi_{44}\sigma_{31} \end{aligned} \tag{1.61}$$

The resistivity in the case of P-type semiconductors is of the order of 500 $\mu\Omega m$, while for the constantana is 0.5 $\mu\Omega m$.

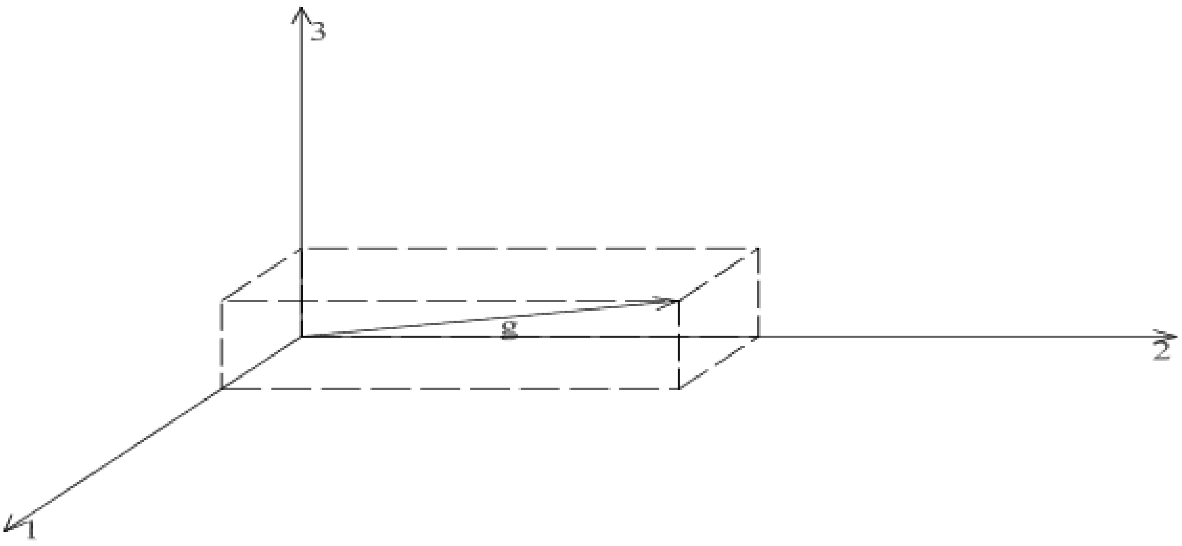

Figure 1.5: Geometry of a generic axial semiconductor (unit vector g)

By substituting 1.61 in 1.55 we have:

$$\begin{aligned} E_{c1}/\rho &= j_1\left[1+\pi_{11}\sigma_{11}+\pi_{12}\left(\sigma_{22}+\sigma_{33}\right)\right]+\pi_{44}\left(i_2\sigma_{12}+i_3\sigma_{31}\right) \\ E_{c2}/\rho &= j_2\left[1+\pi_{11}\sigma_{22}+\pi_{12}\left(\sigma_{33}+\sigma_{11}\right)\right]+\pi_{44}\left(i_3\sigma_{23}+i_1\sigma_{12}\right) \\ E_{c3}/\rho &= j_3\left[1+\pi_{11}\sigma_{33}+\pi_{12}\left(\sigma_{11}+\sigma_{33}\right)\right]+\pi_{44}\left(i_1\sigma_{31}+i_2\sigma_{23}\right) \end{aligned} \tag{1.62}$$

These relationships show that the tension on the element depends on the current density, on the stress and on the piezoresistive coefficients. If we now refer to an element in an arbitrary position with respect to the axes of a cubic crystal, Fig. 1.5, indicated by the unit vector g of directional cosines l, m, n, we have the relationship:

$$\begin{aligned} j_1 &= l\ i_g \\ j_2 &= m\ i_g \\ j_3 &= n\ i_g \end{aligned} \tag{1.63}$$

where i_g is the current in the element. Similarly, the stresses along the crystal axes, expressed in terms of the stress σ_g in the reference system of the element, are:[4]

$$\begin{aligned} \sigma_{11} &= l^2\ \sigma_g\ ;\ \sigma_{22} = m^2\ \sigma_g\ ;\ \sigma_{33} = n^2\ \sigma_g \\ \sigma_{12} &= l\ m\ \sigma_g\ ;\ \sigma_{23} = m\ n\ \sigma_g\ ;\ \sigma_{31} = n\ l\ \sigma_g \end{aligned} \tag{1.64}$$

we have also:

$$E \cdot g = E_g = l\ E_{c_1} + m\ E_{c_2} + n\ E_{c_3} \tag{1.65}$$

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[4] If U makes it possible to go from the components of a vector in the reference having the vector g as third axis to its components 1 2 3, then $T = UT_gU^T$, being $T_g = \begin{bmatrix} 0,0,0 \\ 0,0,0 \\ 0,0,\sigma_g \end{bmatrix}$ ed U having last column equal to (lmn)

by substituting 1.62 and 1.64 in 1.65, we have:

$$E_g/\rho \;=\; i_g\left[1+\sigma_g\left(\pi_{11}+2\left(\pi_{12}+\pi_{44}-\pi_{11}\right)\left(l^2m^2+m^2n^2+n^2l^2\right)\right)\right] \tag{1.66}$$

that can be written as:

$$E_g/\rho \;=\; i_g\left(1+\pi_g\sigma_g\right) \tag{1.67}$$

where π_g is the element stress sensitivity and is related to the semiconductor piezoresistive coefficients by the relations:

$$\pi_g \;=\; A+B\left(l^2m^2+m^2n^2+n^2l^2\right) \tag{1.68}$$

with $A=\pi_{11}$ and $B=2\left(\pi_{12}+\pi_{44}-\pi_{11}\right)$.
From 1.68 it can be seen that the stress sensitivity can be varied wheter by acting on the semiconductor orientation (i.e. by varying the directional cosines l, m, n) or on the extent and type, P or N, of impurities.
The conditions for optimizing the direction g can be obtained by calculating the derivatives of the sensitivity π_g with respect to the two independent directional cosines, i.e. with the relations:

$$\begin{aligned}\partial\pi_g l \;&=\; l\left(2-4l^2-2m^2\right) \;=\; 0\\ \partial\pi_g m \;&=\; m\left(2-4m^2-2l^2\right) \;=\; 0\end{aligned} \tag{1.69}$$

From 1.67 we see that the difference of E_g before and after the stress is given by:

$$\Delta E_g/\rho \;=\; i_g\left(1+\pi_g\sigma_g\right)-i_g \;=\; i_g\pi_g\sigma_g \tag{1.70}$$

normalizing 1.70 with respect to $E_g=i_g\rho$, we get:

$$\Delta E_g/E_g \;=\; \Delta R_g/R_g \;=\; \pi_g\sigma_g \tag{1.71}$$

In the semiconductor there is a uniaxial stress state with:

$$\sigma_g \;=\; E_s\,\varepsilon \tag{1.72}$$

where E_s is the silicon elasticity modulus and ε it is the deformation that is transmitted to the semiconductor by the structure under test. From 1.72 and 1.71, we have:

$$\Delta R_g/R_g \;=\; \pi_g E_s\varepsilon \;=\; K_{sc}\varepsilon \tag{1.73}$$

where K_{sc} can assume very high values up to about 200. As seen from 1.54 the extensometer calibration factor is given by:

$$K=(1+2\nu)+K_{sc} \tag{1.74}$$

where the term $(1+2\nu)$ is due to dimensional variations and has a value between 1.6 and 2.0; due to the very high value of K_{sc}, the contribution of this term is relatively unimportant.
The response of the semiconductor extensometer is highly non-linear, with respect to the deformation itself and temperature. A relationship of the type below can be considered:

$$K(T,\varepsilon) \;=\; \left(T_0/T\right)K_0+C_1\left(T_0/T\right)^2\varepsilon+C_2\left(T_0/T\right)^3\varepsilon^2+... \tag{1.75}$$

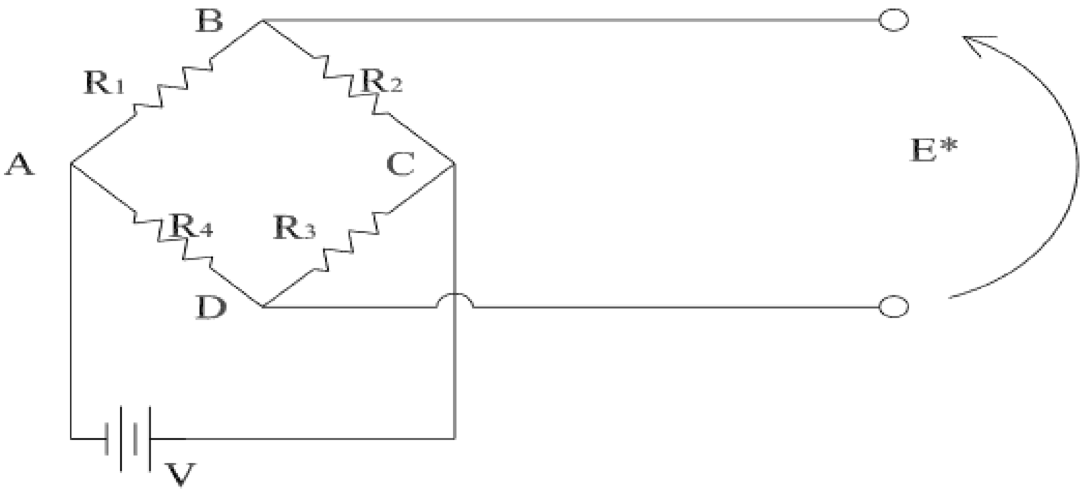


Figure 1.6: Wheatstone bridge with constant voltage scheme

where K_0 is the extensometer factor corresponding to the reference temperature, T_0, and zero strain, T and ϵ are respectively the working temperature and deformation, C_1 and C_2 are constants which depend on the semiconductor characteristics.
A temperature compensation effect can be achieved by using two strain gauges, exploting the effect of positive and negative doping. In this way, K factor can reach up values of 250 with an apparent deformation limited to only 0.5 $\mu s/^oC$ in a temperature range between 10 and 50 ^oC.
From 1.75 we see the non-linear relation between the response of a semiconductor extensometer and the deformation; for low concentration levels the non-linearity is significant while the trend is practically linear with an appropriate choice of the concentration value.
The fatigue life of semiconductor strain gauges is generally more limited than that of resistance strain gauges, and can reach up to 10^7 cycles for a strain level deformation of the order of 500 μs; a wide selection of P-type strain gauges is available, with K values ranging between 50 and 150, as well as of N-type strain gauges, with K values ranging between -100 and -150. Dimensions are between 0.5 mm and a few mm.
As mentioned above, semiconductor strain gauges can be used in miniature transducers, such as accelerometers or pressure transducers, which are characterized by high sensitivity and good frequency response.
The maximum current that can flow through the semiconductor strain gauge is limited by the heat that can be dissipated; this quantity depends on the element size, on the connective adhesive characteristics and on the properties of the structure under test. Indicatively, the power that can be dissipated per unit of length is of the order of 4-8 W/m, which leads to a dissipable value of 0.01 W for a single semiconductor extensometer.

