NUMERICAL EXAMPLES

• Example 2.1

B written with respect to I :

$$\begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{\underset{B\leftarrow I}{R}} \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \end{bmatrix}$$

F written with respect to I :

$$\begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \end{bmatrix}}_{\underset{F\leftarrow I}{R}} \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \end{bmatrix}$$

(a) Find $\underset{F\leftarrow B}{R}$

$$\begin{tikzcd} I \arrow[rr, "\underset{F\leftarrow I}{R}"] \arrow[dr, "\underset{B\leftarrow I}{R}"'] & & F \\ & B \arrow[ur, "\underset{F\leftarrow B}{R}"'] & \end{tikzcd}$$

$$\underset{F\leftarrow B}{R}\ \underset{B\leftarrow I}{R} = \underset{F\leftarrow I}{R}$$

$$\Downarrow$$

$$\underset{F\leftarrow B}{R} = \underset{F\leftarrow I}{R}\ \underset{I\leftarrow B}{R} = \underset{F\leftarrow I}{R}\ \underset{B\leftarrow I}{R}^{-1} = \underset{F\leftarrow I}{R}\ \underset{B\leftarrow I}{R}^{T} =$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \end{bmatrix}$$

This means that

$$\begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \end{bmatrix} = \underbrace{\underset{F\leftarrow I}{R} \; \underset{B\leftarrow I}{R}^T}_{\underset{F\leftarrow B}{R}} \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \end{bmatrix} \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix}$$

The rows of $\underset{F\leftarrow B}{R}$ report the components of $\hat{f}_i$ along $\{\hat{b}_j\}$

- Example 2.2

B written with respect to I as the result of 3 rotations

3-1-3 : $\psi = 150$ deg $\quad \theta = 120$ deg $\quad \phi = -80$ deg

F written with respect to I as the result of 3 rotations

3-2-1 : $\psi = -100$ deg $\quad \theta = -80$ deg $\quad \phi = 110$ deg

(a) Find the Euler angles (3,1,3) that describe the attitude of F with respect to B

$$\underset{B\leftarrow I}{R} = R_3(\phi)\, R_1(\theta)\, R_3(\psi) \qquad \underset{F\leftarrow I}{R} = R_1(\phi) R_2(\theta) R_3(\psi)$$

$$
\begin{tikzcd}
I \arrow[rr, "R_{B\leftarrow I}"] \arrow[dr, "R_{F\leftarrow I}"'] & & B \arrow[dl, "R_{F\leftarrow B}"] \\
& F &
\end{tikzcd}
$$

$$\underset{F\leftarrow B}{R}\ \underset{B\leftarrow I}{R} = \underset{F\leftarrow I}{R}$$

$$\Downarrow$$

$$\underset{F\leftarrow B}{R} = \underset{F\leftarrow I}{R}\ \underset{I\leftarrow B}{R} = \underset{F\leftarrow I}{R}\ \underset{B\leftarrow I}{R}^{T}$$

After evaluating $\underset{F\leftarrow B}{R}$, one recognizes that

$$\underset{F\leftarrow B}{R} = R_3(\phi')\,R_1(\theta')\,R_3(\psi') =$$

$$= \begin{bmatrix} \circ & \circ & S_{\phi'}S_{\theta'} \\ \circ & \circ & C_{\phi'}S_{\theta'} \\ S_{\theta'}S_{\psi'} & -S_{\theta'}C_{\psi'} & C_{\theta'} \end{bmatrix}$$

~~From~~

$$\theta' = \operatorname{acos}\left[\underset{F\leftarrow B}{R}\right]_{33} = 121.52 \text{ deg}$$

$$\begin{cases} S_{\phi'} = \dfrac{\left[\underset{F\leftarrow B}{R}\right]_{13}}{S_{\theta'}} \\[2ex] C_{\phi'} = \dfrac{\left[\underset{F\leftarrow B}{R}\right]_{23}}{S_{\theta'}} \end{cases} \qquad \phi' = 2\operatorname{atan}\frac{S_{\phi'}}{1+C_{\phi'}} = -48.02 \text{ deg}$$

$$\begin{cases} S_{\psi'} = \dfrac{\left[\underset{F\leftarrow B}{R}\right]_{31}}{S_{\theta'}} \\[2ex] C_{\psi'} = -\dfrac{\left[\underset{F\leftarrow B}{R}\right]_{32}}{S_{\theta'}} \end{cases} \qquad \psi' = 2\operatorname{atan}\frac{S_{\psi'}}{1+C_{\psi'}} = 144.25 \text{ deg}$$

