Chapter 1

Experimental techniques of modal analysis

1.1 Comparison between different modal analysis techniques

The experimental determination of modal parameters, i.e. natural frequencies, modal damping coefficients and modal deformations, is of great importance for various aspects in the dynamic study of aeronautical and spatial structures as well as in the evaluation of the response of the structure of the structure, in the control problems and in the detection of damages through the repeated modal analysis of the structure.
Some considerations on modal analysis techniques based on the experimental determination of FRFs using a single stress point have been examined. These techniques offer the possibility to operate experimentally with very short data acquisition times compared to classical methods based on the resonance of the single modes of the structure, which are generally referred to as modal appropriation methods.
However, there are several experimental issues related to the need for a very small frequency resolution, the effects of leakage and, more generally, to the difficulty of obtaining great precision in the experimental data as it may instead be necessary in several cases, such as in the study of high-modal density structures.
A comparison with the other techniques that can be used in the field of experimental dynamic analysis is reported in summary form.

1.1.1 Resonance techniques, modal appropriation

Techniques that are based on the excitation of the structure under resonant conditions on the single mode allow, in principle, a great precision to be achieved in the evaluation of modal parameters but require a more complex experimental apparatus and a longer time for the experimentation itself. Moreover, problems may arise which are connected with the onset of non-linearity and also with possible damage that may occur during the experimentation: in particular in the case of structures that are characterized by very small values of the modal damping coefficients and that may present very large responses under resonance conditions.
In this approach all the available energy is used on the single mode and a very favorable signal-to-noise ratio is obtained; the complexity of the measure is linked to the search for the excitation vector that allows the correct excitatation of the mode; this research must be done for all modes

of interest and can be preceded by the preliminary use of techniques based on the determination of FRFs that allow a first indication on modal deformations.
The complexity of the experimental apparatus depends not only on the need to use a relatively large number of excitement systems, but also on the fact that they must be controlled in *amplitude, frequency and phase*.
It should be noted that the modal damping coefficients must be obtained separtely from specific free decay tests: the damping measurement tends to be overestimated also beacause of the effects due to the presence of the excitation systems which are connected with the structure under test and therefore alter its behavior.
The dynamic study of the *Galileo* satellite was an example, of particular importance in the aerospace field, in which different methods of dynamic analysis were used and represents a very interesting case for comparison between different modal analysis techniques.

1.1.2 FRF multi-input techniques

Also in the framework of the approach based on the determination of the FRFs, it is possible to use excitation techniques with several inputs: among these we can distinguish the technique that uses two input channels and allows the simultaneous measurement of two columns of the FRF matrix. This fact can help, for example with the linear combination of the matrix columns, to separate the modes and, in the case of symmetrical excitation, allows, by means of additions and subtractions of the columns, to highlight symmetrical and antisymmetric modes separately.
In the case of several excitation channels, a more uniform energy distribution can be obtained over the structure. This increases the signal-to-noise ratio at all measurement points and allows the use of input signals even at very low levels, thus limiting the presence of local non-linearity phenomena. The simultaneous measurement of two or more columns of the FRF matrix is thus obtained allowing to highlight the single modes of the structure, through a linear combination of the matrix columns.
This technique makes it possible to combine the positive aspects linked to modal appropriation methods with pure sinusoidal excitation with the typical advantages of the broadband excitation techniques.

1.1.3 Time domain techniques

In order to overcome the limitations of the techniques operating in the frequency domain, attention was turned to a different approach that operates instead in the time domain. Time domain techniques operate directly on the response functions as they are recorded in the experimental tests.
The working princilple is very simple, starting from the equations of motion:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}(t) \tag{1.1}$$

the modal parameters of the system can be obtained if displacements, velocities and accelerations can be measured. Thus the free response of the system can be expressed with as:[1]

$$\mathbf{x}(t) \;=\; \sum_{i=1}^{2n} \phi^{(i)} e^{\mu_i t} + \mathbf{n}(t) \tag{1.3}$$

where $\phi^{(i)}$ indicate the modal deformations, and μ_i indicate the characteristic roots or poles of the mechanical system that are related to the natural frequencies and to the damping coefficients. For example in the case of damping that can be modelled as viscous, from:

$$\mu_i = -\zeta_i \omega_{n_i} \pm j\omega_{n_i}\sqrt{1-\zeta_i^2} \tag{1.4}$$

(see App. **??** parr. **??** and **??**) and $\mathbf{n}(t)$ indicates the noise that is associated with the measurement. The problem defined by Eq. 1.3 is to evaluate μ_i and $\phi^{(i)}$ given the measurement in time of the free response $\mathbf{x}(t)$.
This is a non-linear problem where the effect of noise on the measurement is very important; also the number of modes acting in the free response measured is unknown.
Several methods have been proposed to solve the problem but in all cases there are some fundamental difficulties:

- the problem is non-linear and therefore requires the use of iterative algorithms for the solution;
- the number of modes present in the free response that is experimentally measured is unknown;
- the error due to noise is important.

