Chapter 1

Mathematical Models in Structural Dynamics

1.1 Descriptions of the dynamics of a structure

The dynamic model of a structure is given by a multi-degree-of-freedom system; however, this model is an approximation of the actual situation, which is that of a continuous structure and therefore characterised by an infinite number of degrees of freedom. For a multi-degree-of-freedom system one has:[1]

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}(t) \tag{1.1}$$

where $\mathbf{x}$ is a vector, with n components, which includes the degrees of freedom chosen for the representation of the structure (from the experimental point of view they can be the measurement points), $\mathbf{f}(t)$ is the vector of the forces acting on the structure, $\mathbf{M}$ the is mass matrix, $n \times n$, and similarly $\mathbf{C}$ and $\mathbf{K}$ are the viscous damping and stiffness matrices, $n \times n$. The model defined by the 1.1 is the spatial model consisting of the mass, damping and stiffness matrices, and is normally constructed by a numerical procedure (e.g. using the finite element method) and is therefore not generally accessible by the experimental approach. From the study of free vibration, which can be obtained numerically by determining the eigenvalues and eigenvectors of the system 1.1 (with $\mathbf{f(t)}$ null), or from experimentation, n natural pulsations, ω_{n_n}, n damping coefficients, ζ_n, and n deformed modals, $\phi^{(n)}$, can be obtained; these matrices of eigenvectors, $\mathbf{\Phi} = \left[\phi^{(1)}|\phi^{(2)}|, ..., \phi^{(n)}\right]$ and of eigenvalues $\mathbf{\Omega}^2 = \begin{bmatrix} \omega_{n_1}^2 & & & \\ & \omega_{n_2}^2 & & \\ & & \ddots & \\ & & & \omega_{n_n}^2 \end{bmatrix}$ constitute the *modal model*.

From an experimental point of view, there are strong limitations in obtaining a large number of fundamental modes of a structure and also in carrying out the measurement on a very large number of measuring points. From the evaluation of the frequency response functions of the structure, the *frequency response* model is determined; this is directly derived from the classical

[1] In the following, the matrices will be represented in bold uppercase letters and the vectors in lowercase bold characters.

experimental approach in modal analysis and can also be derived from the numerical approach. In summary, three different models can be defined for studying the dynamics of a structure:

- space;
- modal;
- of frequency response functions.

They constitute different but equivalent ways of representing the dynamic behavior of a structure and can be determined numerically or experimentally. The comparison between numerical and experimental results can also be carried out on the basis of these models.

1.2 Model with a single degree of freedom - SDOF

The single degree-of-freedom model cannot represent the behavior of a structural element, but its characteristics are important because those of the multi-degree of freedom model come from them.
Let us consider the system characterized by a mass, m, and a spring of stiffness k (*non-damped model*).

- The equation of motion in the case of free vibration is:

$$m\ddot{x} + k\ x \ = \ 0 \tag{1.2}$$

  If the above characteristic equation is considered, i.e. solutions of the type are sought:

$$x(t) \ = \ x^* e^{st} \tag{1.3}$$

  then:

$$(s^2 m + k) \ = \ 0 \qquad \longrightarrow \qquad s_{1,2} = \pm j\sqrt{\frac{k}{m}} = \pm j\omega_n \tag{1.4}$$

  thus, there is a free response of the type $x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} = x_0 \cos(\omega_n t) + \frac{\dot{x}_0}{\omega_n} \sin(\omega_n t)$. In this case, the "modal model" consists of a vibration mode whose natural pulsation is given by:

$$\omega_n \ = \ \sqrt{k/m} \tag{1.5}$$

  and the modal deformation is given by a constant.

- In the case of *forced vibration*, an input function $f(t)$ of harmonic type with ω pulse is considered :[2]

$$f(t) \ = \ f^* e^{j\omega t} \tag{1.6}$$

[2] Eq. 1.6 is actually an equation in the complex field, since $Re(f^* e^{j\omega t}) = f^* \cos(\omega t)$ and $Im(f^* e^{j\omega t}) = f^* \sin(\omega t)$ and the system in question is linear. By considering respectively the real or the imaginary part of the output (Eq. 1.7), we would obtain the response (in real field) at steady state to the input $f^* \cos(\omega t)$ and to the input $f^* \sin(\omega t)$.

By imposing:

$$x(t) \;=\; x^* e^{j\omega t} \tag{1.7}$$

we have:

$$(-\omega^2 m + k)x^* e^{j\omega t} \;=\; f^* e^{j\omega t} \tag{1.8}$$

hence the frequency response function is:

$$\frac{x^*}{f^*} \;=\; \frac{1}{k - \omega^2 m} \;=\; H(\omega) \tag{1.9}$$

which can be interpreted as the relationship between displacement and harmonic input force and therefore represents a *dynamic flexibility* also referred to as *receptivity* or *admittance*; we note that the response function $H(\omega)$ does not actually depend on the type of input function and therefore constitutes an intrinsic characteristic of the system. The module of the $H(\omega)$ is given by:

$$|H(\omega)| \;=\; \frac{1}{\sqrt{(k - \omega^2 m)^2}} \tag{1.10}$$

The presence of a *viscous damping term* with damping coefficient c is now considered.

- The equation of motion for *free vibration* is:

$$m\ddot{x} + c\dot{x} + kx \;=\; 0 \tag{1.11}$$

in the search for characteristic exponents, let us impose:

$$x(t) \;=\; x^* e^{st} \tag{1.12}$$

and we have:

$$ms^2 + cs + k \;=\; 0 \tag{1.13}$$

from which:

$$s_{1,2} \;=\; -\,\omega_n \zeta \pm i\omega_n \sqrt{1 - \zeta^2} \tag{1.14}$$

considering that:

$$\omega_n^2 \;=\; \frac{k}{m} \qquad\qquad \zeta \;=\; \frac{c}{c_0} \;=\; \frac{c}{2\sqrt{km}} \tag{1.15}$$

a solution of this type is obtained:

$$x(t) \;= c_1 e^{s_1 t} + c_2 e^{s_2 t} =\; c_1 e^{-\sigma t} e^{j\omega_n' t} +\; c_2 e^{-\sigma t} e^{-j\omega_n' t} \tag{1.16}$$

that is a mode of vibration with a complex natural frequency characterized by an imaginary part:

$$\omega_n' \;= \omega_n \sqrt{1 - \zeta^2} \tag{1.17}$$

and a real part:

$$\sigma = \omega_n \zeta \tag{1.18}$$

Due to the damping term, the ω_n' pulsation is different from the ω_n pulsation of the non-damped system. However, for aerospace structural elements (in the case of multi degrees-of-freedom), the numerical difference terms is very limited since the values of the dimensionless damping coefficients are very small, typically of the order of percent or lower. Let us now consider a general aviation aircraft represented by a $SDOF$ model with mass $m = 2000\ Kg$ in pilot-only take-off onditions and with mass $m^* = 3000\ Kg$ in the maximum take-off weight configuration. The displacement of the elastic system of the trolley under load conditions with only the pilot was measured at $x = 0.045\ m$.
If the system is required to work with a value of the damping coefficient $\zeta = 0.9$,[3] the value of the viscous damping coefficient, c, is evaluated. In the static case $K\ x = f$, therefore:

$$K = \frac{f}{x} = \frac{2000 \times 9.81}{0.045} = 4.36 \times 10^5\ N/m \tag{1.19}$$

thus, the pulse is:

$$\omega = \sqrt{\frac{K}{m}} = \sqrt{218} = 14.76\ rad/s \tag{1.20}$$

and therefore $f = \dfrac{\omega}{2\pi} = 2.35\ Hz$. For the system to work with $\zeta = 0.9$, it must be

$$\zeta = c/2\omega m \quad c = 2\omega m\zeta = 2 \times 14.76 \times 0.9 \times 2000 = 5.31 \times 10^4\ Kg/s \tag{1.21}$$

In the case of a fully loaded aircraft, i.e. with stiffness and viscous damping values equal to those previously evaluated but with a larger mass, we have:

$$\omega^* = \sqrt{\frac{K}{m^*}} = 12.05\ rad/s \tag{1.22}$$

The damping coefficient becomes:

$$\zeta^* = c/2\omega^* m^* = 0.734 \tag{1.23}$$

therefore the aircraft under fully loaded conditions behaves differently from the pilot-only aircraft and it is not possible to obtain the same oscillatory behavior, i.e. the same value of ζ, in the two configurations.

- In the case of forced vibration of harmonic type with pulsation ω, let us consider again:

$$f(t) \ = f^* e^{j\omega t} \tag{1.24}$$

therefore:

$$(-\omega^2 m + j\omega c + k)x^* e^{j\omega t} = f^* e^{j\omega t} \tag{1.25}$$

[3] It should be noted that the damping coefficient is very high in this case because the oscillatory behaviour of the system is to be reduced.

Thus the frequency response function, dynamic flexibility, is obtained from:

$$H(\omega) = \frac{x^*}{f^*} = \frac{1}{k - \omega^2 m + j\omega c} \tag{1.26}$$

In this case it is a complex quantity whose modulus is given by:

$$|H(\omega)| = \frac{1}{\sqrt{(k - \omega^2 m)^2 + (\omega c)^2}} \tag{1.27}$$

and whose phase is given by:

$$\tan H(\omega) = \frac{-\omega c}{k - \omega^2 m} \tag{1.28}$$

An examination of the actual behavior of the structures also suggests a different model for representing the damping characteristics; in particular the frequency dependence of structural characteristics can be represented by a damping that varies with frequency according to:

$$c = \frac{h}{\omega} \tag{1.29}$$

This is the *structural damping or hysteresis model* that corresponds to the equation (written in a mixed time-frequency notation):

$$m\ddot{x} + (k + jh)x = f(t) \tag{1.30}$$

In the case of forced response, the following frequency response function occurs:

$$H(\omega) = \frac{x^*}{f^*} = \frac{1}{k - \omega^2 m + jh} \tag{1.31}$$

which can be written as follows:

$$H(\omega) = \frac{1/k}{1 - (\omega/\omega_n)^2 + jh^*} \tag{1.32}$$

where $h^* = h/k$ is the structural loss factor. Therefore, the module of $H(\omega)$ is:

$$|H(\omega)| = \frac{1}{\sqrt{(k - \omega^2 m)^2 + h^2}} \tag{1.33}$$

The reasons for the introduction of this type of damping are related to the fact that if we consider the energy dissipated by a force of a viscoelastic component $f_d = c\dot{x}$ in a cycle of a harmonic motion with $x(t) = \sin(\omega t)$ of period $T = 2\pi/\omega$, we have:

$$\mathcal{E}_d = \int_0^T f_d \dot{x} dt = \int_0^{2\pi/\omega} c\omega^2 \cos^2(\omega t) dt = \pi c\omega \tag{1.34}$$

the energy dissipated depends linearly on the frequency of the motion, a fact that does not have experimental evidence. The characterization of the hysteretic dissipating force, as introduced above, is evidently able to overcome this modelling error.

1.2.1 Frequency response functions for the model SDOF

A frequency response function $H(\omega)$ has been defined as the ratio between the displacement, x^*, and the force, f^*. Of course it is also possible to choose a different frequency response function to describe the system: for example, with reference to the speed $v(t) = \dot{x}(t) = v^* e^{j\omega t}$ as the output quantity, we can define a frequency response function, indicated with *mobility*, with the:

$$Y(\omega) = \frac{v^*}{f^*} \tag{1.35}$$

By considering the following relations:

$$x(t) = x^* e^{j\omega t} \quad \longrightarrow \quad v(t) = \dot{x}(t) = v^* e^{j\omega t} = j\omega x^* e^{j\omega t} \tag{1.36}$$

we have:

$$Y(\omega) = j\omega \, \frac{x^*}{f^*} = j\omega H(\omega) \tag{1.37}$$

with the following relations for the module:

$$|Y(\omega)| = \omega |H(\omega)| \tag{1.38}$$

and for the phase:

$$\theta_Y = \theta_H + 90^\circ \tag{1.39}$$

Reference can also be made to the acceleration $a(t) = \ddot{x}(t) = a^* e^{\omega t}$ as the output quantity, thus defining the FRF known as *accelerance*:

$$A(\omega) = \frac{a^*}{f^*} = -\omega^2 H(\omega) \tag{1.40}$$

As we have said, $FRFs$ are complex functions and therefore and therefore cannot be represented directly on a Cartesian plane; the classic types of representation are:

- module (normally expressed in *decibels*, dB) as a function of pulsation (in logarithmic scale) and phase as a function of pulsation (in logarithmic scale), *Bode diagram*;
- real part as a function of frequency (or pulse) and imaginary part as a function of frequency (or pulsation); with reference to the case of viscous damping one has

$$Re[H(\omega)] = \frac{k - \omega^2 m}{(k - \omega^2 m)^2 + \omega^2 c^2} \quad \text{e} \quad Im[H(\omega)] = \frac{-\omega c}{(k - \omega^2 m)^2 + \omega^2 c^2} \tag{1.41}$$

  with the trends reported in Fig. 1.1 and Fig. 1.2.