- Example 2.3

Spacecraft attached to B

Sensor attached to F

$\left(R_{B \leftarrow I} = R_3(70\text{deg})R_1(100\text{deg})R_3(-70\text{deg}) \right)$

Given the attitude $R_{B \leftarrow I}$ of B with respect to I

and the (internal) orientation of the sensor with respect to B, $R_{F \leftarrow B} = R_1(-100\text{deg})R_2(-10\text{deg})R_3(120\text{deg})$

(a) Find the sensor attitude with respect to I

$$R_{F \leftarrow I} = R_{F \leftarrow B}\, R_{B \leftarrow I}$$

(b) Write $R_{F \leftarrow I}$ in terms of sequence (3-1-3) (Euler angles)

(c) Write $R_{F \leftarrow I}$ in terms of sequence (3-2-1) (Bryant angles)

Solution :

(b) $\psi = 87.29$ deg $\quad \theta = 158.52$ deg $\quad \phi = -32.85$ deg

(c) $\psi = 118.29$ deg $\quad \theta = 11.45$ deg $\quad \phi = 161.71$ deg

- Example 2.4

The spacecraft is attached to B, whose attitude is specified by the following sequence of Bryant angles:

$$\psi = 10 \text{ deg} \qquad \theta = 25 \text{ deg} \qquad \phi = -15 \text{ deg}$$

~~One obtain~~

(a) Find the eigenaxis and the rotation angle

Step 1 Find $\underset{B \leftarrow I}{R} = R_1(\phi) R_2(\theta) R_3(\psi)$

then

$$\Phi = \operatorname{acos}\left\{ \frac{1}{2}\left[\left(\underset{B\leftarrow I}{R}\right)_{11} + \left(\underset{B\leftarrow I}{R}\right)_{22} + \left(\underset{B\leftarrow I}{R}\right)_{33} - 1 \right] \right\} = 31.77 \text{ deg}$$

$$a_1 = \frac{1}{2 S_\Phi} \left\{ \left(\underset{B\leftarrow I}{R}\right)_{23} - \left(\underset{B\leftarrow I}{R}\right)_{23} \right\} = -0.532$$

$$a_2 = \frac{1}{2 S_\Phi} \left\{ \left(\underset{B\leftarrow I}{R}\right)_{31} - \left(\underset{B\leftarrow I}{R}\right)_{13} \right\} = 0.740$$

$$a_3 = \frac{1}{2 S_\Phi} \left\{ \left(\underset{B\leftarrow I}{R}\right)_{12} - \left(\underset{B\leftarrow I}{R}\right)_{21} \right\} = 0.411$$

The second principal rotation angle is

$$\Phi' = 2\pi - \Phi$$

• Example 4.1

Given the following inertia matrices:

(A) $\underset{\sim}{I}^{(B)} = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 4 \end{bmatrix}$ (B) $\underset{\sim}{I}^{(B)} = \begin{bmatrix} 3 & 0 & \sqrt{2} \\ 0 & 5 & 0 \\ \sqrt{2} & 0 & 4 \end{bmatrix}$

Find the principal axes of inertia and the principal inertia moments

Procedure (a) find eigenvalues and eigenvectors associated with $\underset{\sim}{I}^{(B)}$

(b) make the eigenvectors unit vectors

(c) order the unit vectors such that $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ is right-handed
i.e. order the rows of matrix $\underset{E \leftarrow B}{R}$ such that $\det \underset{E \leftarrow B}{R} = +1$

Solution

(A) $$\det \left\{ \begin{bmatrix} 3-I_i & 0 & 1 \\ 0 & 5-I_i & 0 \\ 1 & 0 & 4-I_i \end{bmatrix} \right\} = (5-I_i)\left\{(3-I_i)(4-I_i) - 1\right\} = 0$$

which has solutions $I_1 = \frac{7+\sqrt{5}}{2}$ $I_2 = 5$ $I_3 = \frac{7-\sqrt{5}}{2}$

