# Chapter 1

# Vibration measurements

## 1.1 General considerations: vibrations and comfort

Let us consider the case of a vibration consisting of a simple harmonic motion of amplitude $A$ and pulsation $\omega$; between displacement, $x$, speed $\dot{x}$, and acceleration, $\ddot{x}$, we have the relationships:

$$\begin{aligned} x(t) &= A\sin(\omega t) \\ \dot{x}(t) &= \omega A\cos(\omega t) \\ \ddot{x}(t) &= -\omega^2 A\sin(\omega t) \end{aligned} \tag{1.1}$$

it is worthnoting that while the frequency remains unchanged, there is a $90^o$ phase shift between speed and displacement, and between speed and acceleration.
The relations between the amplitudes are linked through the pulsation $\omega$; if reference is made to the amplitude relative to the velocity, $|A_{\dot{x}}| = \omega|A_x|$, we can write:

$$\begin{aligned} |A_x| = |A_{\dot{x}}|/\omega \rightarrow \lg A_x = \lg A_{\dot{x}} - \lg\omega \\ |A_{\ddot{x}}| = \omega|A_{\dot{x}}| \rightarrow \lg A_{\ddot{x}} = \lg A\dot{x} + \lg\omega \end{aligned} \tag{1.2}$$

therefore, with reference to the amplitude in velocity, the amplitude in displacement is inversely proportional to $\omega$ and that in acceleration is directly proportional to $\omega$.
For high frequencies, the measurement of amplitude relative to acceleration is the most easily obtainable with a measuring instrument; while for low frequencies it is the one relative to displacement; it can also be seen that for the amplitude relative to the displacement there is an attenuation of the high frequency components.
A classic way of presenting displacement, velocity and acceleration as a function of frequency is to use a graph with the frequency on the abscissa and the velocity on the ordinate, both in a logarithmic scale. The lines with a 45° slope (+1) indicate lines of constant displacement while the lines with a $-45°$ (-1) slope indicate lines with constant acceleration (Fig. **??**).
In general, the best index for assessing possible structural damage due to vibration is the velocity amplitude, while the acceleration amplitude is the one to which humans are most sensitive.
The ISO (International Organization for Standardization) defines a standard for acceptable vibration levels in different situations. These standards are expressed in terms of mean quadratic values of the signal $x(t)$, with the symbology in use RMS (Root Mean Square), defined by:

$$x_{RMS} = \left[\lim_{T\to\infty} \frac{1}{T}\int_0^T x^2(t)dt\right]^{1/2} \tag{1.3}$$

The classic approach to representing these limits is with a graphical representation of the relationships linking displacement, velocity and acceleration in the case of a SDOF (Single Degree Of Freedom) system.
Fig. **??** shows the limits for human sensitivity, structural damage and vibration of mechanical systems.
Let us evaluate the effect that a ground irregularity of the order of 0.2 mm has on an aircraft moving on the ground. If the aircraft is represented with an undamped SDOF model with mass $m$ and stiffness $K$ characteristics of value:
$m = 10000\ Kg$
$K = 5 \times 10^6\ N/m$
then:

$$\omega = \sqrt{\frac{K}{m}} = \sqrt{\frac{5 \times 10^6}{10^4}} = \sqrt{500} = 22.36\ rad/s$$
$$f = \frac{\omega}{2\pi} = 356\ Hz$$

Let us now consider the irregularity height as the mean square value of the displacement. Considering the frequency value calculated above and an amplitude of 0.2 $mm$, the corresponding point in the diagram is within the passenger's sensitive zone with an acceleration level of 0.05 $m/s^2$.
In order to move the operating point away from that zone, once the ground irregularity has been fixed, the frequency of the system must be reduced: this can be achieved by increasing the mass, which however is obviously related to the category of the aircraft, or by decreasing the stiffness, which depends on the characteristics of the suspension such as tire pressure. Assuming:
$m = 10000\ Kg$
$K = 1 \times 10^6\ N/m$
we have:

$$\omega = \sqrt{\frac{K}{m}} = \sqrt{100} = 10\ rad/s$$
$$f = 1.59\ Hz$$

It can be seen from the diagram that the operating point moves outside the sensitivity zone with an acceleration level of 0.01 $m/s^2$.
Of course this procedure is only indicative of the problem: the SDOF model may be insufficient to estimate the dynamics of the aircraft in its ground motion, and the variation of the suspension stiffness characteristics may be incompatible with other aircraft design choices.
With regard to the amplitudes of the vibrations of interest, these vary by several orders of magnitude depending on the problems: in the case of optical benches or medical instruments, they may be of the order of $10^{-4}$ $mm$ for frequencies between 0.1 $Hz$ and 1 $Hz$, while for mechanical vibrations the frequencies of interest vary between 10 $Hz$ and $10,000$ $Hz$ and the amplitudes range between a few tenths of a millimeter and several centimeters.
Let us resume the study of the aircraft, previously examined with the $SDOF$ model, with the following characteristics:
$m = 10000\ Kg$
$K = 5 \times 10^6\ N/m$
to which a viscous damping coefficient is added:

$c = 10^5\ Ns/m$
and let us consider another *SDOF* system, representing a turntable, with the following parameters:
$m = 1\ Kg$
$K = 500\ N/m$
$c = 10\ Ns/m$
The dynamic characteristics of the two systems are evaluated.
For the aircraft we have:

$$\omega_1 = \sqrt{\frac{K_1}{m_1}} = \sqrt{\frac{5 \times 10^6}{10^4}} = \sqrt{500} = 22.36\ rad/s \tag{1.4}$$

the dimensionless damping coefficient is given by:

$$\zeta_1 = \frac{c_1}{2m_1\omega_1} = \frac{10^5}{2 \times 10^4 \times 22.36} = 0.223 \tag{1.5}$$

and the damped pulse is therefore:

$$\omega_{s_1} = \omega_1\sqrt{1 - \zeta_1^2} = 21.80\ rad/s \tag{1.6}$$

For the second system we have:

$$\omega_2 = \sqrt{\frac{K_2}{m_2}} = \sqrt{\frac{500}{1}} = 22.36\ rad/s \tag{1.7}$$

the dimensionless damping coefficient is given by:

$$\zeta_2 = \frac{c_2}{2m_2\omega_2} = \frac{10}{2 \times 1 \times 22.36} = 0.223 \tag{1.8}$$

and the damped pulse is therefore:

$$\omega_{s_2} = \omega_2\sqrt{1 - \zeta_2^2} = 21.80\ rad/s \tag{1.9}$$

The two systems, which are physically completely different, have the same natural frequencies and the same damping coefficient. However, although equivalent in this respect, the two systems are different in terms of response.
The accelerometer is the basic transducer for evaluating the dynamic behavior of a system. It allows to identify the local level of stress and the vibrational characteristics of a structural system, therefore its response characteristics.

## 1.2 Seismic transducer and accelerometer

Vibration measurement can be performed with a seismic transducer. It consists of a mass system, a spring and a damper connected to a structure, as shown in Fig.1.1. In this case the dynamics equation is:

$$m\ddot{x} + c(\dot{x} - \dot{u}) + k(x - u) = 0 \tag{1.10}$$

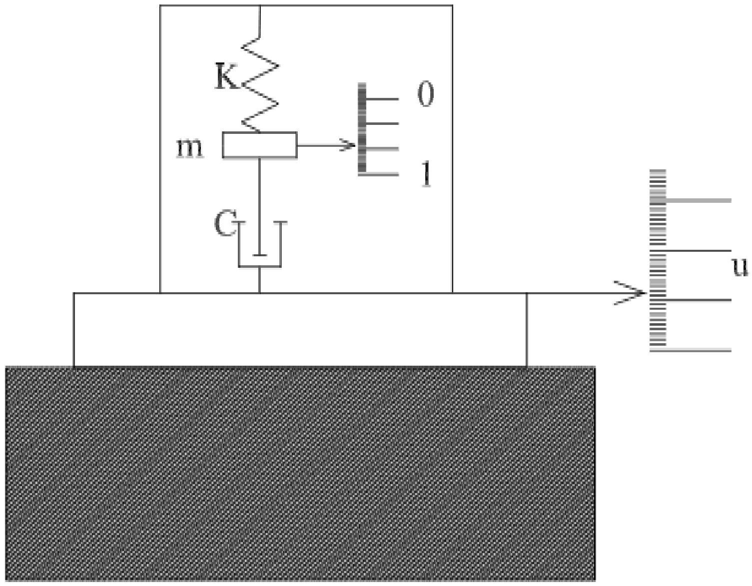


Figure 1.1: Functional diagram of an accelerometer.