- real part and imaginary part on a polar diagram, with frequency as a parameter, *Argand or Nyquist diagram.*[4]

[4] In the case of hysteretic damping, Eq. 1.31, real and imaginary part functions are:

$$H_R = \frac{k - \omega^2 m}{(k - \omega^2 m)^2 + h^2} \quad H_I = \frac{-h}{(k - \omega^2 m)^2 + h^2}$$

from which the curve in the implicit form is obtained:

$$H_R^2 + H_I^2 = -\frac{H_I}{h}$$

which obviously represents a circumference passing through the origin.

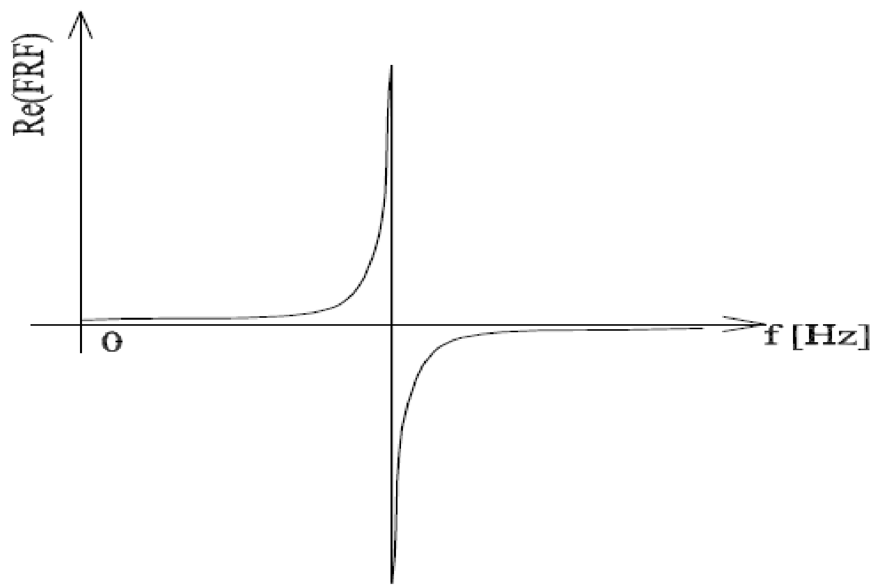

Figure 1.1: Real part of a FRF of a system with a degree of freedom.

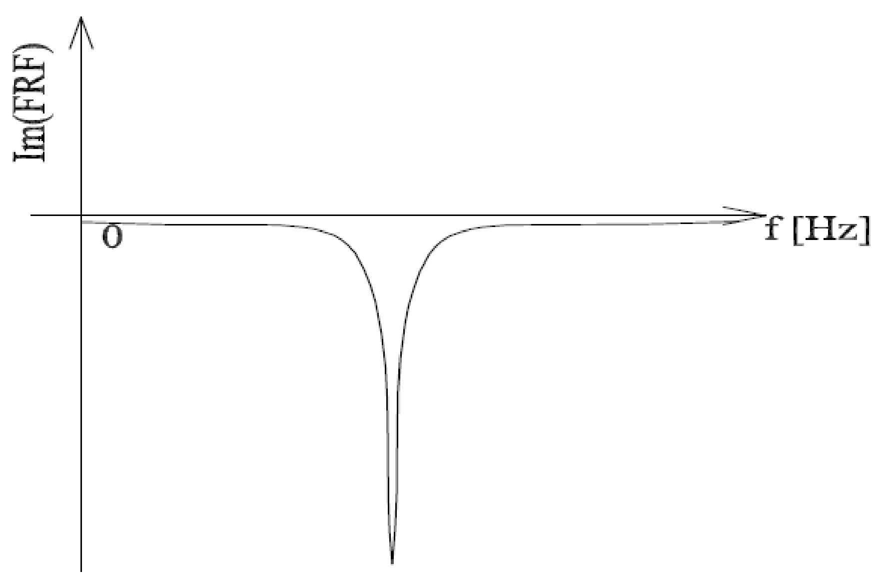

Figure 1.2: Imaginary part of a FRF of a system with a degree of freedom.

The Argand diagram is widely used for its particular effectiveness in presenting in detail the area of the FRF in the neighborhood of the resonance frequency, while the points that are far from the resonance are shifted around the origin of the diagram. In particular, if we consider the FRF relative to the velocity $Y(\omega)$, we see that the modulus as a function of frequency has a symmetrical diagram with respect to the resonance frequency for small damping.
Argand diagrams related to the FRF of velocity, $Y(\omega)$, in the case of viscous damping or to the FRF of displacement, $H(\omega)$, in the case of structural damping are circumferences: this feature is very useful for the " curve fitting " procedure which can be used for the evaluation of modal parameters.

1.3 Model with multiple degrees of freedom - MDOF

We now move on to extend the considerations seen in the case of a single-degree-of-freedom model, $SDOF$, to the greater practical interest case of a multi-degree-of-freedom model, $MDOF$ (Multi Degree of Freedom).

1.3.1 Undamped case: free vibration, vibration modes and frequencies

For the multi-degrees-of-freedom model the equations of motion, in the non-damped case, are:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{f}(t) \tag{1.42}$$

where $\mathbf{M}$ and $\mathbf{K}$ are the mass and stiffness matrices, with dimensions $n \times n$ if n degrees-of-freedom are considered in the system 1.42, $\mathbf{x}(t)$ and $\mathbf{f}(t)$ are the vectors of the displacements and of the applied, again with n components. The mass matrix is a positive definite matrix and the stiffness matrix is a positive semi-definite matrix by virtue of the well-known properties of the homonymous energies associated to these matrices. In particular, for a generic non-zero vector $\mathbf{u}$, it is observed that:

$$\mathbf{u^T M u} > \mathbf{0} \qquad \mathbf{u^T K u} \geq \mathbf{0} \tag{1.43}$$

It is possible to associate these matrices with the eigenvalues problem : [5]

$$(\mathbf{K} - \lambda_n \mathbf{M})\phi^{(n)} = 0 \tag{1.44}$$

in which, as usual, the eigenvalues are calculated by solving the characteristic equation:

$$det(\mathbf{K} - \lambda^2 \mathbf{M}) = 0 \tag{1.45}$$

and the eigenvectors from the corresponding homogeneous problems given by Eq. 1.44. If Eq. 1.44 is written once with reference to the $n-mo$ eigenvalue and once to the $m-mo$ eigenvalue and if relations obtained are respectively premultiplied by ${\phi^{(m)}}^T$ and ${\phi^{(n)}}^T$, we have:

$$ {\phi^{(m)}}^T \mathbf{K} \phi^{(n)} = \lambda_n {\phi^{(m)}}^T \mathbf{M} \phi^{(n)} \tag{1.46}$$
$$ {\phi^{(n)}}^T \mathbf{K} \phi^{(m)} = \lambda_m {\phi^{(n)}}^T \mathbf{M} \phi^{(m)} \tag{1.47}$$

[5] In Appendix **??**, free response problems for multi-degree-of-freedom systems are directly addressed from an algebraic point of view as naturally associated with a standard eigenvalue problem (see Parr. **??** and **??**).