Using I_1 the linear system $\underset{\sim}{I}^{(B)} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ yields

$$w_2 = 0 \qquad w_1 + \left(\frac{1-\sqrt{5}}{2}\right) w_3 = 0 \qquad w_3 - \left(\frac{\sqrt{5}+1}{2}\right) w_1 = 0$$

where the last 2 relations are equivalent, thus only 1 is needed

Choose now $w_1 = 1$ and one has $w_3 = \frac{\sqrt{5}+1}{2}$

Make now $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ a unit vector

$$\tilde{w}_i = \frac{w_i}{\sqrt{w_1^2 + w_2^2 + w_3^2}} \Rightarrow \quad \tilde{w}_1 = \frac{1}{\sqrt{1 + \left(\frac{\sqrt{5}+1}{2}\right)^2}} \quad \tilde{w}_2 = 0 \quad \tilde{w}_3 = \frac{\frac{\sqrt{5}+1}{2}}{\sqrt{1 + \left(\frac{\sqrt{5}+1}{2}\right)^2}}$$

The latter ones are the components of the first eigenvector

Using I_2 one obtains $\quad -2w_1 + w_3 = 0 \quad w_1 - w_3 = 0 \Rightarrow w_1 = w_3 = 0$

one chooses $w_2 = 1 \Rightarrow w_1 = 0,\ w_2 = 1,\ w_3 = 0 \longrightarrow \tilde{w}_1 = 0,\ \tilde{w}_2 = 1,\ \tilde{w}_3 = 0$

Using I_3 one obtains $\left(-\frac{1}{2} + \frac{\sqrt{5}}{2}\right)w_1 + w_3 = 0\ ,\quad w_2 = 0,\quad w_1 + \left(\frac{\sqrt{5}}{2} + \frac{1}{2}\right)w_3 = 0$

one chooses $w_3 = +1 \Rightarrow \tilde{w}_1 = -\dfrac{\sqrt{5}+1}{2\sqrt{1+\left(\frac{\sqrt{5}+1}{2}\right)^2}},\ \tilde{w}_2 = 0,\ \tilde{w}_3 = +\dfrac{1}{\sqrt{1+\left(\frac{\sqrt{5}+1}{2}\right)^2}}$

and, finally,
$$\underset{E\leftarrow B}{R} = \begin{bmatrix} \dfrac{1}{\sqrt{1+\left(\frac{\sqrt{5}+1}{2}\right)^2}} & 0 & \dfrac{\sqrt{5}+1}{2\sqrt{1+\left(\frac{\sqrt{5}+1}{2}\right)^2}} \\ 0 & 1 & 0 \\ \dfrac{-\frac{\sqrt{5}+1}{2}}{\sqrt{1+\left(\frac{\sqrt{5}+1}{2}\right)^2}} & 0 & \dfrac{+1}{\sqrt{1+\left(\frac{\sqrt{5}+1}{2}\right)^2}} \end{bmatrix}$$

This matrix has correct sequence (right-handed) $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ because $\det \underset{E\leftarrow B}{R} = 1$

(B) The characteristic equation is $(5 - I_i)^2 (I_i - 2) = 0$

Using $I_{1,2} = 5$ one obtains $-2w_1 + \sqrt{2}\,w_3 = 0$ and $\sqrt{2}\,w_1 - w_3 = 0$, which are equivalent. Thus, one chooses

$w_1 = 0,\ w_2 = 1 \Rightarrow w_1 = 0,\ w_2 = 1,\ w_3 = 0$
$w_1 = 1,\ w_2 = 0 \Rightarrow w_1 = 1,\ w_2 = 0,\ w_3 = \sqrt{2}$

Using $I_3 = 2$ one obtains $w_1 + w_3\sqrt{2} = 0,\ w_2 = 0,\ \sqrt{2}\,w_1 + 2w_3 = 0$

where the first and third equations are equivalent. Thus, one chooses

$w_3 = 1 \Rightarrow w_1 = -\sqrt{2},\ w_2 = 0,\ w_3 = 1 \longrightarrow \tilde{w}_1 = -\sqrt{\frac{2}{3}},\ \tilde{w}_2 = 0,\ \tilde{w}_3 = \frac{1}{\sqrt{3}}$

and, finally,
$$\underset{E\leftarrow B}{R} = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \sqrt{\frac{2}{3}} \\ 0 & 1 & 0 \\ -\sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{3}} \end{bmatrix}$$
where the first and second eigenvector have been changed in position (rows) so that $\det \underset{E\leftarrow B}{R} = 1$ and $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ are right-handed