where $m$, $c$, $k$ are the characteristics of mass, damping and stiffness of the system, $x$ indicates the displacement of the mass $m$ and $u$ indicates the displacement of the connection base, Fig. 1.1.
Let us indicate with $z = x - u$ the relative motion of the mass $m$ with respect to the structure, we have:

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{u} \tag{1.11}$$

in the case of simple harmonic motion $u(t) = u \cos \omega t$ of the connection base, we have the classic expression of the damped system with harmonic input.
Let us denote by $\zeta$ the dimensionless damping coefficient, which is the ratio between the damping coefficient $c$ of the system and the critical damping value:

$$\zeta = c/c_c = c/2\sqrt{mk} \tag{1.12}$$

We obtain the following solution for the system equation 1.10:

$$|z/u| = \frac{(\omega/\omega_n)^2}{\sqrt{\left(1 - (\omega/\omega_n)^2\right)^2 + 4\zeta^2 (\omega/\omega_n)^2}} \tag{1.13}$$

with:

$$\phi = -\arctan\left[2\zeta(\omega/\omega_n)/\left(1 - (\omega/\omega_n)^2\right)\right] \tag{1.14}$$

Let us take a look to the trend of Fig. 1.2 in which $\zeta$ appears as a parameter; $|z/u|$ tends to 1 as the ratio $\omega/\omega_n$ increases, whatever the system damping characteristic is (i.e. the value of the dimensionless damping coefficient $\zeta$). This means that the higher the oscillation frequency is

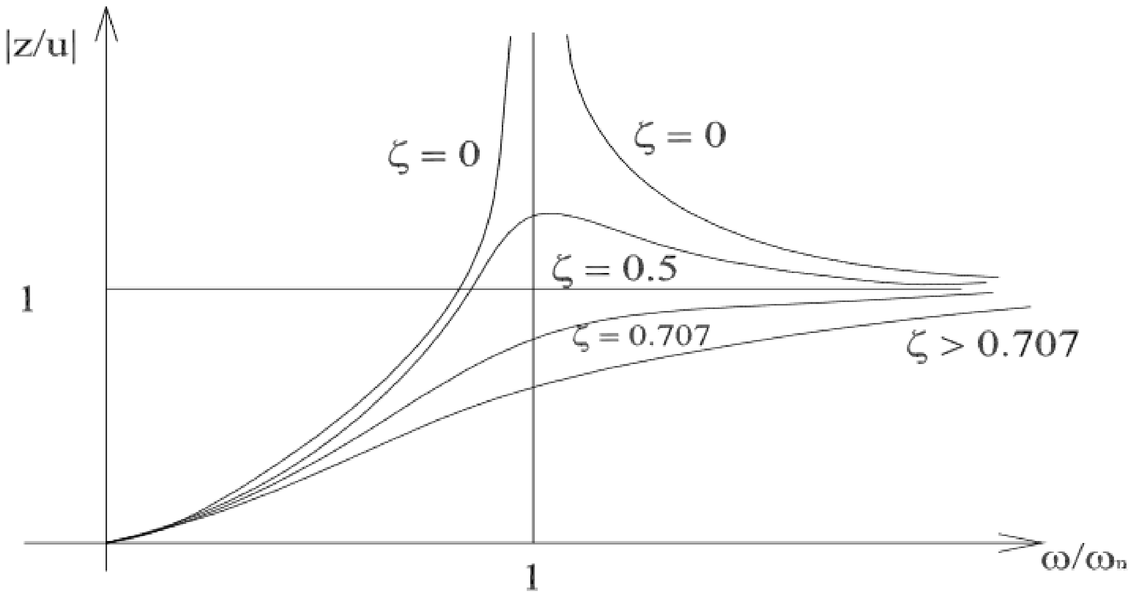


Figure 1.2: Module of the response curve of a seismic sensor.