1.4 The Wheatstone bridge with constant voltage

Let us consider the Wheatstone bridge in the configuration with only resistances on its branches, Fig. 1.6. By indicating with V the voltage applied to the terminals A C, we have:

$$
\begin{aligned}
V_{AB} &= \frac{VR_1}{R_1 + R_2} && (1.76)\\
V_{AD} &= \frac{VR_4}{R_3 + R_4} && (1.77)\\
E^* &= V_{BD} = V_{AB} - V_{AD} && (1.78)\\
E^* &= \frac{(R_1R_3 - R_2R_4)V}{(R_1 + R_2)(R_3 + R_4)} && (1.79)
\end{aligned}
$$

The voltage at the nodes BD, denoted by E^*, is zero, and therefore the bridge is in equilibrium, if the following condition is verified:

$$R_1R_3 = R_2R_4 \tag{1.80}$$

Therefore, 1.80 is known as balancing condition of the bridge. It is used for static strain gauge measurements by placing the measuring extensometer in a bridge branch: the bridge is brought into equilibrium before the application of strain so that $E^* = 0$. Then, the deformation applied to the extensometer can be determined from the unbalanced bridge voltage value.
Let us consider variations in resistance: ΔR_1, ΔR_2, ΔR_3, ΔR_4. The output voltage is obtained from 1.79:

$$\Delta E^* = \left(\frac{A}{B}\right) V \tag{1.81}$$

if second-order terms are neglected and if the balancing condition was satisfied before loading, A and B in 1.81 are:

$$
\begin{aligned}
A &= R_1R_3\left(\frac{\Delta R_1}{R_1} - \frac{\Delta R_2}{R_2} + \frac{\Delta R_3}{R_3} - \frac{\Delta R_4}{R_4}\right)\\
B &= \left(\frac{R_1R_3}{R_1R_2}\right)(R_1 + R_2)^2 && (1.82)
\end{aligned}
$$

then:

$$\Delta E^* = \frac{R_1R_2}{(R_1 + R_2)^2}\left(\frac{\Delta R_1}{R_1} - \frac{\Delta R_2}{R_2} + \frac{\Delta R_3}{R_3} - \frac{\Delta R_4}{R_4}\right) V \tag{1.83}$$

if we indicate with:

$$r = \left(\frac{R_2}{R_1}\right) \tag{1.84}$$

Eq. 1.83 can be written as:

$$\Delta E^* = \frac{r}{(1 + r)^2}\left(\frac{\Delta R_1}{R_1} - \frac{\Delta R_2}{R_2} + \frac{\Delta R_3}{R_3} - \frac{\Delta R_4}{R_4}\right) V \tag{1.85}$$

Eq. 1.85 is the fundamental equation for strain measurements, as long as the deformation is limited to a maximum value of a few per cent.

A synthetic demonstration of Eq. 1.85 is herein presented. Starting from the initial balancing condition:

$$(R_1 R_3 = R_2 R_4) \tag{1.86}$$

and perturbing the resistance R_1 with $R_1 + \Delta R_1$, we get:

$$\Delta E^* = E - 0 = \frac{(R_1 + \Delta R_1)R_3 - R_2 R_3}{(R_1 + \Delta R_1 + R_2)(R_3 + R_4)} V = \frac{\Delta R_1}{\frac{(R_1+\Delta R_1)(R_3+R_4)}{R_3} + \frac{R_2(R_3+R_4)}{R_3}} V \tag{1.87}$$

considering another balanced bridge in which $R_1 = R_2$ (for the equilibrium condition, it must be $R_3 = R_4$), we have:

$$\Delta E^* = V \frac{\Delta R_1}{2(R_1 + \Delta R_1) + 2R_1} = \frac{\frac{\Delta R_1}{2R_1}}{2 + \frac{\Delta R_1}{R_1}} \tag{1.88}$$

which can be linearized with respect to the variable $\Delta R_1/R_1$, thus obtaining:

$$\frac{\Delta E^*}{V} = \frac{1}{4} \frac{\Delta R_1}{R_1}. \tag{1.89}$$

By repeating the same procedure considering R_2 and R_4, similar results are obtained expect for a change in sign, i.e.:

$$\frac{\Delta E^*}{V} = -\frac{1}{4} \frac{\Delta R_2}{R_2}. \tag{1.90}$$

Having linearized, it is possible to consider the superposition principle:

$$\frac{\Delta E^*}{V} = \frac{1}{4} \left(\frac{\Delta R_1}{R_1} - \frac{\Delta R_2}{R_2} + \frac{\Delta R_3}{R_3} - \frac{\Delta R_4}{R_4} \right) \tag{1.91}$$

which is Eq.1.85 for $r = 1$.
The sensitivity of the bridge, to which the strain gauge has been connected, can be calculated with:

$$S = \frac{\Delta E^*}{\varepsilon} = \frac{V}{\varepsilon} \frac{r}{(1+r)^2} \left(\frac{\Delta R_1}{R_1} - \frac{\Delta R_2}{R_2} + \frac{\Delta R_3}{R_3} - \frac{\Delta R_4}{R_4} \right) \tag{1.92}$$

if we consider a bridge with multiple sensors ($n = 1, 2, 3, 4$), whose output is added, we have:

$$\sum_n \frac{\Delta R_n}{R_n} = n \frac{\Delta R}{R} \tag{1.93}$$

indicating by K_g the single extensometer factor, defined by:

$$\frac{\Delta R}{R} = K_g \varepsilon \tag{1.94}$$

Eq. 1.93 becomes:

$$\sum_n \frac{\Delta R_n}{R_n} = n K_g \varepsilon \tag{1.95}$$

substituting Eq. 1.95 into Eq.1.92, we get:

$$S = V\frac{r}{(1+r)^2}nK_g \tag{1.96}$$

This expression for the sensitivity is valid if the bridge supply voltage is constant, i.e. if V is independent on the current flowing through the extensometer. The sensitivity depends on the number n of the active branches of the bridge, on the extensometer K_g factor, on the bridge supply voltage V and on the ratio r between resistances. From the trend of $r/(1+r)^2$, it can be seen that the maximum sensitivity value is obtained for $r = 1$, i.e. when $R_1 = R_2$; under these conditions and in the case of four active branches of the bridge, we have:

$$S = K_g V \tag{1.97}$$

which is the maximum sensitivity value of the bridge. This value is reduced to a quarter if only one active branch of the bridge is available.
If the supply voltage is chosen in such a way that the extensometer works at the maximum dissipable power, the situation changes depending on the number of the active branches of the bridge and their position.

- In case of only one active element in branch 1, e.g. for measurements where temperature compensation is not required, $R_1 = R_g$ and the other three resistances can be chosen so as to maximize the sensitivity as long as the equilibrium condition is satisfied ($R_1R_3 = R_2R_4$). The supply voltage depends on the dissipated power P_g and is given by:

  $$V = I_g(R_1 + R_2) = I_gR_g(1 + r) = (1 + r)\sqrt{P_gR_g} \tag{1.98}$$

  considering that $I_g = \sqrt{P_g/R_g}$, from 1.96 and 1.97 it follows:

  $$S = \frac{r}{(1+r)}K_g\sqrt{P_gR_g} \tag{1.99}$$

  The sensitivity value depends on the circuit efficiency, $r/(1+r)$, and on the extensometer, whose contribution is represented by $K_g\sqrt{P_gR_g}$. The circuit efficiency increases as r increases, but r cannot be too high so as not to raise the supply voltage too much. For example, for $r = 9$ (90% circuit efficiency), a bridge supply voltage of $V = 42.4\ V$ is required for a strain gauge of resistance $R_g = 120\ \Omega$ and dissipable power $P_g = 0.15\ W$.

- In the case of an active extensometer in branch 1, $R_1 = R_g$, and a temperature compensation extensometer in branch 2, $R_2 = R_g$, the value of the supply voltage is given by:

  $$V = 2I_gR_g = 2\sqrt{P_gR_g} \tag{1.100}$$

  Substituting Eq. 1.100 into Eq. 1.96, with $n = 1$, $r = 1$, we have:

  $$S = \frac{1}{2}K_g\sqrt{P_gR_g} \tag{1.101}$$

  In this case the circuit efficiency is set at 0.5 since the condition $R_1 = R_2 = R_g$ imposes $r = 1$. It is therefore evident that the use of the temperature compensation extensometer limits the measurement efficiency: the choice of the extensometer, i.e. the product $K_g\sqrt{P_gR_g}$, determines the achievable sensitivity.

- In the case of an active extensometer in branch 1, $R_1 = R_g$, and a temperature compensation strain gauge in branch 4, $R_4 = R_g$, we have:

  $$V = I_g(R_1 + R_2) = I_g R_g(1 + r) = (1 + r)\sqrt{P_g R_g} \tag{1.102}$$

  Replacing Eq.1.102 in Eq. 1.96, we get:

  $$S = \frac{r}{(1 + r)} K_g \sqrt{P_g R_g} \tag{1.103}$$

  The circuit sensitivity is the same as in the case of a single active branch, thus it is possible to work with temperature compensation without a decrease in circuit sensitivity.