Example 4.2

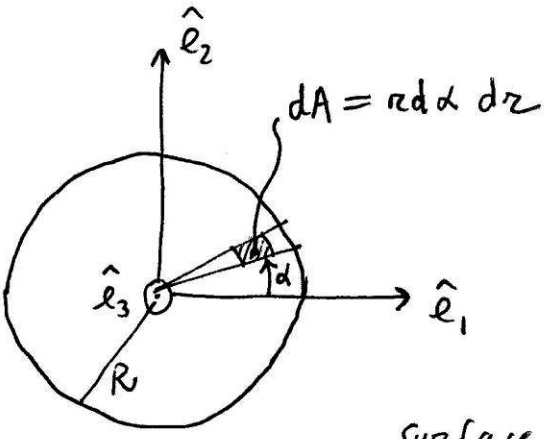
A spacecraft is modeled as a uniform circular disk. It is observed to wobble in a way such that its symmetry axis describes a cone with half angle 60°, and completes a single cone in 1 sec.

(a) Determine the principal moments of inertia as a function of the disk mass M and radius R.

(b) Determine the spin rate $\dot{\phi}$

(c) Determine the disk angular velocity in inertial coordinates.

Solution



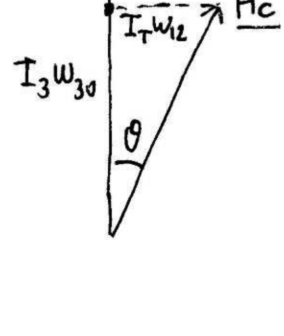
(a) $$I_3 = \int_0^R \int_0^{2\pi} \sigma r^2 \, r \, d\alpha \, dr = \frac{MR^2}{2}$$

$$I_1 = I_2 = \int_0^R \int_0^{2\pi} \sigma (r c_\alpha)^2 \, r \, d\alpha \, dr =$$

$$= \sigma \frac{R^4}{4} \int_0^{2\pi} \frac{1 - C_{2\alpha}}{2} d\alpha = \frac{MR^2}{4}$$

$\sigma = \dfrac{M}{\pi R^2}$ is the surface mass density

(b) For an axisymmetric body



$$H_c = \sqrt{(I_T w_{12})^2 + (I_3 w_{30})^2}$$

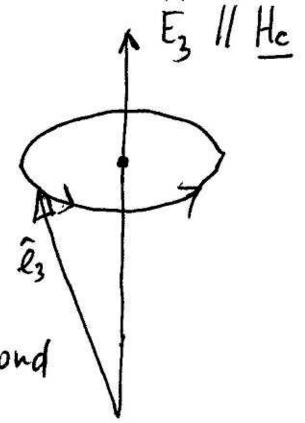
where $w_{12} = \text{const}$ and $w_{30} = \text{const}$

and $\underline{H_c}$ has projection $I_3 w_{30}$ on $\hat{e}_3$

From the previous figure $\quad \dfrac{I_T \omega_{12}}{I_3 \omega_{30}} = \tan\theta$

Then, $\quad \dot{\psi} = \dfrac{I_T \omega_{12}^2}{H_c S_\theta^2} \underset{\uparrow}{=} \dfrac{\omega_{12}}{S_\theta}$.

$$\frac{I_T \omega_{12}}{H_c} = S_\theta$$



As $\hat{e}_3$ completes a cone around $\hat{E}_3$ in 1 second

$$T_\psi = \frac{2\pi}{\dot{\psi}} = 1 \text{ sec} \Rightarrow \frac{\omega_{12}}{S_\theta} = \frac{2\pi}{T_\psi} \Rightarrow \omega_{12} = \frac{2\pi}{T_\psi} S_\theta = \pi\sqrt{3} \text{ sec}^{-1}$$

Moreover, from the previous relation

$$\frac{I_3 \omega_{30}}{I_T \omega_{12}} = \frac{1}{\tan\theta} \longrightarrow \omega_{30} = \frac{I_T}{I_3} \frac{\omega_{12}}{\tan\theta} \underset{\uparrow}{=} \frac{1}{2} \frac{\pi\sqrt{3}}{\sqrt{3}} \text{sec}^{-1} = \frac{\pi}{2} \text{sec}^{-1}$$

$$\frac{I_T}{I_3} = \frac{1}{2}$$

The spin rate for this body is

$$\dot{\phi} = \omega_{30}\left(1 - \frac{I_3}{I_T}\right) = -\omega_{30} = -\frac{\pi}{2} \text{ sec}^{-1}$$