If we subtrac one from each other and by virtue of the symmetry of the two matrices, it follows:

$$0 = (\lambda_n - \lambda_m)\phi^{(n)^T}\mathbf{M}\phi^{(m)} \tag{1.48}$$

that is, if the eigenvalues λ_n and λ_m are distinct then it must be $\phi^{(m)^T}\mathbf{M}\phi^{(m)} = 0$, otherwise , when they are equal, the product $\phi^{(m)^T}\mathbf{M}\phi^{(m)}$ is a posivie quantity, denoted by m_n, due to the *positivity* of the matrix $\mathbf{M}$. Therefore:

$$\phi^{(n)^T}\mathbf{M}\phi^{(m)} = \delta_{mn}m_n \quad \text{i.e.} \quad \mathbf{\Phi^T M \Phi} = \begin{bmatrix} \ddots & & \\ & m_n & \\ & & \ddots \end{bmatrix} \tag{1.49}$$

where ϕ is the matrix having the eigenvectors as columns $\phi^{(n)}$.
Let us now consider, instead of Eq. 1.44, its equivalent

$$\left(\frac{1}{\lambda_n}\mathbf{K} - \mathbf{M}\right)\phi^{(n)} = 0 \tag{1.50}$$

By repeating the previous procedur, the following orthogonality relation is obtained:

$$\phi^{(n)^T}\mathbf{K}\phi^{(m)} = \delta_{mn}k_n \quad \text{i.e.} \quad \mathbf{\Phi^T K \Phi} = \begin{bmatrix} \ddots & & \\ & k_n & \\ & & \ddots \end{bmatrix} \tag{1.51}$$

where the generalized stiffnesses k_n can only be positive (or null) by virtue of the semi-positivity of $\mathbf{K}$.
From Eq. 1.44 premultiplied by $\phi^{(n)^T}$, it is also observed that:

$$\lambda_n = \frac{\phi^{(n)^T}\mathbf{K}\phi^{(n)}}{\phi^{(n)^T}\mathbf{M}\phi^{(n)}} = \omega_n^2 > 0 \tag{1.52}$$

i.e. that the eigenvalue λ is positive and will therefore be indicated in the following with ω_n^2.
Lt us now show that the vectors $\phi^{(n)}$ and constants ω_n, just defined, take on the meaning of *natural (angular) modes and frequencies of vibration*. If so, by physical definition of natural modes and frequencies of vibration, the free problem is:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = 0 \tag{1.53}$$

$$\mathbf{x}(0) = \phi^{(m)} \tag{1.54}$$

$$\dot{\mathbf{x}}(0) = 0 \tag{1.55}$$

should provide the solution:

$$\mathbf{x}(t) = \phi^{(m)}\ \cos(\omega_n t) \tag{1.56}$$

If we use the coordinate change $\mathbf{x} = \mathbf{\Phi q}$ and if we premultiply Eq. 1.53 by $\mathbf{\Phi^T}$, we obtain a series of ordinary differential equations all decoupled, whose nth is:

$$m_n\ddot{q}_n + k_n q_n = 0$$

whose solution is

$$q_n(t) = q_{0_n}\cos(\omega_n t) + \frac{\dot{q}_{0_n}}{\omega_n}\sin(\omega_n t)$$

being $\mathbf{x}(0) = \mathbf{\Phi}\mathbf{q}_0$ and $\dot{\mathbf{x}} = \mathbf{\Phi}\dot{\mathbf{q}}_0$. Reconstructing the original solution and according to the initial conditions of the problem 1.54 and 1.55 we have Eq. 1.56. Therefore, in the following we will identify ω_n and $\phi^{(n)}$ directly as (angular) frequencies and modes of the structure under consideration (although in the finite element discretization of the structure, they would represent a disc).
The complete solution of the problem is represented by the matrices of the eigenvalues, indicated with $\mathbf{\Omega^2}$ which is a diagonal matrix cointaining on the main diagonal the squared natural pulsations, and by that of the eigenvectors, indicated with $\mathbf{\Phi}$ that contains, positioned for columns, the modal deformations $\phi^{(\mathbf{n})}$.
With numerical procedures based on the resolution of Eq. 1.44 it is possible to pass from spatial matrices, of mass $\mathbf{M}$ and stiffness $\mathbf{K}$, to the matrices representing the modal model, indicated with $\mathbf{\Omega^2}$ e ϕ. It is to be remembered that the matrix of natural pulsations, $\mathbf{\Omega^2}$, is univocally defined while the matrix of modal deformations, $\mathbf{\Phi}$, is not, since the single deformed $\phi^{(n)}$ are defined, but a constant, as self-solutions of the homogeneous problem 1.44.
Several procedures can be used for the *normalization* of modal deformations, the most significant being that of normalization with respect to mass; in this case the eigenvectors, indicated in the matrix ϕ^*, are defined by the relations:

$$\mathbf{\Phi}^{*\mathbf{T}}\mathbf{M}\mathbf{\Phi}^* = \mathbf{I} \tag{1.57}$$

$$\mathbf{\Phi}^{*\mathbf{T}}\mathbf{K}\mathbf{\Phi}^* = \mathbf{\Omega^2} \tag{1.58}$$

where $\mathbf{I}$, $\mathbf{\Omega^2}$ are respectively the unitary diagonal matrix and the diagonal matrix of the natural pulsations; the relationship between the generic normalized mode k and its corresponding non-normalized mode is given by:

$$\phi^{*(k)} = \frac{1}{\sqrt{m_k}}\phi^{(k)} \tag{1.59}$$

1.3.2 Undamped case: forced response

Let us now consider, again for the non-damped model, the forced case, in which there is an input force vector characterized by components all at the same pulsation ω, but with different amplitude and phase, defined by the:[6]

$$\mathbf{f}(t) = \mathbf{f}^*\, e^{j\omega t} \tag{1.63}$$

[6] This way of proceeding has also been used in parr. **??**, **??** and 1.2, it is equivalent to considering the Fourier transform of Eq. 1.42: following this approach the vectors $\mathbf{x}^*$ and $\mathbf{f}^*$ would represent respectively the Fourier transform of the output vector $\mathbf{x}(t)$ and that of the inputs $\mathbf{f}(t)$. In fact, if we consider for example a sinusoidal input (causal) $f(t) = \sin(\Omega t)$, the Laplace trasform would be $\tilde{f}(s) = \frac{\Omega}{s^2+\Omega^2}$ and therefore the response would be:

$$\tilde{x}(s) = H(s)\frac{\Omega}{s^2+\Omega^2} = \tilde{x}_s(s) + \tilde{x}_r(s) \simeq \frac{\rho}{s-j\Omega} + \frac{\rho^*}{s+j\Omega} \tag{1.60}$$

whre $\tilde{x}_s$ indicates the part of the response connected with the (stable) poles of the system and $\tilde{x}_r$ the part linked to the input (steady-state response) that we want to consider here. If we look for the residual ρ and ρ^*, we get:

$$\rho = \frac{H(j\Omega)}{2j} \qquad \rho^* = \frac{H(-j\Omega)}{-2j} = \frac{\bar{H}(j\Omega)}{-2j} = \bar{\rho} \tag{1.61}$$

In this case, the solution of the system 1.42 is of the type:

$$\mathbf{x}(t) = \mathbf{x}^* \; e^{j\omega t} \tag{1.64}$$

where $\mathbf{f}^*$ and $\mathbf{x}^*$ are vectors with n components and complex amplitudes. The equation of motion 1.42 becomes:

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{x}^* e^{j\omega t} = \mathbf{f}^* e^{j\omega t} \tag{1.65}$$

a *dynamic flexibility matrix* can thus be defined, which constitutes a response model in the field of frequency response functions, FRF, with the:

$$\mathbf{H}(\omega) = (\mathbf{K} - \omega^2\mathbf{M})^{-1} \tag{1.66}$$

The generic element of the dynamic flexibility matrix can be defined as follows:

$$H_{jk}(\omega) = \frac{x_j^*}{f_k^*} \tag{1.67}$$

where $f_m^* = 0$ if m is different from k. As it is evident from Eq. 1.66 it is possible to calculate the values of the dynamic flexibility matrix, $\mathbf{H}(\omega)$, for each pulsation ω, if the matrices $\mathbf{M}$ and $\mathbf{K}$, belonging the spatial model, are known. This procedure requires the inversion of a matrix, generally of large dimensions, for each value of ω; this has several limitations because of the computational effort required if the number of degrees of freedom increases. In addition, the matrix $\mathbf{H}(\omega)$ must be calculated all at once and no information about the properties of the individual FRF is obtained.
A different approach can be used, which is generally convenient, whereby the dynamic flexibility matrix $\mathbf{H}(\omega)$ is calculated as a function of the modal model instead of the spatial model. From Eq. 1.66 we have:

$$(\mathbf{K} - \omega^2\mathbf{M}) = \mathbf{H}(\omega)^{-1} \tag{1.68}$$

If we pre-multiply by the transposed matrix of eigenvectors, normalized with respect to mass, and post-multiply by the matrix of normalized eigenvectors, we have:

$$\mathbf{\Phi}^{*T}(\mathbf{K} - \omega^2\mathbf{M})\mathbf{\Phi}^* = \mathbf{\Phi}^{*T} \; \mathbf{M}(\omega)^{-1} \; \mathbf{\Phi}^* \tag{1.69}$$

Using the orthogonality properties, Eq. 1.57 and 1.58, Eq. 1.69 becomes:

$$\begin{bmatrix} \ddots & & \\ & \omega_{n_k}^2 - \omega^2 & \\ & & \ddots \end{bmatrix} = \mathbf{\Phi}^{*\mathbf{T}}\mathbf{H}(\omega)^{\mathbf{-1}}\mathbf{\Phi}^* \tag{1.70}$$

It the domain of time, we would have: $(H(-j\Omega) \equiv \bar{H}(j\Omega))$

$$x_r(t) = \mathcal{L}^{-1}\left[\frac{H(j\Omega)}{2j(s - j\Omega)} - \frac{\bar{H}(j\Omega)}{2j(s + j\Omega)}\right] = |H(j\Omega)| \sin\left(\Omega t + H(j\Omega)^{\angle}\right) \tag{1.62}$$

where $H(j\Omega)^{\angle}$ indicates the phase of the complex number $H(j\Omega)$. *The frequency response function $H(j\omega)$ represents in a (complex) modulus and phase the module and phase of the system's response to a simple harmonic input of pulse Ω and unit amplitude and zero phase.*

from which if we reverse, pre-multiply by the matrix $\mathbf{\Phi}^*$ and post-multiply by the matrix $\mathbf{\Phi}^{*\mathbf{T}}$, we get:

$$\mathbf{H}(\omega) = \mathbf{\Phi}^* \begin{bmatrix} \ddots & & \\ & \frac{1}{\omega_{n_k}^2 - \omega^2} & \\ & & \ddots \end{bmatrix} \mathbf{\Phi}^{*\mathbf{T}} \tag{1.71}$$

From relation 1.71 we see that the dynamic flexibility matrix is a symmetrical matrix, in fact it results from the product of the matrix $\mathbf{\Phi}^*$,for a diagonal matrix, $\left(\mathbf{\Omega}^2 - \omega^2\mathbf{I}\right)^{-1}$, for its transpose, $\mathbf{\Phi}^{*\mathbf{T}}$; on the other hand, the matrix $\mathbf{H}(\omega)$ must be symmetrical according to the principle of reciprocity (Betti's theorem):

$$H_{jk}(\omega) = \frac{x_j^*}{f_k^*} = H_{kj}(\omega) = \frac{x_k^*}{f_j^*} \tag{1.72}$$

Eq. 1.71 allows to calculate the single element of the dynamic flexibility matrix from:

$$H_{jk}(\omega) = \sum_{r=1}^{n} \frac{\phi_j^{(r)}\phi_k^{(r)}}{m_r(\omega_{n_r}^2 - \omega^2)} \tag{1.73}$$

where the symbol $\phi_k^{(r)}$ indicates the $k-th$ component of the $r-th$ mode; the single element of the flexibility matrix can therefore also be written synthetically as:

$$H_{jk}(\omega) = \sum_{k=1}^{n} \frac{A_{jk}^{(r)}}{\omega_{n_r}^2 - \omega^2} \tag{1.74}$$

where $A_{jk}^{(r)}$ is the modal constant of the $r-th$ mode relative to j and k degrees of freedom.