- If one active strain gauge is considered for each branch of the bridge (as is the case with strain gauges on a bending beam), the signals of the individual strain gauges are added together and the value of n in Eq. 1.96 is set to four. The supply voltage is given by:

  $$V = 2I_g R_g = 2\sqrt{P_g R_g} \tag{1.104}$$

  The resistance is the same for all branches $R_1 = R_2 = R_3 = R_4 = R_g$, thus $r = 1$. Substituting Eq. 1.104 in Eq. 1.96, we get:

  $$S = \frac{2 \cdot 4\ K_g \sqrt{P_g R_g}}{(1 + r)^2} \tag{1.105}$$

  then:

  $$S = 2K_g \sqrt{P_g R_g} \tag{1.106}$$

  In the case of a bridge with four active extensometers, the sensitivity is more than double that of a circuit with only one active element, and a temperature compensation effect is also achieved.

- In the case of a circuit with two active elements on branches 1 and 4 , $R_1 = R_4 = R_g$, we get:

  $$V = I_g(R_g + R_2) = I_g R_g(1 + r) = (1 + r)\sqrt{P_g R_g} \tag{1.107}$$

  Replacing Eq. 1.107 in Eq. 1.96, with $n = 2$ (two active elements), we have:

  $$S = \frac{r}{(1 + r)} 2K_g I_g R_g = \frac{2rK_g}{(1 + r)} \sqrt{P_g R_g} \tag{1.108}$$

  The sensitivity is very close to that of the bridge with four active elements.

1.5 The Wheatstone bridge with constant current

Previously, we have dealt with bridge circuits in which the supply voltage remains constant when the circuit resistance varies. As we have seen, if $\Delta R/R$ is large, there is a non-linear effect. This non-linearity severely limits the use of this type of bridge with semiconductor strain

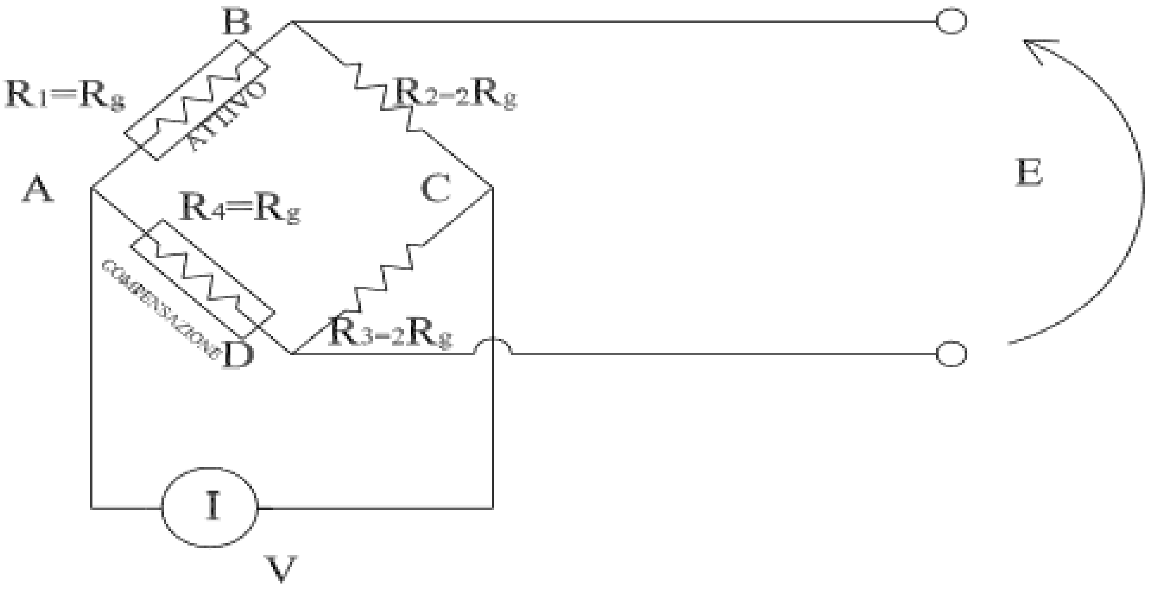


Figure 1.7: Wheatstone bridge with constant current scheme.

gauges which have very high sensitivities and therefore very large ratio values $\Delta R/R$.
A different situation occurs if bridge circuits with constant current generators are used, whose development is more recent than that of the classic constant voltage bridge. These devices have high impedance (from 1 to 10 $M\Omega$) which vary the output voltage with the resistance so as to keep the current constant, Fig. 1.7. At point A of the bridge we have:

$$I = I_1 + I_2 \tag{1.109}$$

and the voltage across the resistor R_1 is:

$$V_{AB} = I_1 R_1 \tag{1.110}$$

while the voltage across the resistor R_4 is:

$$V_{AD} = I_2 R_4 \tag{1.111}$$

therefore the voltage at terminals BD is given by:

$$E^* = V_{BD} = V_{AB} - V_{AD} = I_1 R_1 - I_2 R_4 \tag{1.112}$$

thus, the equilibrium condition of the bridge is:

$$I_1 R_1 = I_2 R_4 \tag{1.113}$$

the voltage V_{AC} is given by:

$$V_{AC} = I_1(R_1 + R_2) = I_2(R_3 + R_4) \tag{1.114}$$

from Eq. 1.114 one has:

$$I_1 = \left(\frac{R_3 + R_4}{R_1 + R_2}\right) I_2 \tag{1.115}$$

recalling Eq. 1.109, it is possible to express the currents I_1 and I_2 as a function of the current of the generator, I. Then:

$$\begin{aligned} I_1 &= \frac{R_3 + R_4}{(R_1 + R_2 + R_3 + R_4)} I \\ I_2 &= \frac{R_1 + R_2}{(R_1 + R_2 + R_3 + R_4)} I \end{aligned} \tag{1.116}$$

Replacing Eq. 1.116 in Eq. 1.112, the bridge output voltage can be expressed as a function of the current I:

$$E^* = \frac{(R_1 R_3 - R_2 R_4)}{(R_1 + R_2 + R_3 + R_4)} I \tag{1.117}$$

From Eq. 1.117, it can be seen that the equilibrium condition of the constant current bridge, corresponding to zero output voltage, is given by:

$$R_1 R_3 = R_2 R_4 \tag{1.118}$$

that is the same condition as in the case of the constant voltage bridge. If we consider variations ΔR_i of the resistances R_i, we have:

$$(E^* + \Delta E^*) = \frac{[(R_1 + \Delta R_1)(R_3 + \Delta R_3) - (R_2 + \Delta R_2)(R_4 + \Delta R_4)]}{\sum_1^4{}_i R_i + \sum_1^4{}_i \Delta R_i} I \tag{1.119}$$

from Eq. 1.119, by imposing the initial balancing condition of the bridge, we get:

$$\begin{aligned} \Delta E^* = \frac{I R_1 R_3}{\sum_i R_i + \sum_i \Delta R_i} \Bigg[& \frac{\Delta R_1}{R_1} - \frac{\Delta R_2}{R_2} + \frac{\Delta R_3}{R_3} - \frac{\Delta R_4}{R_4} \\ & + \frac{\Delta R_1}{R_1}\frac{\Delta R_3}{R_3} - \frac{\Delta R_2}{R_2}\frac{\Delta R_4}{R_4} \Bigg] \end{aligned} \tag{1.120}$$

The variation of the bridge output voltage, ΔE^*, is a non-linear function of the resistance variations ΔR_i (non-linear terms at both numerator and denominator). However, these non-linear effects can be reduced so that, even with the large resistance variations (typical of the semiconductor strain gauges), a linear relationship can be used.
Let us consider the bridge of Fig. 1.8. An active strain gage is placed in branch 1, a compensating strain gage is placed in branch 4 and fixed resistances are placed on branches 3 and 4; we have:

$$\begin{aligned} R_1 &= R_4 = R_g \\ R_2 &= R_3 = r R_g \\ \Delta R_2 &= \Delta R_3 = 0 \end{aligned} \tag{1.121}$$

If there are no significant variations in temperature we have:

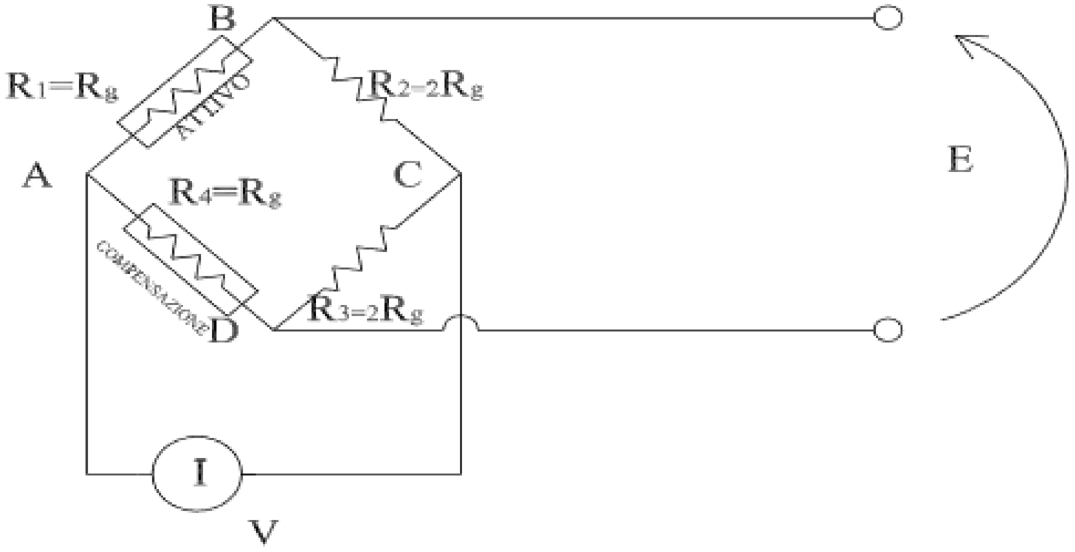


Figure 1.8: Thermal compensation scheme.

$$\begin{aligned}\Delta R_1 &= \Delta R_g \\ \Delta R_4 &= 0\end{aligned} \tag{1.122}$$

and Eq. 1.120 becomes:

$$\Delta E^* = \frac{IR_g r}{2(1+r) + \frac{\Delta R_g}{R_g}} \left(\frac{\Delta R_g}{R_g} \right) \tag{1.123}$$