As far as the effect of non-linearity is concerned, *Prony method* (or *complex exponential method*) has brought the problem back to a polynomial schema whose coefficients are calculated from the system's responses;[2] it is observed that the determination of the roots of a polynomial presents

[1] In reality the free response of the system would be given by:

$$\mathbf{x}(t) \;=\; \sum_{i=1}^{2n} c_i \phi^{(i)} e^{\mu_i t} \tag{1.2}$$

where c_i are complex constants which depend on the initial conditions. Taking into account what will be said later, the presence of these constants, which have the only role of linearly combining the elementary impulsive responses given by the individual terms of Eq. 1.1, does not change the results to be found in the following sections 1.2 and 1.3.

[2] The method is based on the expression of the impulsive response matrix: if in fact the generic term of the FRF matrix can be expressed as:

$$H_{ij} = \sum_{r}^{N_m} \left[\frac{A_{ij}^{(r)}}{j\omega - \mu_r} + \frac{A_{ij}^{*(r)}}{j\omega - \mu_r^*} \right] \tag{1.5}$$

then we have for the impulsive response matrix:

$$h_{ij} = \sum_{r}^{N_m} \left[A_{ij}^{(r)} e^{\mu_r t} + A_{ij}^{*(r)} e^{\mu_r^* t} \right] \tag{1.6}$$

equation in which data over time are known, while residuals and poles are generally unknown.

numerical issues in particular if the polynomial is of high order and if there are neighboring roots.
It should be noted that by over-dimensioning the system, i.e. by searching for more modes than are actually present, it is possible to reduce the effect due to the presence of noise. The use of techniques operating in the time domain remained very limited until the 1970s when the development of new algorithms brought back the attention on this approach. A particular interest, for its potentialities of use, is presented by the technique proposed by Ibrahim, called Ibrahim Time Domain (ITD), which is explained in details in the following paragraph.

1.2 Technique in the time domain of Ibrahim, ITD

It is a method of structural identification that "works" in the time domain and allows us to determine, starting from the free response of the structure, the modal parameters, i.e. natural frequencies, modal damping and modal deformations.
The method can operate even if the signal-to-noise ratio is low, in fact the structure model can be "over-sized" and the "apparent modes" that are added allow "filtering" the effect of the noise.
The equation of the motion of a non-forced multiple degrees-of-freedom system is:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = 0 \tag{1.7}$$

We have solutions of this type:

$$\mathbf{x}(t) \;=\; \phi e^{\mu t} \tag{1.8}$$

if:

$$\left(\mu_k^2\mathbf{M} + \mu_k\mathbf{C} + \mathbf{K}\right)\phi^{(k)} \;=\; 0 \tag{1.9}$$

which results in the condition:

$$det\left(\mu_k^2\mathbf{M} + \mu_k\mathbf{C} + \mathbf{K}\right) = 0 \tag{1.10}$$

Eq. 1.10 is an algebraic equation with real coefficients of degree $2n$, with n being the number of degrees-of-freedom of the system equation 1.7. Thus we have $2n$ eigenvalues which are conjugate complexes. Each real eigenvalue corresponds to a real eigenvector and to each pair of complex conjugated eigenvalues corresponds a pair of complex conjugated eigenvectors.
If one imagines that the experimentally detected free response corresponds to the excitation of m modes, with $m \leq n$ then with a number of modes actually present in the free response that is less than the number of degrees-of-freedom of the system 1.7, one has:

$$x_i(t) \;=\; \sum_{k=1}^{2m} \phi_i^{(k)} e^{\mu_k t} \qquad i = 1, ..., n \;;\; k = 1, ..., 2m \tag{1.11}$$

where with $x_i(t)$ is the free response which corresponds to the $i-th$ measuring point, i.e. to the $i-th$ degree-of-freedom.
Let us assume that we have $2m$ measurement points, i.e. a number of measurement points equal to the number of modes actually present in the free response, we have then:

$$x_i(t_j) \;=\; x_{i_j} \;=\; \sum_{k=1}^{2m} \phi_i^{(k)} e^{\mu_k t_j} \tag{1.12}$$

where x_{i_j} is the free answer relative to the $i-th$ measureming point evaluated at the time t_j. Let us denote by NT the total number of "time instants" that are available for each free response and let us define the following matrices:

- Response matrix:

$$\Phi_x = \begin{bmatrix} x_1(t_1) & x_1(t_2) & ... & x_1(t_{NT}) \\ x_2(t_1) & x_2(t_2) & ... & x_2(t_{NT}) \\ ... & ... & ... & ... \\ x_{2m}(t_1) & x_{2m}(t_2) & ... & x_{2m}(t_{NT}) \end{bmatrix} \tag{1.13}$$

  The response matrix has been indicated with Φ_x and has size $2m \times 2m$ if NT is equal to $2m$. This matrix cointain the " time history " (evaluated at the instants $t_1, ..., t_{NT}$) of the free answer of the $2m$ points of measurement.

- Modal matrix:

$$\Psi_\phi = \begin{bmatrix} \phi_1^{(1)} & \phi_1^{(2)} & ... & \phi_1^{(2m)} \\ ... & ... & ... & ... \\ \phi_{2m}^{(1)} & \phi_{2m}^{(2)} & ... & \phi_{2m}^{(2m)} \end{bmatrix} \tag{1.14}$$

  The modal matrix, indicated with Ψ_ϕ, has dimensions $2mx2m$. It contains for colums the modal deformations, thus the $k-th$ column is relative to the $k-th$ complex mode.

- Matrix of natural frequencies:

$$\mathbf{\Lambda} = \begin{bmatrix} e^{\mu_1 t_1} & & e^{\mu_1 t_{NT}} \\ . & . & . \\ e^{\mu_{2m} t_1} & . & e^{\mu_{2m} t_{NT}} \end{bmatrix} \tag{1.15}$$

  it is a matrix cointaing by rows in exponential terms the natural frequencies of the structure.

The free response 1.12 can therefore be written as:

$$\Phi_x = \Psi_\phi \mathbf{\Lambda} \tag{1.16}$$

The free response, again recorded at the same measurement points, but referring to different time instants, since they are shifted in time by an interval Δt_1, can be written as:

$$x_i(t + \Delta t_1) = \sum_{k=1}^{2m} \phi_i^{(k)} e^{\mu_k (t + \Delta t_1)} \tag{1.17}$$

which becomes:

$$x_i(t + \Delta t_1) = \sum_{k=1}^{2m} \phi_i^{(k)} e^{\mu_k \Delta t_1} \; e^{\mu_k t} \tag{1.18}$$

Let us introduce the following matrices:

$$\hat{\Phi}_x = \begin{bmatrix} x_1(t_1 + \Delta t_1) & ... & x_1(t_{NT} + \Delta t_1) \\ . & . & . \\ x_{2m}(t_1 + \Delta t_1) & . & x_{2m}(t_{NT} + \Delta t_1) \end{bmatrix} \tag{1.19}$$

which still has the meaning of response matrix, but, with respect to Φ_x, matrix, is translated in time by an interval Δt_1; and the *modified modal matrix*, which with respect to the modal matrix Ψ_ϕ takes into account the time delay Δt_1:

$$\hat{\Psi}_\phi \;=\; \begin{bmatrix} \phi_1^{(1)} e^{\mu_1 \Delta t_1} & ... & \phi_1^{(2m)} e^{\mu_{2m} \Delta t_1} \\ ... & ... & ... \\ \phi_{2m}^{(1)} e^{\mu_1 \Delta t_1} & ... & \phi_{2m}^{(2m)} e^{\mu_{2m} \Delta t_1} \end{bmatrix} \tag{1.20}$$