1.3.3 Proportional damping

Let us now consider a special case of damping which has the advantage of great simplicity of analysis: the key point is that with this damping model the fundamental modes to be considered are practically the same as those of the non-damped model, in fact the modal deformations are identical and the natural frequencies are numerically very close. Therefore it is possible to derive the modal properties of a structure represented with a proportional damping starting from the study of the non-damped model.
In the presence of damping, the general equation of motion is:[7]

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}(t) \tag{1.76}$$

[7]If we assume that the effect of the damping is to dissipate the elastic and kinetic energy of the vibrating system, then, in the case of free vibration, Eq. 1.76 pre-multiplied for $\mathbf{x}^\mathbf{T}$ becomes:

$$\frac{d}{dt}\left(\frac{1}{2}\dot{\mathbf{x}}^T\mathbf{M}\dot{\mathbf{x}} + \frac{1}{2}\mathbf{x}^T\mathbf{K}\mathbf{x}\right) = -\dot{\mathbf{x}}^T\mathbf{C}\dot{\mathbf{x}} \tag{1.75}$$

since the sum of elastic energy and kinetic energy must decrease over time for each state condition, Eq. 1.75 implies that $\mathbf{C}$ must be *positive definite matrix.*

- Let us consider the damping matrix as proportional to the stiffness matrix, then:

$$\mathbf{C} = \beta \mathbf{K} \tag{1.77}$$

if we premultiply the damping matrix by the transpose of the eigenvectors matrix of the non-damped system and if we postmultiply it by the eigenvectors matrix of the non-damped system, we get:

$$\mathbf{\Phi^T C \Phi} = \beta \mathbf{\Phi^T K \Phi} = \beta \begin{bmatrix} \ddots & & \\ & k_k & \\ & & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & \\ & c_k & \\ & & \ddots \end{bmatrix} \tag{1.78}$$

where the elements c_k are the dampings of the individual modes of the model. The matrix obtained by performing this operation is diagonal as a result of the proportionality condition, 1.77; this indicates that the modal deformations of the non-damped system can be used for the system with proportional damping.
Let us consider the system 1.76 in the case of a *free response*. Multiplying by the transpose of the eigenvector matrix of the non-damped system we have:

$$\mathbf{\Phi^T M \ddot{x}} + \mathbf{\Phi^T C \dot{x}} + \mathbf{\Phi^T K x} = \mathbf{0} \tag{1.79}$$

Then replacing the physical coordinates with the *modal coordinates*, $\mathbf{q}$, with the position:

$$\mathbf{x} = \mathbf{\Phi}\ \mathbf{q} \tag{1.80}$$

we get:

$$\begin{bmatrix} \ddots & & \\ & m_k & \\ & & \ddots \end{bmatrix} \ddot{\mathbf{q}} + \begin{bmatrix} \ddots & & \\ & c_k & \\ & & \ddots \end{bmatrix} \dot{\mathbf{q}} + \begin{bmatrix} \ddots & & \\ & k_k & \\ & & \ddots \end{bmatrix} \mathbf{q} = \mathbf{0} \tag{1.81}$$

in which $c_k := \beta k_k$. For the $k - th$ mode, it becomes:

$$m_k \ddot{q}_k + c_k \dot{q}_k + k_k q_k = 0 \tag{1.82}$$

this is the equation of a single-degree-of-freedosystem (see par. 1.3) which has a complex natural frequency with an oscillatory part given by:

$$\omega'_{n_k} = \omega_{n_k} \sqrt{1 - \zeta_k^2} \tag{1.83}$$

where ω_{n_k} is the natural damping of the $k - th$ mode, given by $\omega_{n_k}^2 = k_k/m_k$, and ζ_k is the dimensionless damping coefficient of the $k - th$ mode, given by $\zeta_k = c_k/2\sqrt{k_k m_k}$. The exponential decay is given by:

$$\sigma_k = \zeta_k \omega_{n_k} \tag{1.84}$$

In analogy to what was seen for the non-damped case, in par. 1.3.2, the following expression for the flexibility matrix or frequency response function matrix for the forced system is obtained:

$$\mathbf{H}(\omega) = \left[\mathbf{K} + j\omega \mathbf{C} - \omega^2 \mathbf{M}\right]^{-1} \tag{1.85}$$

and therefore the generic term of the flexibility matrix is:

$$H_{jk}(\omega) = \sum_{r=1}^{n} \frac{\phi_j^{(r)}\phi_k^{(r)}}{k_r - m_r\omega^2 + j\omega c_r} = \sum_{r=1}^{n} \frac{\phi_j^{(r)^*}\phi_k^{(r)^*}}{\omega_r^2 - \omega^2 + j\omega\omega_r\zeta_r} \tag{1.86}$$

with $\zeta_r = c_r/2\sqrt{k_r m_r}$, which is quite similar to the analogous expression 1.73 obtained for the non-damped case, although in this case the term $H_{jk}(\omega)$ is complex.

- A special case of a damping matrix proportional to the stiffness matrix has been considered. Actually, an equivalent situation occurs if the damping matrix is proportional to the mass matrix of the system, according to the following:

$$\mathbf{C} = \alpha\mathbf{M} \tag{1.87}$$

More generally, in the case of proportional damping, it is considered that the damping matrix may be proportional to a linear combination of mass and stiffness matrices:

$$\mathbf{C} = \beta\mathbf{K} + \alpha\mathbf{M} \tag{1.88}$$

from which

$$c_k = \beta k_k + \alpha m_k \tag{1.89}$$

and the damped system will still have eigenvalues of the type 1.83 and eigenvectors that are equal to those of the corresponding non-damped system. This model of proportional damping, in addition to the advantage of simplicity, is of practical interest since the physical damping mechanisms are actually related to the *stiffness characteristics* of the structure, as far as the internal damping of the material is concerned, and to the *mass characteristics*, as far as friction damping is concerned.
However, it should be observed that in the aforementioned cases of damping proportional to stiffness and mass, the poles of the damped system (see also App. B.3) are given by:

$$s_{n_{1,2}} = -\zeta_n\omega_n \pm j\omega_n\sqrt{1-\zeta_n^2} \tag{1.90}$$

where $\zeta_r = (\alpha m_r + \beta k_r)/2\sqrt{k_r m_r}$, which holds also for single-degree-of-freedom systems with viscous damping.