Eq. 1.123 is non-linear due to the term $\Delta R_g / R_g$ at denominator. To assess the magnitude of the non-linear effect, Eq. 1.123 can be written:

$$\Delta E^* = \frac{IR_g r}{2(1+r)} \frac{\Delta R_g}{R_g} (1 - \eta) \tag{1.124}$$

where the non-linear term, η, is given by:

$$\eta = \frac{(\Delta R_g / R_g)}{2(1+r) + \frac{\Delta R_g}{R_g}} = \frac{K_g \varepsilon}{2(1+r) + K_g \varepsilon} \tag{1.125}$$

where K_g is the strain gauge factor; if reference is made to semiconductor strain gauges, the K_g value is of the order of 100 or more and the maximum deformation is limited in the range from 1000 μs to 2000 μs. The importance of the non-linear term η can be assessed considering the following numerical values:

$$\begin{aligned}K_g &= 100 \\ \varepsilon &= 2000 \mu s \\ r &= 10\end{aligned} \tag{1.126}$$

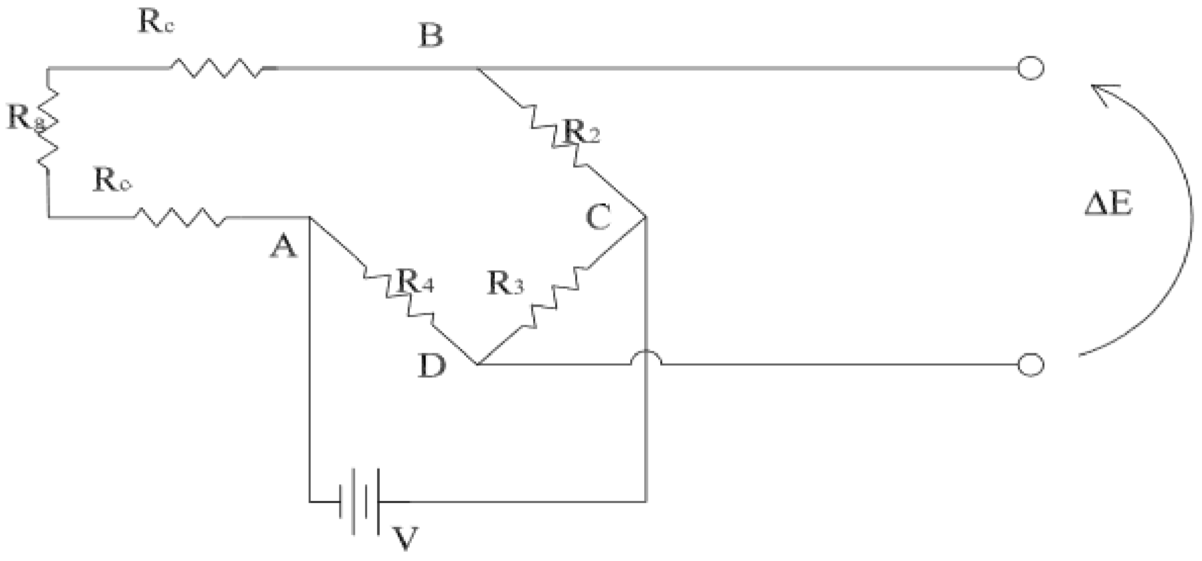


Figure 1.9: Connecting cables effects.

where r is the ratio between the "fixed" resistances on branches 2 and 3 and the extensometer resistance. From Eq. 1.125, we have:

$$\eta = \frac{100 \times 2 \times 10^{-3}}{2 \times 11 + 100 \times 2 \times 10^{-3}} = \frac{0.2}{22 + 0.2} = 0.009 \tag{1.127}$$

therefore, the non-linear term value is less than one percent. In practice, we have $\eta << 1$ unless K_g is very high and r is very small; if $K_g = 250$ and $r = 1$, for $\epsilon = 2000\ \mu s$, we have $\eta = 0.11$.

1.6 Effect of connecting cables

Given that the resistance variations to be measured with strain gauges are very small, the effects of resistance changes related to other elements of the measurement system, such as the connecting cables leading from the extensometer to the Wheatstone bridge, can be important. Considering a connection with an active strain gauge on branch 1 of the bridge, let R_g denote the extensometer resistanceand 2 R_c the cables resistance, Fig. 1.9. There are three main consequences:

1. signal attenuation;
2. effect on balance;
3. effect on temperature compensation.

The first effect, which leads to an attenuation of the resistance variation, is evaluated by considering $R_1 = R_g + 2R_c$. It follows:

$$\frac{\Delta R_1}{R_1} = \frac{\Delta R_g}{R_g + 2R_c} = \frac{\Delta R_g/R_g}{1 + 2R_c/R_g} \simeq \frac{\Delta R_g}{R_g}\left(1 - 2\frac{R_c}{R_g}\right) \tag{1.128}$$

Therefore:

$$\frac{\Delta R_1}{R_1} = \frac{\Delta R_g}{R_g}(1 - \mathcal{L}) \tag{1.129}$$

where $\mathcal{L}$ is the signal attenuation factor due to the connecting cables. If $2R_c << R_g$, it holds:

$$\mathcal{L} = \frac{2R_c/R_g}{1 + 2R_c/R_g} \simeq 2\frac{R_c}{R_g} \tag{1.130}$$

In order to keep the loss factor as small as possible, the cable resistance must be limited with respect to the extensometer resistance.
In order to have $\mathcal{L} \leq 0.01$, it must be $R_c/R_g \leq 0.005$. Therefore, if test conditions require long cables, it may be necessary to switch from strain gauges with $R_g = 120\ \Omega$ to strain gauges with $R_g = 350\ \Omega$. Of course, the conductor resistance depends (in addition to the material) on the cable cross-section. Indicatively, for copper calbes of 30.5 m (100 ft) length, resistance values ranging from a few tenths to several tens of ohms are possible ($0.2 \leq R_{L,100} \leq 100\ \Omega$).
The second effect is related to the balancing of the bridge; if:

$$R_1 = R_g + 2R_c \qquad R_2 = R_3 = rR_g \qquad R_4 = R_g \tag{1.131}$$

$$\Delta R_2 = \Delta R_3 = 0 \tag{1.132}$$

the initial balancing condition $R_1R_3 = R_2R_4$ becomes:

$$(R_g + 2R_c)\, rR_g = rR_gR_g \tag{1.133}$$

therefore, the initial balancing condition is no longer satisfied. The balance can be obtained with a variable resistance in parallel with R_2. However, if the ratio R_c/R_g is greater than a few percent, bridge balancing may not be possible.
The third effect concerns the temperature compensation obtained with a strain gauge on branch 4 of the bridge;

$$\Delta E^* = V\frac{r}{(1+r)^2}\left(\frac{\Delta R_1}{R_1} - \frac{\Delta R_4}{R_4}\right) \tag{1.134}$$

If the two strain gauges are subjected to the same ΔT while only the extensometer on branch 1 is subjected to the strain, we have:

$$\Delta E^* = V\frac{r}{(1+r)^2}\left[\left(\frac{\Delta R_g}{R_g + 2R_c}\right)_{\varepsilon} + \left(\frac{\Delta R_g}{R_g + 2R_c}\right)_{\Delta T} + \left(\frac{2\Delta R_c}{R_g + 2R_c}\right)_{\Delta T} - \left(\frac{\Delta R_g}{R_g}\right)_{\Delta T}\right] \tag{1.135}$$

Temperature compensation is no longer achieved because of two different reasons: on the one hand, terms 2 and 4 are not equal and, on the other hand, term 3, due to the temperature effect of cable resistance, can be important.
These effects can be drastically reduced by placing the two strain gauges on branches 1 and 4 at a distance, with a connection of the type shown in Fig. 1.10.

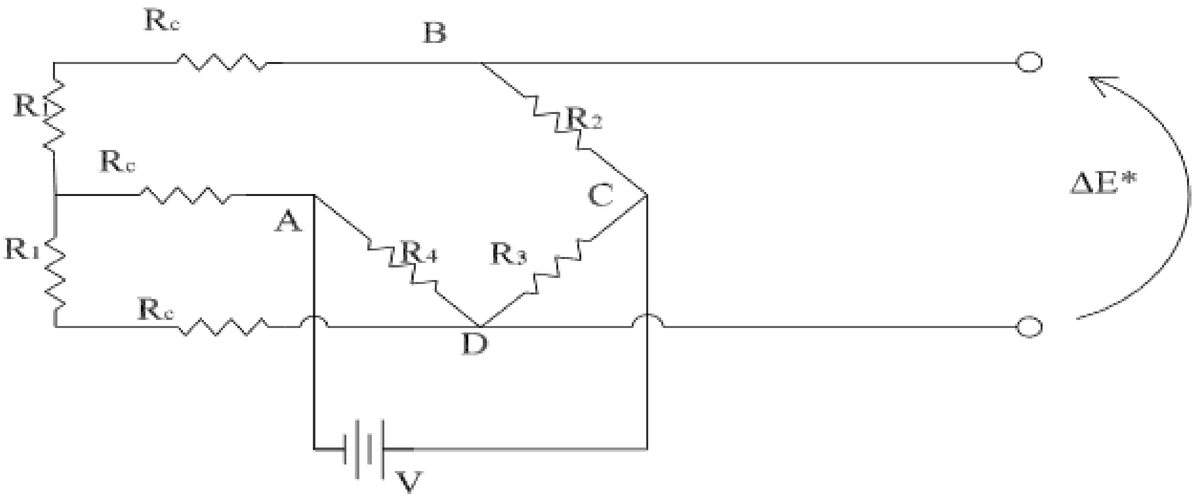


Figure 1.10: Compensation scheme for cable effects.