(c) The angular rate in body axes has components $(\omega_1, \omega_2, \omega_3)$, related to the angular rates $\dot{\psi}, \dot{\theta}, \dot{\phi}$ through the Kinematics equations for Euler angles, considering also that $\dot{\theta} = 0$

$$\begin{cases} \omega_1 = \dot{\psi}\, S_\phi S_\theta \\ \omega_2 = \dot{\psi}\, C_\phi S_\theta \\ \omega_3 = \dot{\psi}\, C_\theta + \dot{\phi} \end{cases} \qquad \underline{\omega} = \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}$$

$$\psi(t) = \psi_0 + \frac{\omega_{12}}{S_\theta} t$$

$$\theta(t) = \theta_0 \quad (= 60 \text{ deg})$$

$$\phi(t) = \phi_0 - \omega_{30} t$$

Then, the body frame $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ is the result of 3 elementary rotations, starting from $(\hat{E}_1, \hat{E}_2, \hat{E}_3)$

$$\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} = \underbrace{R_3(\phi)\, R_1(\theta)\, R_3(\psi)}_{\underset{B\leftarrow I}{R}} \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \end{bmatrix}$$

Therefore, the angular velocity in the inertial frame has components $(\Omega_1, \Omega_2, \Omega_3)$

$$\underline{w} = [w_1 \;\; w_2 \;\; w_3] \underset{B\leftarrow I}{R} \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \end{bmatrix} = [\Omega_1 \;\; \Omega_2 \;\; \Omega_3] \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \end{bmatrix}$$

i.e. $$\begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix} = \underset{B\leftarrow I}{R}^{T} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

The initial values ψ_0 and ϕ_0 can be identified as follows

a) ψ_0 depends on the choice of $\hat{E}_1$, that must be orthogonal to $\hat{E}_3$ of course; as this choice is arbitrary, one can select $\hat{E}_1$ such that $\psi_0 = 0$

b) ϕ_0 is related to the initial angular velocity components ω_{10} and w_{20}. In fact

$$\left.\begin{aligned} S_{\phi_0} S_{\theta_0} &= \frac{I_T w_{10}}{H_C} \\ C_{\phi_0} S_{\theta_0} &= \frac{I_T w_{20}}{H_C} \end{aligned}\right. \longrightarrow \quad \begin{aligned} S_{\phi_0} &= \frac{\omega_{10}}{\sqrt{\omega_{10}^2 + \omega_{20}^2}} = \frac{\omega_{10}}{\omega_{12}} \\ C_{\phi_0} &= \frac{\omega_{20}}{\sqrt{\omega_{10}^2 + \omega_{20}^2}} = \frac{\omega_{20}}{\omega_{12}} \end{aligned}$$

If $\omega_{10} = \pi \frac{\sqrt{3}}{2} \sec^{-1}$ and $\omega_{20} = 3\pi \frac{1}{2} \sec^{-1}$ one obtains

$$\begin{cases} S\phi_0 = \dfrac{\pi\sqrt{3}}{2\sqrt{3}\,\pi} = \dfrac{1}{2} \\ C\phi_0 = \dfrac{3\pi}{2\pi\sqrt{3}} = \dfrac{\sqrt{3}}{2} \end{cases} \longrightarrow \phi_0 = 30 \text{ deg}$$

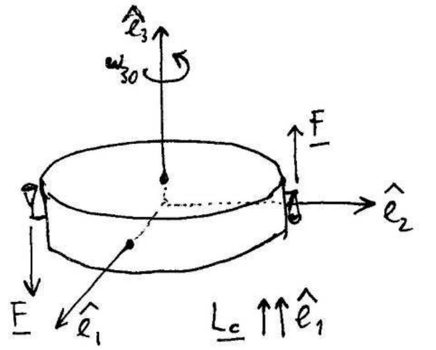
- Example 6.1

A spacecraft is axisymmetric and is spinning about its axis, which is also the axis of maximal inertia. The moments of inertia are