Between the $k-th$ columns of the matrices Ψ_ϕ and $\hat{\Psi}_\phi$, there is a relation:

$$\hat{\psi}_\phi^{(k)} \;=\; \psi_\phi^{(k)} e^{\mu_k \Delta t_1} \tag{1.21}$$

then Eq. 1.18 can be expressed, in synthetic form, by the matrix relation:

$$\hat{\Phi}_x \;=\; \hat{\Psi}_\phi \mathbf{\Lambda} \tag{1.22}$$

In this way, two response matrices have been constructed, indicated with Φ_x and $\hat{\Phi}_x$ which contain the whole " time history " of the available measurement points evaluated in the same time intervals but with a time shift Δt_1 between the two response matrices.
From Eq. 1.16 and Ew. 1.22 we can write:

$$\mathbf{\Lambda} = \Psi_\phi^{-1} \Phi_x \tag{1.23}$$

$$\hat{\Phi}_x \;=\; \hat{\Psi}_\phi \Psi_\phi^{-1} \Phi_x \tag{1.24}$$

Eq. 1.24 is a relationship that connects the two matrices that are related to the data over time that are experimentally detected; if a new matrix is introduced, indicated with $\mathbf{A}$, defined by the relation:

$$\mathbf{A} \;:=\; \hat{\Psi}_\phi \Psi_\phi^{-1} \quad \longrightarrow \quad \Psi_\phi \mathbf{A} - \hat{\Psi}_\phi = 0 \tag{1.25}$$

from Eq. 1.24, it follows:

$$\hat{\Phi}_x \;=\; \mathbf{A} \Phi_x \quad \longrightarrow \quad \mathbf{A} = \hat{\Phi}_x \Phi_x^{-1} \tag{1.26}$$

Of course from the 1.26 we see that it is possible to derive the matrix $\mathbf{A}$ containing the unknowns of the problem in terms of eigenvectors and eigenvectors, once the two response matrices Φ_x and $\hat{\Phi}_x$ are determined by experimental data.
From 1.25 we have:

$$\hat{\Psi}_\phi \;=\; \mathbf{A} \Psi_\phi \tag{1.27}$$

But from 1.21, which expresses the link between $\hat{\Psi}_\phi$ e Ψ_ϕ, the matricial Eq. 1.27 written for each of its columns $k-th$ becomes:

$$\phi^{(k)} e^{\mu_k \Delta t_1} \;=\; \mathbf{A} \phi^{(k)} \tag{1.28}$$

from which:

$$\left(\mathbf{A} - e^{\mu_k \Delta t_1} \mathbf{I} \right) \phi^{(k)} = 0 \qquad k = 1, ..., 2m \tag{1.29}$$

From the relation 1.29 we see the the eigenvalues and eigenvectors of the matrix $\mathbf{A}$ are linked to those of the dynamic system in the sense that the eigenvectors of $\mathbf{A}$ are coincident with the modal deformations and the eigenvalues of $\mathbf{A}$ (which are $e^{\mu_k \Delta t_1}$) are connected to the roots μ_k of Eq. 1.9.
All the elements of matrix $\mathbf{A}$ are real, since matrix $\mathbf{A}$ is obtained from two real matrices Φ_x and $\hat{\Phi}_x$, and therefore the eigenvalues of A are real or complex conjugates and the corresponding eigenvectors are also real or complex conjugates.
The generic eigenvalue of $\mathbf{A}$ is complex:

$$\begin{aligned} e^{\mu_k \Delta t_1} &= a_k + j b_k \\ \mu_k &= \sigma_k + j\omega_k \end{aligned} \tag{1.30}$$

and so:

$$a_k + j b_k = e^{\sigma_k \Delta t_1} \left[\cos(\omega_k \Delta t_1) + j \sin(\omega_k \Delta t_1)\right] \tag{1.31}$$

from which:

$$\begin{aligned} a_k &= e^{\sigma_k \Delta t_1} \cos(\omega_k \Delta t_1) \\ b_k &= e^{\sigma_k \Delta t_1} \sin(\omega_k \Delta t_1) \end{aligned} \tag{1.32}$$

and ultimately, damping coefficients and damped natural frequencies are:

$$\begin{aligned} \sigma_k &= \frac{1}{2\Delta t_1} \ln\left(a_k^2 + b_k^2\right) \\ \omega_k &= \frac{1}{\Delta t_1} \arctan\left(b_k / a_k\right) \end{aligned} \tag{1.33}$$