1.7 Strain gauges as sensors in transducers

1.7.1 Load cell

The load cell can be built from a tensile element with four strain gauges, placed as shown in Fig. 1.11, and connected as a full bridge, according to the scheme in Fig. 1.12. The traction load applied to the specimen leads to axial and transverse deformations given by the relations:

$$\begin{aligned} \varepsilon_a &= \frac{F}{A_p E} \\ \varepsilon_t &= \frac{-\nu F}{A_p E} \end{aligned} \tag{1.136}$$

where A_p is the specimen section and E, ν are the elastic constants of the material. With the strain gauges arranged as shown in Fig. 1.12, $r = 1$ and the following equation:

$$\frac{\Delta E^*}{V} = \frac{r}{(1+r)^2}\left(\frac{\Delta R_1}{R_1} - \frac{\Delta R_2}{R_2} + \frac{\Delta R_3}{R_3} - \frac{\Delta R_4}{R_4}\right) \tag{1.137}$$

becomes:

$$\frac{\Delta E^*}{V} = \frac{1}{4}\left(\frac{\Delta R_1}{R_1} - \frac{\Delta R_2}{R_2} + \frac{\Delta R_3}{R_3} - \frac{\Delta R_4}{R_4}\right) \tag{1.138}$$

if we indicate with K_g the extensometer factor, we have:

$$\begin{aligned} \frac{\Delta R_1}{R_1} &= \frac{\Delta R_3}{R_3} = K_g \varepsilon_a = \frac{K_g F}{A_p E} \\ \frac{\Delta R_2}{R_2} &= \frac{\Delta R_4}{R_4} = K_g \varepsilon_t = \frac{-\nu K_g F}{A_p E} \end{aligned} \tag{1.139}$$

placing Eq. 1.139 into Eq. 1.138, we have:

$$\frac{\Delta E^*}{V} = \frac{K_g F}{2 A_p E}(1+\nu) = K_F F \tag{1.140}$$

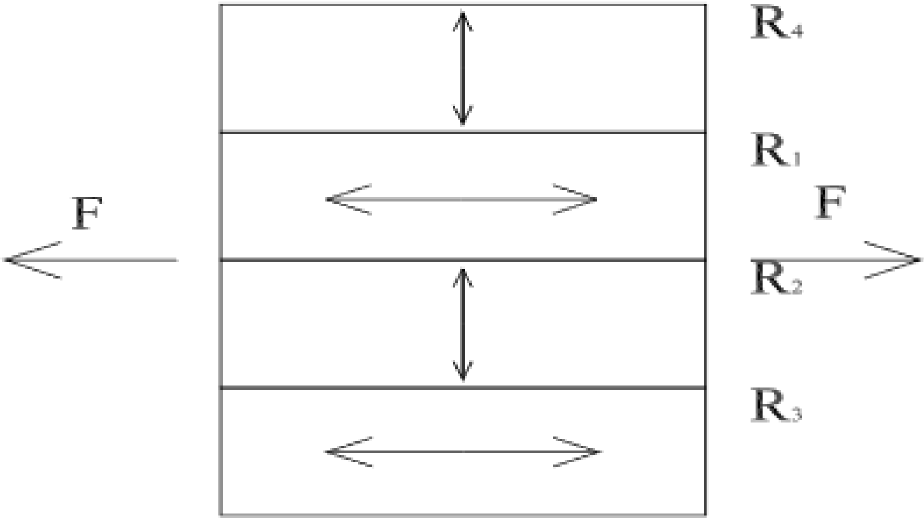


Figure 1.11: Load cell scheme.

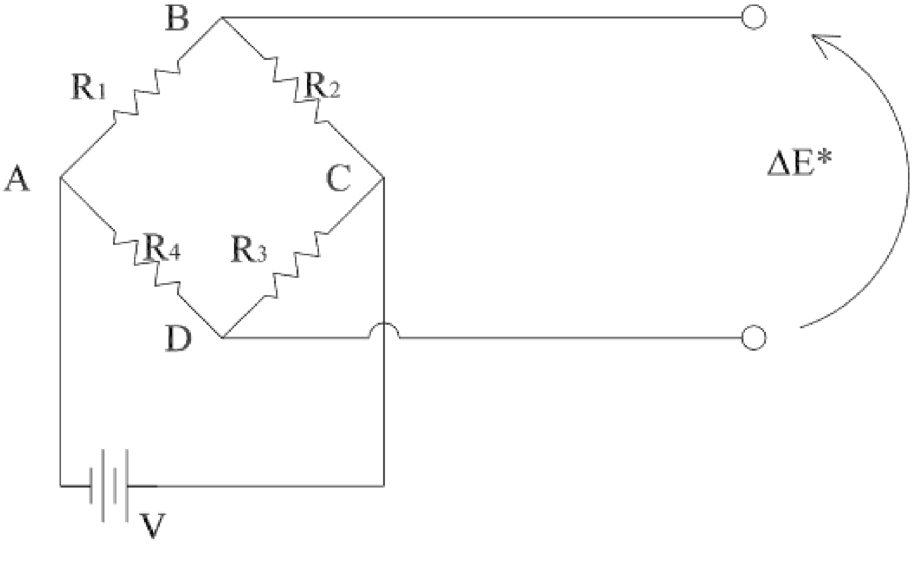


Figure 1.12: Connection between Wheatstone bridge and load cell.

which, if $K_g \simeq 2$, can be approximated with the following relation:

$$K_F = \frac{K_g}{2A_pE}(1+\nu) \simeq \frac{1+\nu}{A_pE} \tag{1.141}$$

From Eq. 1.140, it can be seen that the output is linearly proportional to the force F applied to the specimen. The value of $\Delta E^*/V$ depends on the specimen characteristics, but generally has a value around a few thousandths. The value of the measurable force is given by:

$$F = A_p \frac{\Delta E^*}{V} \frac{E}{(1+\nu)} \tag{1.142}$$

and its maximum value is also conditioned by the fatigue limit of the strain gauges used as sensors. We have:

$$\sigma = \frac{F}{A_p} = \frac{\Delta E^*}{V} \frac{E}{(1+\nu)} \tag{1.143}$$

Let us consider the case of a steel test specimen with $E = 210GPa$, $\nu = 0.32$ and $\Delta E^*/V = 0.003$. From Eq. 1.143, it follows:

$$\sigma_{max} = \frac{0.003 \times 210 \times 10^9}{1 + .32} = 477\ MPa \tag{1.144}$$

which is a value compatible with the fatigue limits of steel. However, the corresponding axial deformation is:

$$\varepsilon_{max} = \frac{\sigma_{max}}{E} = 2271\ \mu s \tag{1.145}$$

which is a high value compared to the fatigue limits of strain gauges.
It should also be noted that the positioning of the strain gauges on the test specimen is such as to cancel out the effect of any bending moments (due to eccentricity of the axial load or to the presence of transverse forces). Indeed, the components M_1 and M_2 of the applied moment $\mathbf{M}$ lead to resistance changes betweenwhich the following relationships exist:

$$\begin{array}{ccc} \left.\frac{\Delta R_2}{R_2}\right|_{M_1} = \left.\frac{-\Delta R_4}{R_4}\right|_{M_1} & ; & \left.\frac{\Delta R_1}{R_1}\right|_{M_1} = \left.\frac{\Delta R_3}{R_3}\right|_{M_1} = 0 \\ \left.\frac{\Delta R_3}{R_3}\right|_{M_2} = \left.\frac{-\Delta R_1}{R_1}\right|_{M_2} & ; & \left.\frac{\Delta R_2}{R_2}\right|_{M_2} = \left.\frac{\Delta R_4}{R_4}\right|_{M_2} = 0 \end{array} \tag{1.146}$$

therefore, such changes do not influence the measurement of the force F. The load cell is also practically insensitive to the effect of torsional loads.

1.7.2 Membrane pressure transducer

A special extensometer is used to obtain the maximum measurement sensitivity. The membrane deformation, which is measured by strain gauges, can be expressed in terms of its radial and circumferential components with the relations:

$$\begin{aligned} \varepsilon_r &= \frac{3(1-\nu^2)}{8Et^2}(R_0^2 - 3r^2)p \\ \varepsilon_\theta &= \frac{3(1-\nu^2)}{8Et^2}(R_0^2 - r^2)p \end{aligned} \tag{1.147}$$

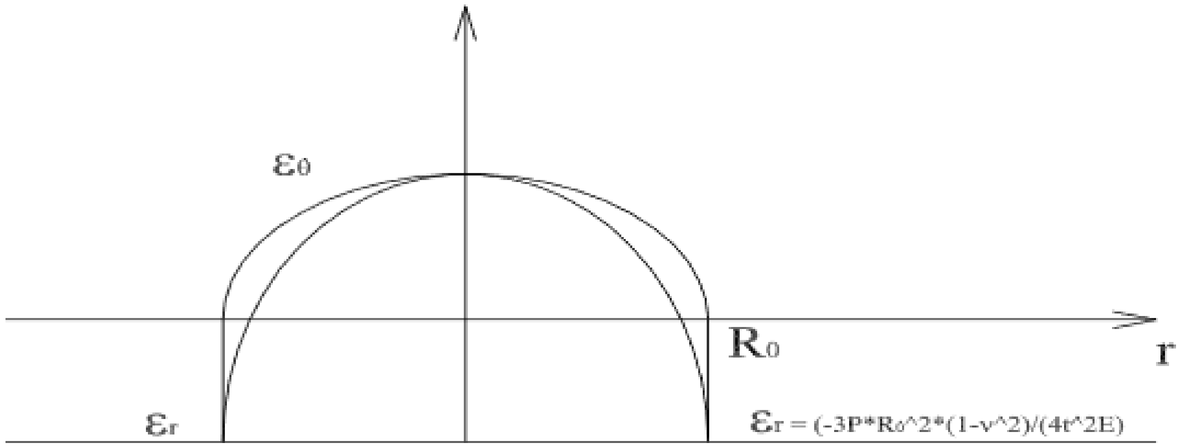


Figure 1.13: Deformation field in the membrane.

where p is the pressure applied to the membrane, and is the quantity to be measured, t is the membrane thickness, R_0 is the external radius and r is the radial position of the extensometer. From Eq. 1.147, it is observed that ε_θ is always positive and reaches its maximum value for $r = 0$, whereas ε_r can be either positive or negative and reaches its maximum positive value for $r = 0$ and its minimum negative value for $r = R_0$, Fig. 1.13. The special extensometer, herein used, is designed to take advantage of this deformation distribution; the output signal is:

$$\frac{\Delta E}{V} = \alpha_p \frac{R_0^2(1-\nu^2)}{t^2 E} p \tag{1.148}$$

Several special strain gauges for pressure transducers are manufactured, with diameters ranging from a few millimeters to a few centimeters and with $\alpha_p = 0.82$. The transducer diaphragm is deformed by the applied pressure and the pressure-deformation relationship is non-linear; however, the behavior can be considered linear if the central displacement of the membrane is small. The central displacement of the membrane is given by:

$$w_c = \frac{3R_0^4(1-\nu^2)}{16t^3 E} p \tag{1.149}$$

if the ratio $w_c/t \leq 0.25$, then the relation 1.148, which expresses the voltage variation as a function of the applied pressure, can be considered valid.