- Quite similar considerations can be developed for a multiple-degrees-of-freedom model, but with hysteresis damping (see par. 1.2); in this case the general equation of motion is written (in the mixed time-frequency form):

$$\mathbf{M}\ddot{\mathbf{x}} + (\mathbf{K} + j\mathbf{H})\mathbf{x} = \mathbf{f} \tag{1.91}$$

Let us consider the hysteresis damping matrix $\mathbf{H}$ as proportional to the mass and stiffness matrices:

$$\mathbf{H} = \beta\mathbf{K} + \alpha\mathbf{M} \tag{1.92}$$

also in this case the eigenvectors of the damped system are equal to those of the non-damped system and the following formula applies to the complex eigenvalues:

$$s_k^2 = -\omega_{n_k}^2(1 + j\eta_k) \tag{1.93}$$

where $\omega_{n_k}^2 = k_k/m_k$ and the loss factors $\eta_k = \beta + \alpha/\omega_{n_k}^2$; the generic term of the dynamic flexibility matrix is:

$$H_{jk}(\omega) = \frac{\sum_{r=1}^{n} \phi_j^{(r)} \phi_k^{(r)}}{k_r - \omega^2 m_r + j\eta_r k_r} \tag{1.94}$$

1.3.4 Hysteresis damping: general case

The model that considers damping as proportional to mass and stiffness is a special case, although very important from a practical point of view; Therefore, the more general case of damping must also be considered in order to better understand the experimental data obtained from tests on structures whose behaviour does not necessarily follow the proportionality model for damping.

Let us now refer to the general case of hysteresis damping:

$$\mathbf{M\ddot{x}} + \mathbf{Kx} + \mathbf{jHx} = \mathbf{f} \tag{1.95}$$

where $\mathbf{H}$ is a symmetrical matrix. In the case of a free response ($\mathbf{f} = 0$), the solution is of the type:

$$\mathbf{x}(t) = \phi\ e^{\mu t} \tag{1.96}$$

we have a self-resolution problem represented by two matrices μ^2 and $\mathbf{\Phi}$ which contain the eigenvalues μ_n and the eigenvectors $\phi^{(n)}$. In this case the two matrices are complex and the modal deformations are represented in complex form; the $k-th$ eigenvalue can be written as:

$$\mu_k^2 = -\omega_{n_k}^2(1 + j\eta_k) \tag{1.97}$$

where ω_{n_k} is close to the natural pulsation of the non-damped system and η_k indicates the loss factor for the $k-th$ mode; it is observed that μ_k that appears in Eq. 1.97 is different from the natural pulsation of the non-damped mode, even if numerically the values are very close.

Modal deformations $\phi^{(k)}$ are complex, this means that the amplitude of each degree of freedom of the system is characterized with modulus and phase, while in the non-damped case or with proportional damping, there is always a phase that can only assume the values of 0 or 180 degrees. In the more general case of damping, and therefore with non-proportional damping, there are complex modes in which the phase varies from one degree of freedom to another and can assume any value.

In the case of complex modes, the orthogonality properties seen in the case of real modes still apply::

$$\mathbf{\Phi^T M \Phi} = \begin{bmatrix} \ddots & & \\ & m_k & \\ & & \ddots \end{bmatrix} \tag{1.98}$$

$$\mathbf{\Phi}^T\left(\mathbf{K}+j\mathbf{H}\right)\mathbf{\Phi}=\begin{bmatrix}\ddots & & \\ & k_k & \\ & & \ddots\end{bmatrix} \tag{1.99}$$

the generalized mass and stiffness, m_k, k_k are naturally complex and depend on the type of normalization chosen for the modal deformations, while the eigenvalues (see Eq. 1.97) are related with the m_k and k_k from:

$$-\mu_k^2 = k_k/m_k \tag{1.100}$$

In the case of forced response, for excitation $\mathbf{f}=\mathbf{f}^*\mathbf{e}^{\mathbf{j}\omega\mathbf{t}}$ and harmonic response $\mathbf{x}=\mathbf{x}^*\mathbf{e}^{\mathbf{j}\omega\mathbf{t}}$ the equation of motion is:

$$(\mathbf{K}+j\mathbf{H}-\omega^2\mathbf{M})\mathbf{x}^*e^{j\omega t}=\mathbf{f}^*e^{j\omega t} \tag{1.101}$$

From Eq. 1.101, the following expression for the dynamic flexibility matrix is obtained:

$$\mathbf{H}(\omega)=\left(\mathbf{K}+j\mathbf{H}-\omega^2\mathbf{M}\right)^{-1} \tag{1.102}$$

Proceeding in analogy to what was seen previously in par. 1.3.2, if we express the dynamic flexibility matrix in terms of the matrices of the modal model, instead of the spatial model matrices as in 1.102, we obtain:

$$\mathbf{H}(\omega)=\mathbf{\Phi}\begin{bmatrix}\ddots & & \\ & \dfrac{1}{-\mu_k^2-\omega^2} & \\ & & \ddots\end{bmatrix}\mathbf{\Phi}^{\mathbf{T}} \tag{1.103}$$

and for the single term of the matrix we have for the general case of non-unitary generalized masses:

$$H_{jk}(\omega)=\sum_{r=1}^{n}\frac{\phi_j^{(r)}\phi_k^{(r)}}{m_r\left(\omega_{n_r}^2-\omega^2+j\eta_r\omega_{n_r}^2\right)} \tag{1.104}$$

In Eq. 1.104, unlike what was seen previously, both the numerator and the denominator are complex, because the eigenvectors are complex. This is the essential difference from the case where damping is considered to be proportional to mass or stiffness matrices.