Relation 1.26 allows us to construct the $\mathbf{A}$ matrix ; by evaluating the transposition of 1.26, we have:

$$\Phi_x^T \mathbf{A}^T = \hat{\Phi}_x^T \tag{1.34}$$

the size of the matrices Φ_x^T and $\hat{\Phi}_x^T$ are $NT \times 2m$ and therefore in case $NT = 2m$ they are square matrices and the relation 1.34 defines a system of linear equations in which Φ_x^T is the matrix of the coefficients and $\hat{\Phi}_x^T$ and the matrix of known terms; From Eq. 1.34 we can derive the matrix $\mathbf{A}^T$ and therefore the matrix $\mathbf{A}$. If instead $NT > 2m$, which is always possible since it is simply a matter of considering for each measuring point, corresponding to a degree of freedom, a sufficiently large number of measurement instants, Eq. 1.34 defines a system of NT equations in $2m$ unknowns. This is an oversized system in which the number of equations is greater than the number of unknowns to be determined. In this case, a "least squares" solution can be found and the $\mathbf{A}$ matrix is obtained by premultiplying Eq. 1.34 by the matrix Φ_x, i.e. from the relation:

$$\Phi_x \Phi_x^T \mathbf{A}^T = \Phi_x \hat{\Phi}_x^T \tag{1.35}$$

where Φ_x has dimensions $2m \times NT$. The system 1.35 is always reduced to a system of size $2m \times 2m$, independently from the size of NT (with $NT \geq 2m$), and therefore it is solved directly.
Of course, if the number of instants of measurement is large $NT > 2m$, more experimental

data available are available than are strictly necessary for the solution of the system 1.34, one obtains consequently a better stability and precision in the determination of the **A** matrix and therefore of the modal characteristics of the structure that depend on the precision with which the eigenvalues and eigenvectors of the matrix **A** are evaluated.
As we have seen, the ITD method is based on the construction of two response matrices whose components are:

$$\begin{aligned} \phi_{x_{ij}} &= x_i(t_j) \qquad\qquad j = 1, ..., NT \\ \hat{\phi}_{x_{ij}} &= x_i(t_j + \Delta t_1) \end{aligned} \tag{1.36}$$

where $x_i(t_j)$ is the free response of the $i-th$ measurement point at the time t_j and Δt_1 represents represents a time shift that can be chosen arbitrarily, but must be different from the sampling time Δt.
Of course if the number of measurement points that are actually available, indicated with p_m, is less than $2m$ then the matrices Φ_x and $\hat{\Phi}_x$ are incomplete, since just a part of the $2m$ lines needed to construct the matrices themselves is available (it is in fact intuitive that we must have $2m$ points of measure corresponding to the number of modes of the structure that are present in the free answer). The modal parameters that must be determined are independent of the initial conditions and therefore one can think of using different intervals of the free response actually measured to "fill" the " empty " rows of the matrices Φ_x and $\hat{\Phi}_x$; indeed, the $(2m-p_m)$ " empty " rows can be completed with " fictitious" measurement points which are obtained from the signals recorded in the "real" measurement points, but shifted in time with different temporal intervals $\Delta t_2, ..., \Delta t_{(2m-p_m)}$.
Let us then consider that the free response is recorded from p_m measuring points and proceed to structural identification with a $2m$ degrees-of-freedom model, with $2m > p_m$. The matrices Φ_x and $\hat{\Phi}_x$ have dimensions $2m \times NT$ and the matrix $\hat{\Phi}_x$ is obtained from Φ_x using the free response records of the structure, shifted by time $\Delta t1$ with respect to those used to construct Φ_x. The first p_m rows of the matrix Φ_x are " built " with the experimental recordings of the available measurement points; then, having chosen an interval Δt_2, the $(2m - p_m)$ other rows are constructed with signal segments shifted of Δt_2 and multiples; in this way, which of course is only one of the possible ways, one can "oversize" the model. The time intervals Δt_1 and Δt_2 are, sometimes, expressed in multiples of the sampling time, Δt:

$$\begin{aligned} \Delta t_1 &= N_1 \Delta t \\ \Delta t_2 &= N_2 \Delta t \end{aligned} \tag{1.37}$$