When the pressure transducer is used for dynamic measurements it is necessary that the "membrane" natural frequency is considerably higher (from five to ten times) than the maximum frequency to be considered in the dynamic analysis.

The membrane natural pulsation can be evaluated by:

$$\omega_n = 2\pi f_n = \frac{10.21t}{R_0^2} \sqrt{\frac{gE}{12(1-\nu^2)\gamma}} \tag{1.150}$$

where γ is the material density and g is the gravitational acceleration. The result is a low-cost, easy-to-use pressure transducer with a high natural frequency.

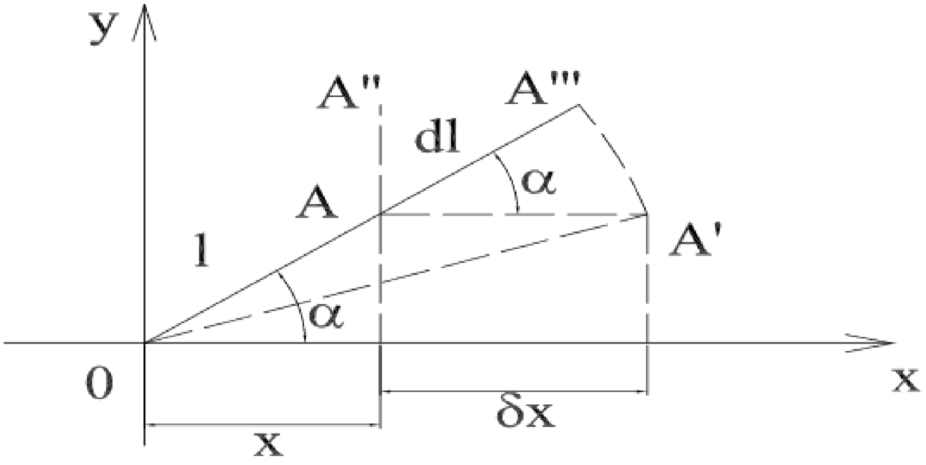


Figure 1.14: Plane deformation.

1.8 Measurement of deformation and stress

1.8.1 Measurement of deformation

Let us consider consider an x, y reference system and a strain gauge, indicated by OA, of length ℓ, placed at an angle α with respect to the x- axis (Fig. 1.14); if deformation occurs in the x-direction, with point A brought to position A', the deformation in the α-direction of the extensometer is:

$$\varepsilon_\alpha = \delta\ell/\ell \tag{1.151}$$

where:

$$\ell \cos\alpha = x \tag{1.152}$$

$$\delta\ell = \delta x \cos\alpha \tag{1.153}$$

then:

$$\varepsilon_\alpha = (\delta x/x)\cos\alpha\cos\alpha = \varepsilon_x \cos^2\alpha \tag{1.154}$$

with $\varepsilon_x = \delta x/x$.
Similarly, if a deformation occurs in y direction, with point A brought to position A'', the deformation in the -αdirection of the extensometer is:

$$\varepsilon_\alpha = \varepsilon_y \sin^2\alpha \tag{1.155}$$

with $\varepsilon_x = \delta y/y$;
Finally, let us look at shear strain γ_{xy}:

$$\varepsilon_\alpha = \gamma_{xy} \sin\alpha\cos\alpha \tag{1.156}$$

If we now consider normal and shear strains together, we have [5]

$$\varepsilon_\alpha = \varepsilon_x \cos^2\alpha + \varepsilon_y \sin^2\alpha + \gamma_{xy} \sin\alpha\cos\alpha \tag{1.158}$$

The deformation ε_α can be measured with the extensometer itself. Therefore, if three deformations are measured at three different angles (α_1, α_2, α_3), then three unknown deformations in xy reference system (ε_x, ε_y, ε_{xy}) can be calculated. If Eq. 1.158 is written in term of the angle 2α, using the trigonometric relationships $2\sin\alpha\cos\alpha = \sin 2\alpha$, $\sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha)$, $\cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$, one gets:

$$\varepsilon_\alpha = (\varepsilon_x + \varepsilon_y)/2 + ((\varepsilon_x - \varepsilon_y)/2)\cos 2\alpha + \gamma_{xy}(\sin 2\alpha)/2 \tag{1.159}$$

By deriving 1.159 with respect to α and by imposing the condition:

$$d\varepsilon_\alpha/d\alpha = 0 \tag{1.160}$$

we find the angle α_p which defines the main axes of deformation:

$$\tan 2\alpha_p = (\gamma_{xy}/(\varepsilon_x - \varepsilon_y)) \tag{1.161}$$

Once α_p is known, the main deformations ε_{max} is obtained using the following formula:

$$\varepsilon_{max} = \varepsilon_x \cos^2\alpha_p + \varepsilon_y \sin^2\alpha_p + \gamma_{xy}\sin\alpha_p\cos\alpha_p \tag{1.162}$$

The main deformation ε_{min} can be evaluated using the same formula at an angle ($\alpha_p \pm 90°$). From the main deformations values, the the shear strain maximum value is obtained:

$$\gamma_{xy_{max}} = \varepsilon_{max} - \varepsilon_{min} \tag{1.163}$$

the direction of the maximum shear strain is inclined by 45° with respect to the main axes.
Let us consider three strain gauges, connected in such a way as to measure the deformation at three different angles at a single point, as shown in Fig. 1.15. The following experimental measurements are obtained:

$$\varepsilon_1 = 850\ \mu s \tag{1.164}$$

$$\varepsilon_2 = -100\ \mu s \tag{1.165}$$

$$\varepsilon_3 = 350\ \mu s \tag{1.166}$$

If we consider an xy reference system, with the x-axis coincident with the first extensometer axis, it is possible to calculate the strains ε_x, ε_y, ε_{xy} from Eq. 1.158 and the inclination of the main axes from Eq. 1.159.
For the first extensometer, we have:

$$850 = \varepsilon_x \cos^2(0) + \varepsilon_y \sin^2(0) + \gamma_{xy}\sin(0)\cos(0) \tag{1.167}$$

---

[5] Note that it is indicated with ε_{xy} what is normally indicated with $\gamma_{xy} \equiv \varepsilon_{xy}$. Furthermore Eq. 1.158 can be obtained directly using the plane deformation tensor in Cartesian coordinates

$$\begin{bmatrix} \varepsilon_x & \varepsilon_{xy}/2 \\ \varepsilon_{xy}/2 & \varepsilon_y \end{bmatrix} \tag{1.157}$$

applying it to the generic normal to the extensometer $\mathbf{n} = \{\cos\alpha, \sin\alpha\}^{\mathbf{T}}$ and subsequently projecting on it the vector obtained.

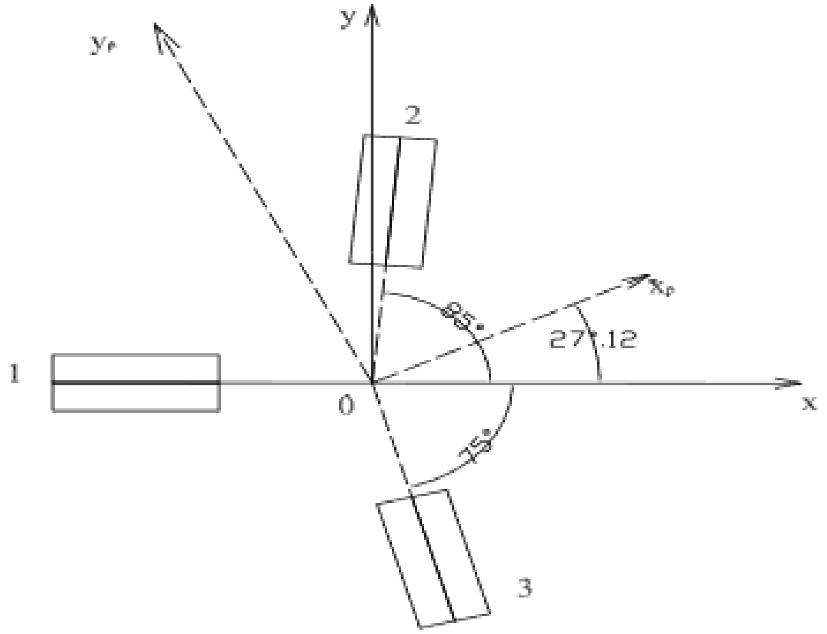


Figure 1.15: Strain gauge rosette.

from which:

$$\varepsilon_x = 850 \; \mu s \tag{1.168}$$

For the second extensometer, we have:

$$-100 = 850 \cos^2(85) + \varepsilon_y \sin^2(85) + \gamma_{xy} \sin(85) \cos 85 \tag{1.169}$$

from which:

$$\varepsilon_y = -107.2 + 0.08752 \; \gamma_{xy} \tag{1.170}$$

For the third extensometer, we have:

$$350 = 850 \cos^2(-75) + \varepsilon_y \sin^2(-75) + \gamma_{xy} \sin(-75) \cos(-75) \tag{1.171}$$

from which:

$$\varepsilon_y = 314.2 - 0.0268 \; \gamma_{xy} \tag{1.172}$$

we get then:

$$\varepsilon_y = -3.4 \; \mu s \tag{1.173}$$

$$\gamma_{xy} = 1185 \; \mu s \tag{1.174}$$

From Eq. 1.159, it follows:

$$\tan 2\alpha_p = 1185/853.4 = 1.39 \tag{1.175}$$

Then:

$$2\alpha_p = 54.24^\circ \tag{1.176}$$
$$\alpha_p = 27.12^\circ \tag{1.177}$$

From Eq. 1.162, ε_{max} can be calculated:

$$\varepsilon_{max} = 1153.4\ \mu s \tag{1.178}$$

ε_{min} occurs at the angle:

$$\alpha_{min} = \alpha_p \pm 90^\circ \tag{1.179}$$

and then:

$$\varepsilon_{min} = -305.3\ \mu s \tag{1.180}$$

Finally, from Eq. 1.163 we have the maximum shear value, relative to a 45° inclined direction with respect to the main axes:

$$\gamma_{xy_{max}} = \varepsilon_{max} - \varepsilon_{min} = 1452.6\ \mu s \tag{1.181}$$