compared to the natural frequency of the system, the closer the output $z$ is to the displacement $u$ of the structure under test.
The frequencies involved in structural problems in aerospace are relatively low, e.g. values between a few $Hz$ and a few hundred $Hz$. In this case, the insturment's natural frequency must be at most a few $Hz$ and therefore its mass must be relatively large, with the conseguent risk of disturbing the measurement with very large insertion errors.
Partly as a result of these considerations, the measurement of the vibrations is generally carried out with the direct measurement of accelerations and not of displacements, as considered above, with the use of transducers known as *accelerometers*: these transducers consist of very small masses which do not influence the behaviour of the structure by their presence. In order to change from acceleration to displacement measurement, it is necessary to perform a double integration on the signal taken from the sensor. Let us consider a sinusoidal input :

$$u = u^* \cos \omega t \tag{1.15}$$

deriving twice, we get:

$$\ddot{u} = -u^* \omega^2 \cos \omega t \tag{1.16}$$

and therefore from 1.13 we obtain:

$$\omega_n^2 |z/\ddot{u}| = \frac{1}{\sqrt{\left(1 - (\omega/\omega_n)^2\right)^2 + 4\zeta^2 (\omega/\omega_n)^2}} \tag{1.17}$$

from this relation it can be seen that the relative displacement $z$ is practically proportional to the acceleration $\ddot{u}$ of the body on which the accelerometer is fixed.
In Fig. 1.3 $\omega_n^2 |z/\ddot{u}|$ as a function of the ratio $\omega/\omega_n$, where $\omega_n$ is the accelerometer undamped

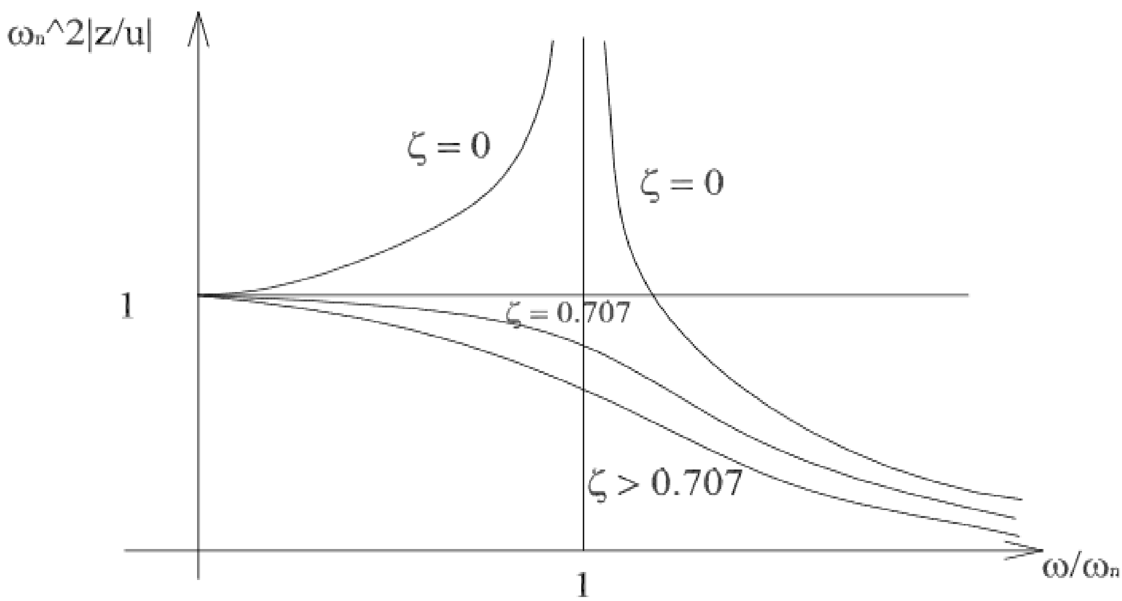

Figure 1.3: Module of the accelerometer response curve.