The temporal duration of the intervals Δt_1 and Δt_2 is not fixed a priori, but has an effect on the identification and of course the time intervals must be different from each other.
In the limit case in which only one measurement point is available, it is still possible to identify the different frequencies and modal damping present in the signal itself by constructing fictitious measurement points with the use of time translations of the recorded signal. It can be observed that this situation is completely analogous to what happens in the frequency domain; if only one frequency response function $\alpha_{ij}(\omega)$ is available, it is still possible to evaluate pulsations and damping of the modes in the band. Of course with this procedure it is not possible to evaluate the modal deformations, because we are always working on the same measurement point, while the determination of the modal deformations requires information on different measurement points. In other words, the components of the modal deformations that can be obtained in the

measurement are obviously equal to the number of "true" measurement points available in the experiment.
As we have seen, the free response is:

$$\mathbf{x}(t) = \sum_{k=1}^{2m} \phi^{(k)} e^{\mu_k t} \qquad k = 1, ..., 2m \tag{1.38}$$

where m represents the number of fundamental modes; of course, in general, the number of modes is an unknown of the problem and 1.38 does not take into account the presence of the noise; In fact:

$$\mathbf{x}(t) = \sum_{k=1}^{2m} \phi^{(k)} e^{\mu_k t} + \mathbf{n}(t) \tag{1.39}$$

If noise is also represented by a linear combination of complex exponential terms, we have:

$$\mathbf{x}(t) = \sum_{k=1}^{2m} \phi^{(k)} e^{\mu_k t} + \sum_{k=2m+1}^{2n} \mathbf{n}^{(k)} e^{\mu_k t} \qquad k = 1, ..., 2n \tag{1.40}$$

In 1.40 a combination of $2m$ structural modes and $2(n-m)$ noise modes is considered; the dimensions of the free response model are "augmented" with respect to the actual dimensions of the dynamic model by introducing new degrees-of-freedom and therefore new modes representing noise. Therefore, it is preferable to choose a very large number of modes to be sure of identifying all the structural modes actually present in the free response and to use the extra ones for noise identification, even if at the cost of a longer calculation time.
Naturally the problem then arises of "distinguishing" the structural modes from the noise modes, which can be done by defining a suitable index indicated with MCF (Modal Confidence Factor).[3] The response matrix Φ_x is constructed for the first m rows with the registrations $x_i(t_j)$ from the m available measurement points and for the other m rows with the registrations $x_i(t_j + \Delta t_3)$ of " fictitious " measurement points therefore, relative to the same measurement points as the first m rows but shifted in time by an interval Δt_3; thus, we have:

$$\Phi_x = \begin{bmatrix} ... & x_1(t_j) & ... \\ ... & ... & ... \\ ... & x_m(t_j) & ... \\ ... & x_1(t_j + \Delta t_3) & ... \\ ... & ... & ... \\ ... & x_m(t_j + \Delta t_3) & ... \end{bmatrix} \qquad j = 1, ..., NT \tag{1.41}$$

then the free response at the instants t_j e $(t_j + \Delta t_3)$ is:

$$\begin{aligned} \mathbf{x}(t_j) &= \sum_{k=1} \phi^{(k)} e^{\mu_k t_j} \\ \mathbf{x}(t_j + \Delta t_3) &= \sum_{k=1} \hat{\phi}^{(k)} e^{\mu_k t_j} \end{aligned} \tag{1.42}$$

[3]The MCF is based on the idea of performing the analysis on two different time segments of the free response $\mathbf{x}(t)$ that are shifted in time by an interval Δt_3 and check if the modes that are identified in a temporal segment are also present in the other. This procedure allows us to identify the structural modes, which are typical of the structure's behaviour and must obviously be present in the two time segments, from the noise modes, which are instead random and therefore do not have to correspond.

where:

$$\hat{\phi}^{(k)} = \phi^{(k)} e^{\mu_k \Delta t_3} \tag{1.43}$$

Let us then consider the matrix $\mathbf{\Lambda}$, whose generic element is:

$$\Lambda_{ij} \;=\; e^{\mu_i t_j} \tag{1.44}$$

from Eq. 1.41 and Eq. 1.42 we have:

$$\begin{bmatrix} \mathbf{x}(t_j) \\ \mathbf{x}(t_j + \Delta t_3) \end{bmatrix} \;=\; \begin{bmatrix} \Psi_\phi \\ \hat{\Psi}_\phi \end{bmatrix} \mathbf{\Lambda} \tag{1.45}$$

therefore:

$$\Phi_x \;=\; \Psi^*_\phi \mathbf{\Lambda} \tag{1.46}$$

Of course, the same procedure applies the matrix $\hat{\Phi}_x$ and we return to the determination of the eigenvalues and eigenvectors of the matrix $\mathbf{A}$ defined by:

$$\hat{\Phi}_x \;=\; \mathbf{A}\Phi_x \tag{1.47}$$

The difference between the problem 1.47 and the one defined previously is that, in this case, the unknowns have the structure:

$$\begin{bmatrix} \Psi^*_\phi \end{bmatrix} \;=\; \begin{bmatrix} \Psi_\phi \\ \hat{\Psi}_\phi \end{bmatrix} \tag{1.48}$$

where the 1.43 defines the relation between the upper and lower component of the vectors for each individual structural mode $\phi^{*^{(k)}}$.
Thus we define the " prediction " values of the $k-th$ mode to be compared with the one obtained; these are complex modes and therefore there are two indexes, respectively for the modulus and for the phase, and so we define the MCF factor whose modulus is the ratio between the moduli and whose phase is the difference between the phases; the optimal value for identification is 100 % for the modulus and 0^o for the phase. With this criterion the structural modes and also the possible forced responses are identified together.
It is observed that Eq. 1.33 determines the modal frequency, $f_k = \omega_k/2\pi$ starting from the real part and from the imaginary part, a_k and b_k, of the $k-th$ eigenvalue of the matrix $\mathbf{A}$:

$$f_k \;=\; (1/2\pi\Delta t_1) \arctan(b_k/a_k) \tag{1.49}$$

where Δt_1 is the time shift between the response matrices Φ_x and $\hat{\Phi}_x$.
But the arctangent function is periodic and therefore the range of values must be limited between $-\pi/2$ and $\pi/2$; therefore the maximum frequency that can be identified is given by:

$$f_{max} \;=\; (1/4\Delta t_1) \tag{1.50}$$

any frequencies higher than f_{max} will be "translated" into frequenciesthat do not actually exist, between 0 and f_{max}, thus introducing a typical aliasing error. It follows:

- the choice of Δt_1 is related to the maximum frequency of the signal to be analyzed and, of course, is limited by the sampling time Δt since it must be $\Delta t_1 \geq \Delta t$;
- the signal must be filtered, consistently with the choice of Δt_1 in order to "cut" the frequencies higher than the value of f_{max}, which corresponds to the limit imposed by Eq. 1.50, and eliminate the aliasing error.

The parameter NT indicates the total number of time instants that are considered in the signal recording: the minimum value is $2m$; in general there is an advantage in the accuracy of the results if $NT > 2m$. The value chosen for the $NT/2m$ ratio is important and it can be verified that its optimal value is given by:

$$(NT/2m)_{ott} \cong 3 \tag{1.51}$$

The parameter Δt_3 indicates the time shift between the data in the two components of the response matrix and allows a comparison between the modes identified by the two "time histories" shifted by Δt_3: the modes common in the two time histories are considered as structural modes and others as modes of noise. On the other hand, the parameter Δt_2 defines the time shift between the actual measurement points of "fictitious" measurement points that are introduced to complete the response matrix; of course, it must be:

$$\Delta t_1 \neq \Delta t_2 \neq \Delta t_3 \tag{1.52}$$

1.3 The logarithmic decrement technique, randomdec, RDT*

This method separates the deterministic and random components of a signal: it can be used in cases where the signal-to-noise ratio is limited and in particular to identify the free response of a structure starting from the recording of the response of the structure to a random solicitation. This is the typical situation in flight tests of aeronautical and space structures. This technique is based on the hypothesis that the excitation signal of the recorded response is random, stationary and zero mean value and allows the identification of the free response of a structure, which is necessary for the use of the Ibrahim method, starting for example from a flight test obtained from an input due to a gust.
Let us denote by $x(t)$ the response signal which has a trend random, but which is made up of a deterministic part, indicated with $s(t)$, and a aleatory part, indicated with $a(t)$, which is considered stationary and zero mean value:

$$x(t) = s(t) + a(t) \tag{1.53}$$

Let us consider as an example the case in which the deterministic part $s(t)$ is given simply by a step of amplitude S_o and, having fixed arbitrarily n temporal instants of equal duration T, n signal samples are recorded:

$$x_k(t) = x(t - t_k) \qquad t_k = kT \qquad k = 1, 2, ..., n \tag{1.54}$$

if we sum these values and divide them by the number n of the samples, we get:

$$(1/n)\sum_k x_k(t) = (1/n)\sum_k (s_k(t - t_k) + a_k(t - t_k)) = S_o + (1/n)\sum_k a(t - t_k) \tag{1.55}$$