If the structure under test is made of a light alloy with the following elastic characteristics:

$$E = 70\ GPa \tag{1.182}$$
$$\nu = 0.32 \tag{1.183}$$

from relations:

$$\sigma_{max} = (E/(1-\nu^2))(\varepsilon_{max} + \nu\varepsilon_{min}) \tag{1.184}$$
$$\sigma_{min} = (E/(1-\nu^2))(\varepsilon_{min} + \nu\varepsilon_{max}) \tag{1.185}$$

we have:

$$\sigma_{max} = 82.33MPa \tag{1.186}$$
$$\sigma_{min} = 4.97MPa \tag{1.187}$$

and from:

$$\sigma_{xy} = (E\gamma_{xy})/(2(1+\nu)) = G\gamma_{xy} \tag{1.188}$$

the maximum shear value is obtained, again with reference to a direction at 45° with respect to the main axes:

$$\sigma_{xy_{max}} = 38.55MPa \tag{1.189}$$

The effect of a positioning error of the strain gauges with respect to the nominal position is now evaluated. Assuming an error of 3 degrees, we have:

$$\varepsilon_x = 850\ \mu s \tag{1.190}$$
$$\varepsilon_y = 6.8\ \mu s \tag{1.191}$$
$$\gamma_{xy} = 893.6\ \mu s \tag{1.192}$$

From Eq. 1.161 we have:

$$\alpha_p \;=\; 23.32^\circ \tag{1.193}$$

From Eq. 1.162 we get:

$$\varepsilon_{max} \;=\; 1042.7\ \mu s \tag{1.194}$$

$$\varepsilon_{min} \;=\; -185.9\ \mu s \tag{1.195}$$

From Eq. 1.163 we have then:

$$\gamma_{xy_{max}} \;=\; \varepsilon_{max} - \varepsilon_{min} \;=\; 1228.6\ \mu s \tag{1.196}$$

Again in the case of light alloy, the following stresses values are obtained:

$$\sigma_{max} \;=\; 76.67\ MPa \tag{1.197}$$

$$\sigma_{min} \;=\; 11.51\ MPa \tag{1.198}$$

$$\sigma_{xy} \;=\; 32.58\ MPa \tag{1.199}$$

As can be seen from the numerical example, the effects of the strain gauge positioning error are considerable; it is therefore clear that photoetched “rosettes” should be used, since the error of the relative position of the strain gauges is practically negligible.

1.8.2 Stress and sliding measurement for isotropic materials

It is possible to take advantages of the transverse sensitivity of a strain gauge, S_t, to obtain a stress sensor. A strain gauge can designed in such a way as to have an output proportional to the stress along its axis, thanks to an appropriate choice of S_t; indeed:

$$\frac{\Delta R}{R} = K_a(\varepsilon_a + S_t\varepsilon_t) \tag{1.200}$$

with:

$$S_t = K_t/K_a \tag{1.201}$$

while the link $\sigma - \varepsilon$, in the case of plane strain, is given by:

$$\begin{aligned} \varepsilon_a &= \frac{1}{E}(\sigma_a - \nu\sigma_t) \\ \varepsilon_t &= \frac{1}{E}(\sigma_t - \nu\sigma_a) \end{aligned} \tag{1.202}$$

from Eq. 1.202 and Eq. 1.200, we have:

$$\frac{\Delta R}{R} = \frac{K_a}{E}(\sigma_a - \nu\sigma_t) + \frac{S_tK_a}{E}(\sigma_t - \nu\sigma_a) = \tag{1.203}$$

$$= \frac{\sigma_a K_a}{E}(1 - \nu S_t) + \frac{\sigma_t K_a}{E}(S_t - \nu) \tag{1.204}$$

if:

$$S_t = \nu \tag{1.205}$$

then:

$$\frac{\Delta R}{R} = \frac{K_a(1-\nu^2)}{E}\sigma_a = K_\sigma \sigma_a \tag{1.206}$$

the resistance variation of the extensometer, $\Delta R/R$, is independent on σ_t and proportional to σ_a. Once the material of the extensometer and that of the specimen have been chosen, K_σ is a constant; therefore the resistance variation is proportional to σ_a according to K_σ proportionality factor.

Generally, a strain sensor consists of two extensometers arranged in a V -shape. Usually the sensor is at a generic position in the strain field and the elements of the grid are at an angle ϑ with respect to the sensor axis. If we denote by $\varepsilon_{x-\vartheta}$ and $\varepsilon_{x+\vartheta}$ the deformations measured by the two extensometers, we have:

$$\sigma_{xx} = \frac{E}{2(1-\nu)}(\varepsilon_{x+\vartheta} + \varepsilon_{x-\vartheta}) \tag{1.207}$$

Indeed:

$$\varepsilon_{x+\vartheta} = \varepsilon_x \cos^2\vartheta + \varepsilon_y \sin^2\vartheta + \gamma_{xy}\sin\vartheta\cos\vartheta \tag{1.208}$$

$$\varepsilon_{x-\vartheta} = \varepsilon_x \cos^2\vartheta + \varepsilon_y \sin^2\vartheta - \gamma_{xy}\sin\vartheta\cos\vartheta \tag{1.209}$$

so

$$\varepsilon_{x+\vartheta} + \varepsilon_{x-\vartheta} = 2\left(\varepsilon_x \cos^2\vartheta + \varepsilon_y \sin^2\vartheta\right) \tag{1.210}$$

that is

$$\varepsilon_{x+\vartheta} + \varepsilon_{x-\vartheta} = 2\cos^2\vartheta\left(\varepsilon_x + \frac{\sin^2\vartheta}{\cos^2\vartheta}\varepsilon_y\right) \tag{1.211}$$

If the angle θ is chosen so that:

$$\tan^2\vartheta = \nu \tag{1.212}$$

then:

$$\cos^2\vartheta = \frac{1}{1+\nu} \tag{1.213}$$

we have:

$$\varepsilon_{x+\vartheta} + \varepsilon_{x-\theta} = \frac{2}{(1+\nu)}\left(\varepsilon_x + \nu\varepsilon_y\right) \tag{1.214}$$

but:

$$\sigma_{xx} = \frac{E}{(1-\nu^2)}\left(\varepsilon_x + \nu\varepsilon_y\right) \tag{1.215}$$

therefore:

$$\sigma_{xx} = \frac{E}{2(1-\nu)}\left(\varepsilon_{x+\vartheta} + \varepsilon_{x-\vartheta}\right) \tag{1.216}$$

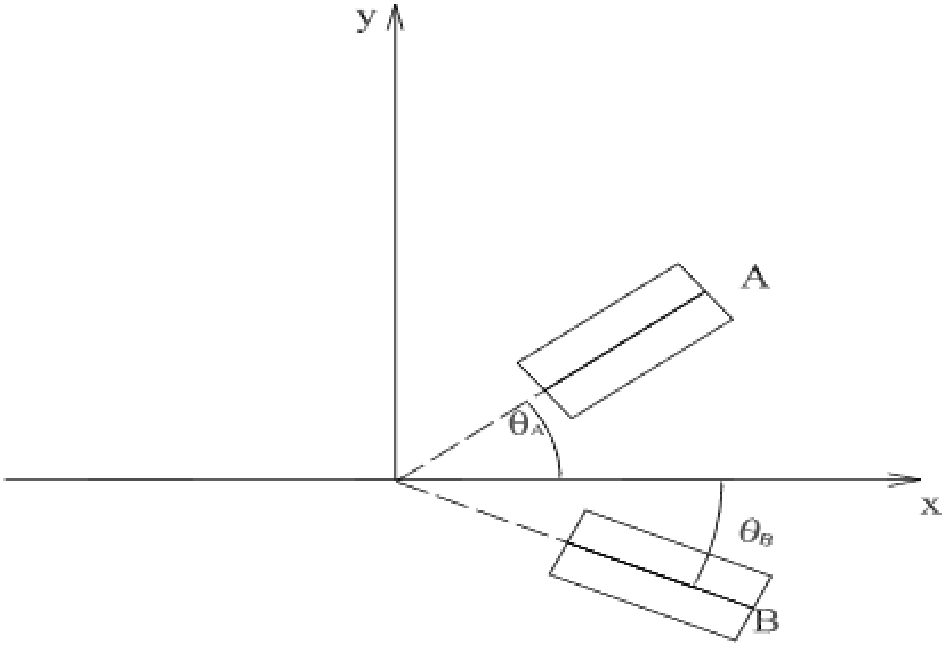


Figure 1.16: Diffraction extensometer.

The sensor reading directly provides the half-sum: $(\varepsilon_{x-\vartheta} + \varepsilon_{x+\vartheta})/2$. If the condition $\tan^2 \vartheta = \nu$ is verified, then it is sufficient to know the elastic characteristics of the material, E and ν, to get directly σ_{xx}, i.e. the strain along the sensor axis If $\nu = 0.3$, then $\vartheta \simeq 30°$.