pulsation, is reported. For $\omega/\omega_n$ tending to zero, the ratio $\omega_n^2|z/\ddot{u}|$ tends to one whatever the value of $\zeta$ is, while for $\omega/\omega_n$ tending to infinity, the ratio $\omega_n^2|z/\ddot{u}|$ tends to zero whatever the value of $\zeta$ is. From Fig. 1.3, it can be seen that $z$ tends to the value of $\ddot{u}$ as the pulsation $\omega$ is smaller than $\omega_n$ and that the most suitable value for the dimensionless damping coefficient is $\zeta = 0.707 = 1/\sqrt{2}$.
The higher the natural frequency, the smaller the displacement for the same acceleration; if a very high bandwidth is to be achieved, the natural $\omega_n$ pulsation must be increased and therefore the mass of the accelerometer must be decreased and its stiffness increased (for the natural pulsation holds $\omega_n = \sqrt{k/m}$). This reduces the effect of disturbance of the transducer on the structure and thus limits the insertion error, but also reduces the transducer sensitivity. Piezoelectric accelerometers can have very high natural frequencies, for example $f_n = 10^5\ Hz$; if we consider a bandwidth limited to 20 % of $f_n$, they can be used up to $f_n = 2 \times 10^4\ Hz$.

## 1.3 Piezoelectric accelerometers

Piezoelectric accelerometers can be made of crystals which, when stressed in one direction, exhibit charges proportional to the stress in a direction other than the direction of stress.
All transducers consist of a base connected to the structure, a piezoelectric crystal, and a mass that are contained within a protective casing.
Let us consider a quartz crystal stressed with a force $F$, fig. 1.4. As a result of the application of the force, $+Q$ and $-Q$ charges are presented on surfaces other than the stress surface:

$$Q = d_{ij}F \tag{1.18}$$

where $d_{ij}$ (with characteristic values around $10^{-12}\ C/N$) indicates a piezoelectric constant that provides the amount of charge that the quartz presents due to stress.

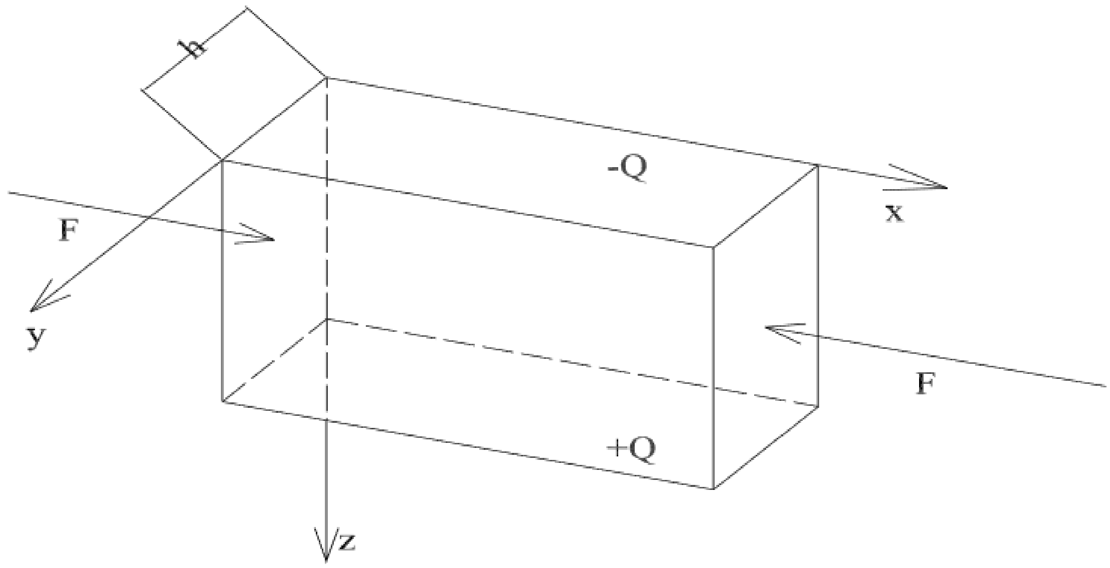


Figure 1.4: Elementary scheme of a piezoelectric accelerometer.