$$I_1 = I_2 = I_T = 15 \text{ kg m}^2 \qquad I_3 = 20 \text{ kg m}^2$$

while $\omega_{30} = 2 \text{ sec}^{-1}$

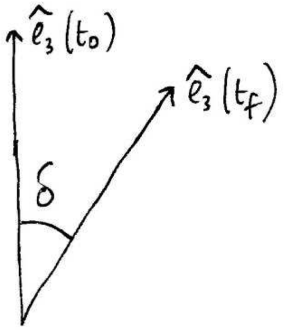
Two thrusters are used to control the spacecraft. Each of them provides 10 N for a duration up to $\Delta t_{max} = 0.3$ sec.



The distance (termed "arm") between $\underline{F}$ and the mass center is $R = 1$ m and the torque is

$$\underline{L_c} = 2 F R \hat{e}_3$$

Determine the two impulsive maneuvers (duration and timing) that allow spacecraft reorientation such that



$\hat{e}_3(t_0)$ and $\hat{e}_3(t_f)$ are displaced by $\delta = 15$ deg

and at t_f $\underline{\omega} = \omega_{30} \hat{e}_3(t_f)$

Solution.

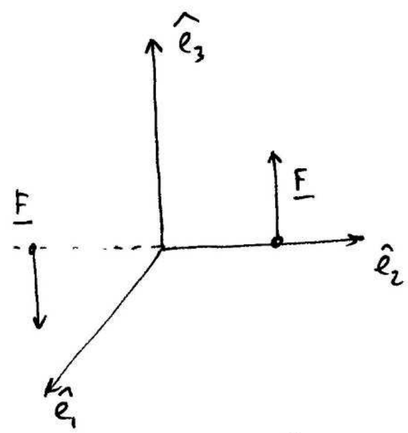
Three steps are needed in order to perform this maneuver

(1) First impulsive torque to change $\underline{H_c}(t_0^-)$ to $\underline{H_c}(t_0^+)$

where $$\underline{H_c}(t_0^+) = \underline{H_c}(t_0^-) + I_1 \Delta\omega_1 \hat{e}_1$$

where $$I_1 \Delta\omega_1 = 2 F R \Delta t$$

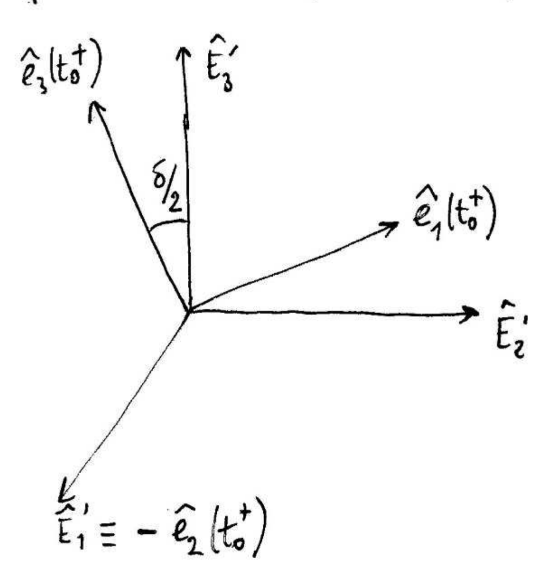
and $$I_1 \Delta\omega_1 = I_3 \omega_{30} \tan\frac{\delta}{2}$$



Thus, one obtains

$$\Delta t = \frac{I_1 \Delta\omega_1}{2FR} = \frac{I_3 \omega_{30}}{2FR} \tan\frac{\delta}{2} = 0.263 \text{ sec}$$

(2) After the 1st "impulsive" torque, retrograde precession starts, and



$$\begin{cases} \dot{\psi} = \dfrac{I_3}{I_T} \dfrac{\omega_{30}}{C_\theta} \\ \dot{\theta} = 0 \\ \dot{\phi} = \omega_{30}\left(1 - \dfrac{I_3}{I_T}\right) \end{cases} \qquad \theta = \frac{\delta}{2}$$

The time at which precession shall end is

$$\bar{t} = \pi \frac{S_\theta}{\omega_{12}} = 1.168 \text{ sec}$$

(3) The third impulse aligns $H_c(t_f^+)$ with $\hat{e}_3(t_f)$ and has the same magnitude and duration of the first one.