If the direction of principal stresses is known, measurement is also possible with only one extensometer as long as it is at a particular position ϑ corresponding to the condition $\tan^2 \vartheta = \nu$; in this case, indeed, if x indicates the main direction, we have:

$$\varepsilon_{x-\vartheta} = \varepsilon_{x+\vartheta} = \varepsilon_{\vartheta} \tag{1.217}$$

and so:

$$\sigma_{xx} = \frac{E}{(1-\nu)} \varepsilon_{\vartheta} \tag{1.218}$$

Let us now consideer two extensometers, indicated with A and B in Fig. 1.16, positioned at the angles ϑ_A and ϑ_B with respect to the x-axis:

$$\begin{aligned} \varepsilon_A &= \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\vartheta_A + \frac{\varepsilon_{xy}}{2} \sin 2\vartheta_A \\ \varepsilon_B &= \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\vartheta_B + \frac{\varepsilon_{xy}}{2} \sin 2\vartheta_B \end{aligned} \tag{1.219}$$

The sliding, ε_{xy}, is:

$$\varepsilon_{xy} = \frac{2(\varepsilon_A - \varepsilon_B) - (\varepsilon_x - \varepsilon_y)(\cos 2\vartheta_A - \cos 2\vartheta_B)}{\sin 2\vartheta_A - \sin 2\vartheta_B} \tag{1.220}$$

if the extensometers are oriented so that it is $\cos(2\vartheta_A) = \cos(2\vartheta_B)$, we have:

$$\varepsilon_{xy} = \frac{2(\varepsilon_A - \varepsilon_B)}{\sin 2\vartheta_A - \sin 2\vartheta_B} = K_{xy}(\varepsilon_A - \varepsilon_B) \tag{1.221}$$

with:

$$K_{xy} = \frac{2}{\sin 2\vartheta_A - \sin 2\vartheta_B} \tag{1.222}$$

The cosine is an even function. Thus, the condition $\vartheta_A = -\vartheta_B$ also satisfies $\cos(2\vartheta_A) = \cos(2\vartheta_B)$. Therefore, the sliding ε_{xy} is proportional to the difference between the deformations ε_A and ε_B. The angle $\vartheta_A = -\vartheta_B$ can be any but if $\vartheta_A = \pi/4$ is chosen, then the sliding is simply given by:

$$\varepsilon_{xy} = \varepsilon_A - \varepsilon_B \tag{1.223}$$

therefore, it can be measured directly with two extensometers oriented at $\pm 45^o$ with respect to the x-axis and appropriately connected to two branches of a Wheatstone bridge. Four-element "rosettes" are often used to double the sensitivity with a full bridge connection.

1.9 Optical measurements of extensometry

The widespread use of lasers as monochromatic light sources has led to the development of various optical strain gauge systems.

1.9.1 Diffraction extensometer

The diffraction extensometer consists essentially of two laminae, which are glued or welded to a structure of length ℓ. They are separated by a distance ,indicated with b, to form an opening. A monochromatic light, produced by a laser source, is sent to the aperture and produces a diffraction effect on a screen placed at a distance R from the opening itself. If the distance R is very large with respect to the aperture size, the distribution of light intensity by diffraction is given by:

$$I \;=\; A_0^2 \sin^2 \beta / \beta^2 \tag{1.224}$$

where A_0 indicates the width of the light on the central line, identified by $\theta = 0$, and β is defined by the relation:

$$\beta \;=\; (\pi b/\lambda) \sin \theta \tag{1.225}$$

where λ is the wavelength of monochromatic light coming from the laser source and θ is the angle as indicated in Fig. 1.17.
If the diffraction analysis is limited to small distances from the central line, indicated with y in Fig 1.17, we have:

$$\sin \theta \;\cong\; y/R \tag{1.226}$$

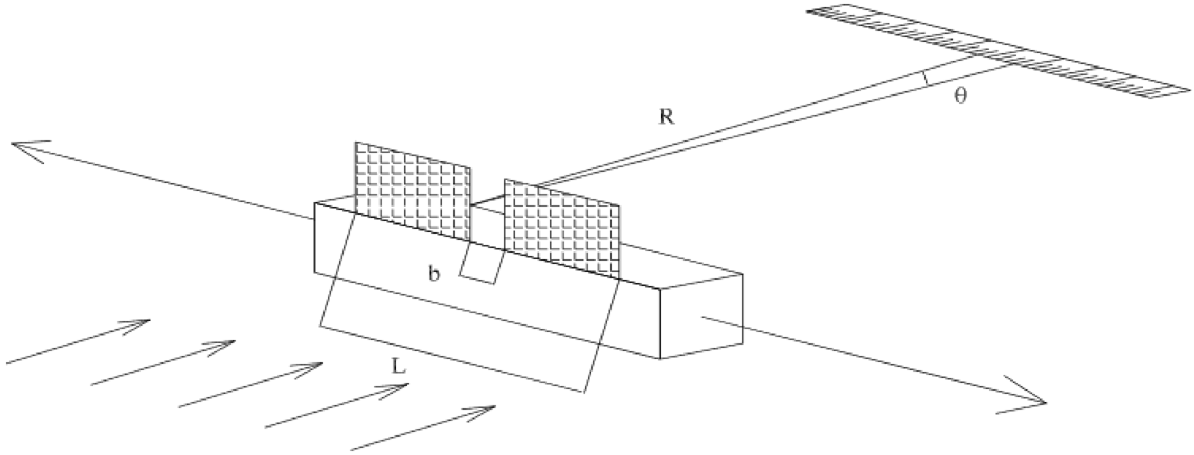


Figure 1.17: Interference fringes.

and then:

$$\beta \;=\; (\pi b/\lambda)(y/R) \tag{1.227}$$

From Eq. 1.224, it is seen that the luminous intensity cancels out if $\sin\beta = 0$ and therefore for $\beta = 0$ or $\beta = n\pi$ (with $n = 1, 2, ...$).

If we consider the points on the screen for which $I = 0$, we can obtain a relation between their position on the screen scale and the aperture size; indeed, we have:

$$b \;=\; (\lambda R n/y) \tag{1.228}$$

where n is the order of null point relative to the point which has position y on the scale. Due to the structure deformation there is a variation in the opening width, which is linked to the deformation by the relationship:

$$\varepsilon \;=\; \frac{\Delta b}{b} \tag{1.229}$$

this variation in opening corresponds to a variation in diffraction.

If we consider the diffraction after the deformation, we have:

$$(b + \Delta b) \;=\; (\lambda R n^*/y_1) \tag{1.230}$$

while before the deformation we had:

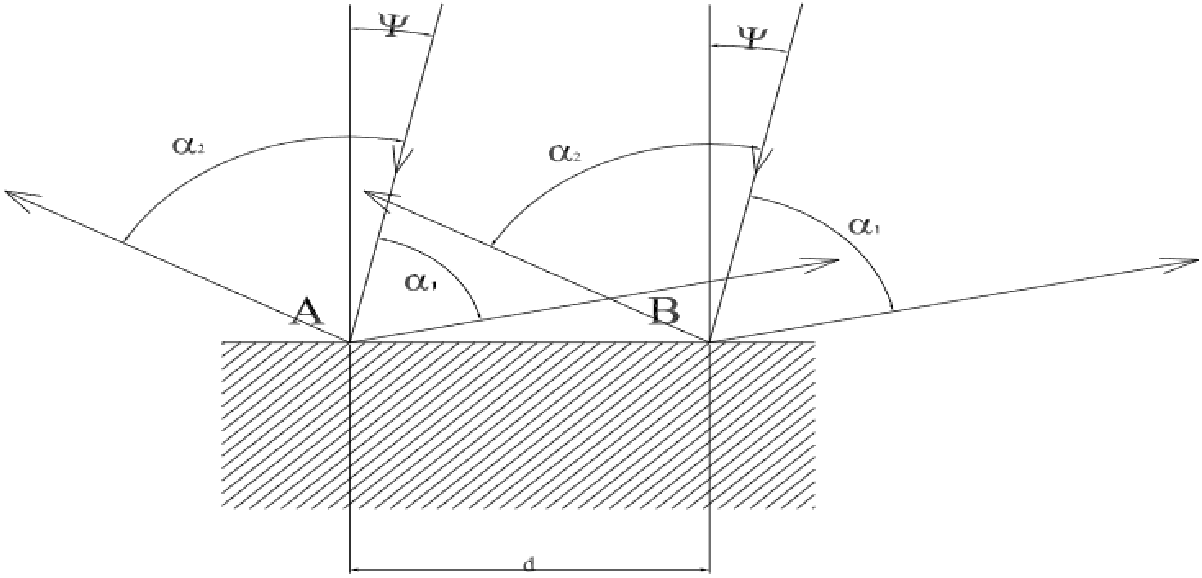
$$b \;=\; (\lambda R n^*/y_0) \tag{1.231}$$

and therefore the deformation ε can be obtained from the relation:

$$\varepsilon \;=\; \Delta b/b \;=\; \left[\frac{(b+\Delta b) - b}{b}\right] = \frac{y_0(y_0 - y_1)}{y_1 y_0} \tag{1.232}$$

The zero point index to which the measurement refers, indicated with n^*, is chosen as high as possible, compatibly with optical constraints, so as to obtain the highest possible values for the distances y_0, y_1.

This type of sensor has several advantages, particularly in the case of high temperature measurements, as it is temperature compensated if the plates are of the same material as the specimen.



1.9.2 Interference extensometer

An optical system can be based on the interference phenomenon. This phenomenon occurs if a monochromatic light, produced by a laser source, is reflected by two V-shaped notches on a high surface finish of a structure, Fig. 1.9.2. The notches are 10^{-3} mm deep and spaced about a tenth of a mm apart and they have a notch angle of 110°. If the notches are small enough with respect to the wavelength to allow light diffraction and if they are also close enough to allow superposition of the diffraction and leading to interference effect, then the intensity of the light is given by:

$$I = 4A_0^2(\sin^2\beta/\beta^2)\cos^2\phi \tag{1.233}$$

where:

$$\begin{aligned} \beta &= (\pi b/\lambda)\sin\theta \\ \phi &= (\pi d/\lambda)\sin\theta \end{aligned}$$

$b =$ notch width
$d =$ notch spacing
$\theta =$ notch angle

The light is reflected by the two sides of the notch resulting in two interference phenomena on two screens generally placed at a distance of about 200 mm from the notches.

The luminous intensity canceals out and, as a result, there are black stripes on the screen at β and ϕ angles given by:

$$\beta = n\pi \qquad (n = 1, 2, ...) \tag{1.234}$$

$$\phi = (m+1)/3\pi \qquad (m = 0, 1, 2, ...) \tag{1.235}$$

Due to the deformation of the specimen, there is a change in the notch spacing and notch width. These effects produce shifts in the interferences, i.e. in the position of the black trails on the screen, which can be connected to the deformation between the notches. We have:

$$\varepsilon \; = \; (\Delta N_1 - \Delta N_2)\lambda/(2d\sin\alpha) \tag{1.236}$$

where ΔN_1, ΔN_2 fringe order changes induced by the deformation and α is the angle between the incident and diffracted light producing the interference. This method also has several advantages: in particular, it can be used for deformation measurements on rotating structures and in extreme environmental conditions and the temperature compensation is direct.