The $+Q$ and $-Q$ charges, which are caused by the presence of the force $F$, are separated by a dielectric, which consists of the quartz crystal itself, thus forming a condenser defined by:

$$Q = C\ V \tag{1.19}$$

where $C$ is the capacitor capacity:

$$C = \epsilon A_q / h \tag{1.20}$$

where $A_q$ is the surface (charged) of the quartz, $h$ is the distance between the capacitor plates, i.e. the crystal thickness, and $\epsilon$ is the dielectric constant. A voltage is thus obtained:

$$V = \frac{d_{ij} F}{C} = \frac{h d_{ij} m}{\epsilon A_q} \ddot{y} = K_q \ddot{y} \tag{1.21}$$

where $\ddot{y}$ is the acceleration along the $y$-direction ($F = m\ddot{y}$) and $m$ is the mass supportive with the quartz crystal.
The voltage $V$ is therefore proportional to the inertia force of the mass $m$ and thus to its acceleration; but the voltage $V$ requires the presence of electric charges which tend to "discharge" through the capacitor, therefore the amplifier which must detect voltage $V$ is likely to have a very high input impedance.
If we consider the capacity of the quartz crystal, denoted $C_q$, that of the connecting cables, $C_c$, and that of the amplifier, $C_a$, then the total capacity, $C_T$, is:

$$C_T = C_q + C_c + C_a \tag{1.22}$$

similarly with regard to the resistances we have:

$$1/R_T = 1/R_q + 1/R_c + 1/R_a = \left(R_q R_a + R_q R_c + R_c R_a\right) / R_q R_a R_c \tag{1.23}$$

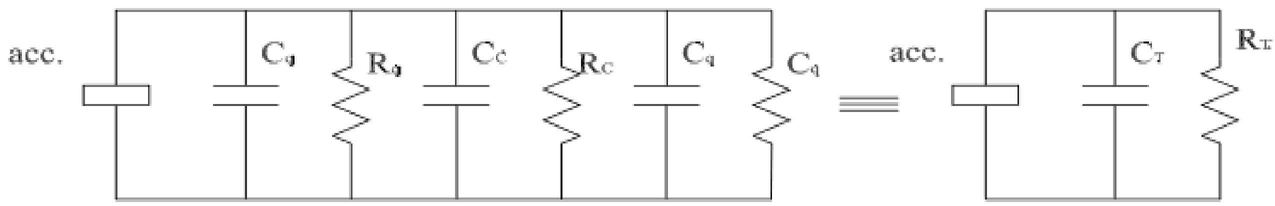


Figure 1.5: Equivalent scheme of a piezoelectric accelerometer.


Figure 1.6: Effective bandwidth of a piezoelectric accelerometer.

and so

$$R_T = \frac{R_q R_c R_a}{R_q R_a + R_q R_c + R_c R_a} \tag{1.24}$$

as shown in the circuits of fig. 1.5. This is therefore a capacitor of $C_T$ capacity, given by 1.22, which is discharged onto a resistor of $R_T$ resistance, provided by 1.23, and the voltage varies over time according to the relation:

$$V(t) = V_0 e^{-t/R_T C_T} = V_0 e^{-t/\tau} \tag{1.25}$$

consequently the time constant $\tau = R_T C_T$ must be much larger than the time it takes to make the measurement, which is related to the signal period. Therefore it will be difficult to measure signals with very long periods, i.e. signals that are slowly variable over time.
Of course, various effects due to connecting cables must also be taken into account: they have small capacities, which vary with their length, and resistances which vary depending on environmental conditions such as temperature.
The overall characteristics of a piezoelectric accelerometer are related to the product of the mechanical transfer function and that due to the electrical circuit. In Fig. 1.6 an indicative trend of the overall transfer function, $H_T(f)$, as a function of frequency, is reported. Three operating regions can be identified:

- at low frequency up to $f_1$[1] the response is determined by the electrical circuit;
- between the frequencies $f_1$ and $f_2$ the response is close to ideal. This is the working region of the accelerometer;
- above $f_2$ the response is determined by the mechanical transfer function; note the resonance peak corresponding to the accelerometer resonance frequency $f_n$.