1.3.5 Viscous damping: general case

The equation of motion for a multi-degree freedom system with viscous damping in the case of free vibration is:

$$\mathbf{M}\ddot{\mathbf{x}}+\mathbf{C}\dot{\mathbf{x}}+\mathbf{K}\mathbf{x}=0 \tag{1.105}$$

with $\mathbf{C}$ symmetric and positive matrix. If we look for solutions of the type:

$$\mathbf{x}(t)=\phi\ e^{st} \tag{1.106}$$

Eq. 1.105 becomes:

$$\left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right)\phi = 0 \tag{1.107}$$

The solution of Eq. 1.107 consists of the solution of an eigenvalue problem in the form:

$$\left(s_k^2\mathbf{M} + s_k\mathbf{C} + \mathbf{K}\right)\phi^{(k)} = 0 \qquad k = 1, 2, ..., n \tag{1.108}$$

which is different from the one considered in the case of hysteresis damping. In fact, as shown by Eq. 1.108, there are $2n$ eigenvalues, if n indicates the number of degrees of freedom in the system 1.105, instead of the N eigenvalues considered in the case of hysteresis; but these $2n$ eigenvalues are pairwise conjugate complexes: of course each eigenvalue corresponds to an eigenvector and the eigenvectors are pairwise conjugate complexes as well. These considerations, as well as the others that follow, are demonstrated in App. **??** (see parr. **??** and **??**) on a purely algebraic basis.

The solution of the system 1.107 is therefore given by $2n$ complex conjugated eigenvalues and by $2n$ complex conjugated eigenvectors indicated with s_k, s_k^* and $\phi^{(k)}$, $\phi^{*^{(k)}}$ respectively. The eigenvalues can be written as:

$$s_k = \omega_{n_k}\left(-\zeta_k + j\sqrt{1-\zeta_k^2}\right) \tag{1.109}$$

where ω_{n_k} is the natural pulsation and ζ_k is the damping of the $k-th$ mode.

Also in this case there are orthogonality properties, but they are different from the classic ones; eigenvalues and eigenvectors satisfy Eq. 1.108, if you pre-multiply this relation by $\phi^{(\mathbf{q})^\mathbf{T}}$ we get:

$$\phi^{(q)^T}(s_k^2\mathbf{M} + s_k\mathbf{C} + \mathbf{K})\phi^{(k)} = 0 \tag{1.110}$$

Eq. 1.108 can be written for the $q-th$ mode as:

$$(s_q^2\mathbf{M} + s_q\mathbf{C} + \mathbf{K})\phi^{(q)} = 0 \tag{1.111}$$

If we calculate the transpose of 1.111, taking into account that the matrices of mass, $\mathbf{M}$, of stiffness, $\mathbf{K}$, of damping, $\mathbf{C}$, are symmetric matrices we get:

$$\phi^{(q)^T}(s_q^2\mathbf{M} + s_q\mathbf{C} + \mathbf{K}) = 0 \tag{1.112}$$

If we post-multiply this expression by $\phi^{(\mathbf{k})}$ and subtract the relation thus obtained from Eq. 1.110, we get:

$$(s_k^2 - s_q^2)\phi^{(\mathbf{q})^\mathbf{T}}\mathbf{M}\phi^{(\mathbf{k})} + (\mathbf{s_k} - \mathbf{s_q})\phi^{(\mathbf{q})^\mathbf{T}}\mathbf{C}\phi^{(\mathbf{k})} = \mathbf{0} \tag{1.113}$$

In the case in which the two roots s_k and s_q are different, a first condition of orthogonality is obtained from this expression:

$$(s_k + s_q)\phi^{(\mathbf{q})^\mathbf{T}}\mathbf{M}\phi^{(\mathbf{k})} + \phi^{(\mathbf{q})^\mathbf{T}}\mathbf{C}\phi^{(\mathbf{k})} = \mathbf{0} \tag{1.114}$$

A second condition of orthogonality can be obtained from 1.108 and 1.111. If we multiply the former by $s_q\phi^{(\mathbf{q})^\mathbf{T}}$ and the latter by $s_k\phi^{(\mathbf{k})^\mathbf{T}}$ and we subtract one from each other, we get:

$$s_k s_q\phi^{(\mathbf{q})^\mathbf{T}}\mathbf{M}\phi^{(\mathbf{k})} - \phi^{(\mathbf{q})^\mathbf{T}}\mathbf{K}\phi^{(\mathbf{k})} = \mathbf{0} \tag{1.115}$$

The conditions 1.114 and 1.115 are the orthogonality conditions in the general case of viscous damping, when the hypothesis of proportional damping is not used; as we can see, these are more complex conditions than the classic ones.
Let us now consider the case in which the modes k and q constitute a pair of conjugate complex modes, we have:

$$s_k = \omega_{n_k}\left(-\zeta_k + j\sqrt{1-\zeta_k^2}\right) s_q = \omega_{n_k}\left(-\zeta_k - j\sqrt{1-\zeta_k^2}\right) \tag{1.116}$$

the corresponding eigenvectors are complex conjugates, $\phi^{(\mathbf{q})} = \phi^{(\mathbf{k})^*}$; considering these relationships in the first orthogonality condition, 1.114, we get:

$$-2\omega_{n_k}\zeta_k\phi^{(\mathbf{k})^{*\mathbf{T}}}\mathbf{M}\phi^{(\mathbf{k})} + \phi^{(\mathbf{k})^{*\mathbf{T}}}\mathbf{C}\phi^{(\mathbf{k})} = \mathbf{0} \tag{1.117}$$

from which the first condition of orthogonality is obtained:

$$2\omega_{n_k}\zeta_k = \frac{\phi^{(\mathbf{k})^{*\mathbf{T}}}\mathbf{C}\phi^{(\mathbf{k})}}{\phi^{(\mathbf{k})^{*\mathbf{T}}}\mathbf{M}\phi^{(\mathbf{k})}} = \frac{c_k}{m_k} \tag{1.118}$$

Proceeding in an similar manner for the second condition of orthogonality, 1.115, we obtain:

$$\omega_{n_k}^2\phi^{(\mathbf{k})^{*\mathbf{T}}}\mathbf{M}\phi^{(\mathbf{k})} - \phi^{(\mathbf{k})^{*\mathbf{T}}}\mathbf{K}\phi^{(\mathbf{k})} = \mathbf{0} \tag{1.119}$$

from which the second condition of orthogonality is obtained:

$$\omega_{n_k}^2 = \frac{\phi^{(\mathbf{k})^{*\mathbf{T}}}\mathbf{K}\phi^{(\mathbf{k})}}{\phi^{(\mathbf{k})^{*\mathbf{T}}}\mathbf{M}\phi^{(\mathbf{k})}} = \frac{k_k}{m_k} \tag{1.120}$$

m_k, k_k and c_k, appearing in the conditions 1.118 and 1.120, are again referred to as the modal mass, stiffness and damping, even if their meaning is different from that corresponding to the case of proportional damping.