because of the hypotheses that the aleatory component must be random, stationary and zero mean value, we must have:

$$(1/n)\sum_k a_k(t-t_k) \;=\; 0 \tag{1.56}$$

and therefore from the sum 1.55, we get:

$$(1/n)\sum_k x_k(t) \;=\; S_o \tag{1.57}$$

It can be seen, therefore, that this averaging procedure has made it possible to isolate the deterministic component of the signal.
Of course this example is extremely simple because of the particular nature (step) of the signal to be identified; there is no problem in the "sampling" of the signal which concerns the time interval T chosen in 1.54 and in evaluating the instant of "trigger".
The dynamics equation of a structure is:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) \;=\; \mathbf{f}(t) \tag{1.58}$$

where $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ are the mass, damping and stiffness matrices of the structure while $\mathbf{x}$ and $\mathbf{f}$ are the vectors of displacements and forces acting on the system. If we substitute the following values in 1.58:

$$t \;=\; t_k + \tau \qquad\qquad k \;=\; 1, ..., n \tag{1.59}$$

we have:

$$\mathbf{M}\ddot{\mathbf{x}}(t_k+\tau) + \mathbf{C}\dot{\mathbf{x}}(t_k+\tau) + \mathbf{K}\mathbf{x}(t_k+\tau) \;=\; \mathbf{f}(t_k+\tau) \tag{1.60}$$

Adding up the n equations 1.60 and posing:

$$\mathbf{h}(\tau) \;=\; (1/n)\sum_{k=1}^{n} \mathbf{x}(t_k+\tau) \tag{1.61}$$

we have:

$$\mathbf{M}\ddot{\mathbf{h}}(\tau) + \mathbf{C}\dot{\mathbf{h}}(\tau) + \mathbf{K}\mathbf{h}(\tau) \;=\; (1/n)\sum_{k=1}^{n} \mathbf{f}(t_k+\tau) \tag{1.62}$$

but if the input vector $\mathbf{f}(t)$ consists of random, stationary and zero-mean value forces, the following condition must be verified:

$$(1/n)\sum_{k=1}^{n} \mathbf{f}(t_k+\tau) \;=\; 0 \tag{1.63}$$

therefore from Eq. 1.64, we have (changing name to the time variable):

$$\mathbf{M}\ddot{\mathbf{h}}(t) + \mathbf{C}\dot{\mathbf{h}}(t) + \mathbf{K}\mathbf{h}(t) \;=\; 0 \tag{1.64}$$

which is the free response of the system 1.58.
The logarithmic decrement technique is based on the hypothesis that the response of a structure subjected to a stationary, random and zero-mean value excitation can be considered as the sum of three components:

1. a *step* response, due to an initial displacement;

2. an *impulsive* response, due to an initial speed;

3. a *random* response, due to a random input.

The *RDT* evaluates the free response with an averaging procedure performed on signals obtained from the input signal with different "trigger" criteria. In particular the trigger criteria generally adopted are:

- choice based on a fixed signal level;

- choice based on zero crossing.

With the first criterion we construct a signal indicated with $h(t)^*$ starting from every instant in which the chosen level is reached, and then the signals thus obtained are summed and averaged. The sum of these signals leads to the cancellation of the *random* response and there remains a signal in time which is the free response of the system corresponding to an initial position equal to the value chosen for the trigger.
The second *triggering* criterion leads to an analogous result, which however refers to an initial speed value, which is positive for the zero crossing points with positive derivative and is negative for the others; the signals obtained for the positive and negative points must then be "added up" with the appropriate sign. Of course, when dealing with the identification of a system with multiple degrees of freedom, the triggering procedure must be simultaneous for all the measurement points; it is therefore evident that a measurement point must be chosen as a " pilot ", i.e. as a reference for the triggering criterion in order to avoid delays, and therefore phase shifts, between the different responses.
It is noted that the *RDT* has some advantages over an autocorrelation procedure which also allows the signal to be "cleaned" from the noise. In short, the logarithmic decrement technique allows the identification of the impulse response starting from an unknown and not measured random input and this can be used both for dynamic tests on aircraft, where for example the input signal is given by the burst, and for dynamic tests of space structures, for example with the use of flight recordings of rockets in which the excitation is given by the propulsion or with recordings obtained in inhabited space systems in which the excitation is given by the movement of the crew.