It can be seen from fig. 1.6 that if the accelerometer is based on a semiconductor strain gauge, rather than a pietzoelectric, the low frequency response, in the field $0 - -f_1$, remains unity up to zero frequency.
The construction details of piezoelectric accelerometers vary depending on the manufacturer and the objectives of the accelerometer. Secondary effects such as temperature, acoustic pressure, bending of the base, magnetic fields, etc. must be taken into account.
The crystal is generally preloaded so as to work with linear characteristics. This preload also serves to work with the crystal in compression for both positive and negative accelerations.
The development of microcircuits has made it possible to incorporate the charge amplifier into the accelerometer itself.
Miniature accelerometers measuring $3 \times 3 \times 3$ $mm$ and with a mass less than half a gram and triaxial accelerometers measuring $7 \times 7 \times 7$ $mm$ with a mass less than one gram can be used.
Trasverse sensitivity is relatively high, in the order of a few percent.
The connection to the measuring structure can be made with wax, magnetic or mechanical devices. The connection leads to a reduction of the natural frequency compared to that of the unconnected acceleromete, the reduction can be of the order of 50%.

## 1.4 Angular acceleration measurements

Several methods have been proposed for angular acceleration measurements, some of which are still under development.
Generally, reference is made to a pair of accelerometers placed at a small fixed and known distance, indicated by $L$ in fig. 1.7. From the measurement of the accelerations of points $A$ and $B$, indicated with $\ddot{x}_A$ and $\ddot{x}_B$, the accelerations $\ddot{x}_0$ and $\ddot{\theta}_0$ can be obtained. In fact:

$$\ddot{x}_0 = (\ddot{x}_A + \ddot{x}_B)/2 \tag{1.26}$$

$$\ddot{\theta}_0 = (\ddot{x}_A - \ddot{x}_B)/L \tag{1.27}$$

It is observed that the measurement of angular acceleration requires the difference between two acceleration values $\ddot{x_A}$ e $\ddot{x_B}$ that are very close to each other; in general this difference is just a few percent of their value (as little as one or two percent) and therefore the error made in the measurement is very large. For example, the sensitivity to the transverse component of acceleration can be of the same order of magnitude as the sensitivity of the measurement; this justifies the uncertainty that still exists this type of measurement.
More recently transducers have been developed, again based on piezoelectric elements, which can measure linear and angular accelerations together with high sensitivity (of the order of 1000 $mV/g$ e 50 $mV/rad/s$) and bandwidth from 0.5 to 2000 $Hz$.

[1]The value of $f_1$ depends on the characteristics of the single accelerometer but is tipically of the order of a few $Hz$.

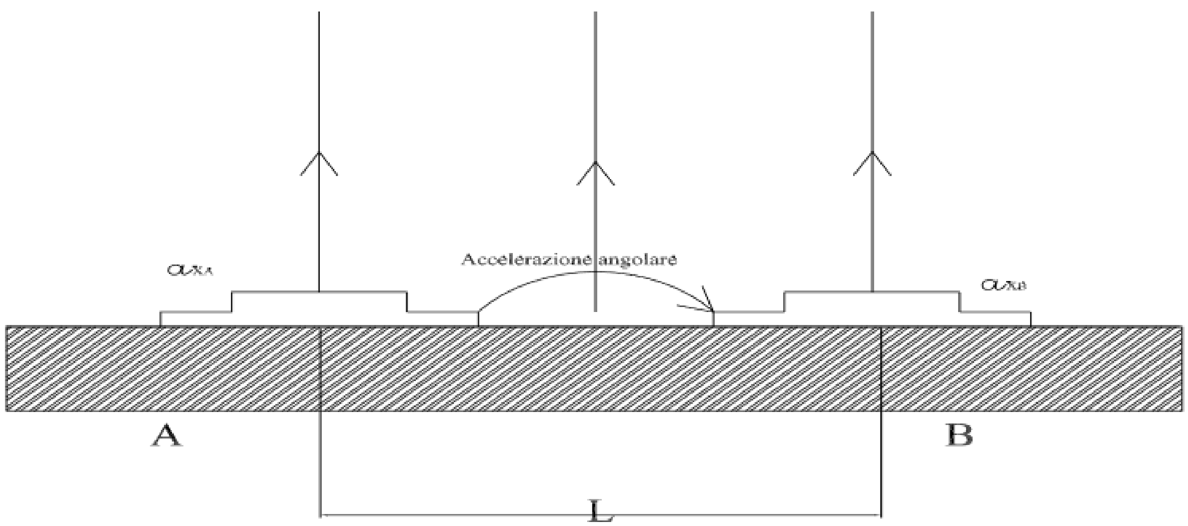


Figure 1.7: Scheme of a rotation sensor.