Spaceflight Mechanics

Exercise set 7

1. A cylindrical, homogeneous satellite has mass of 200 kg, radius 1 m, and height 0.5 m. Its orientation is associated with the following Euler parameters (quaternions):

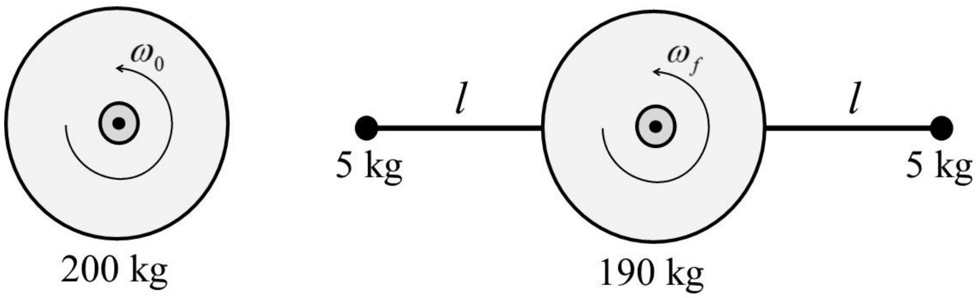
$$q_0 = 0.5 \qquad q_1 = 0.75 \qquad q_2 = \frac{\sqrt{3}}{4} \qquad q_3 = 0$$

(a) Obtain the Bryant's angles (sequence 3-2-1) associated with this orientation.

The satellite has pure spin about the axis of maximal inertia, with angular rate $\omega_0 = 3\ \text{sec}^{-1}$. Then, two appendices are deployed. They are modeled as point masses, and each of them has mass of 5 kg (see figure, taken from the top). The rods that connect these two masses with the cylinder are rigid and massless. After deployment, the angular rate decreases to $\omega_f = 1\ \text{sec}^{-1}$.

(b) Calculate the length of each rod (denoted with *l* in the figure).

Before deployment (top view) *After deployment (top view)*



2. An axisymmetric satellite, with symmetry axis $\hat{e}_3$, is not subject to external torques and no energy dissipation occurs. At a given time t_0, the satellite rotates with angular velocity components given by

$$\omega_{10} = 0.02\ \text{sec}^{-1} \qquad \omega_{20} = -0.05\ \text{sec}^{-1} \qquad \omega_{30} = 0.5\ \text{sec}^{-1}$$

whereas the principal inertia moments are

$$I_1 = I_2 = I_T = 200\ \text{kg}\cdot\text{m}^2 \qquad I_3 = 250\ \text{kg}\cdot\text{m}^2$$

The inertial axis $\hat{E}_3$ is aligned with the angular momentum $\boldsymbol{H}_C$, and the Euler angles (sequence 3-1-3) are employed to describe the instantaneous spacecraft orientation. The precession angle ψ at t_0 equals 60 deg.

(a) Calculate the angular momentum components along the principal axes of inertia at t_0.

(b) Evaluate the nutation angle θ and the spin angle ϕ at t_0.

(c) At t_0, obtain the rotation matrix $\underset{B \leftarrow I}{\mathbf{R}}$, where B $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ is the reference frame associated with the principal axes of inertia and I $(\hat{E}_1, \hat{E}_2, \hat{E}_3)$ is the inertial frame.

At $t_0 + 60 \text{ sec}$, an impulsive torque (with duration of 1 sec) is applied, such that the subsequent attitude motion is pure spin about body axis $\hat{e}_3$.

(d) Determine direction (in the body axes frame $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$) and magnitude of this impulsive torque.

3. A cylindrical axisymmetric spacecraft has radius of 1 m, moments of inertia

$$I_1 = I_2 = I_T = 20 \text{ kg} \cdot \text{m}^2 \qquad I_3 = 35 \text{ kg} \cdot \text{m}^2$$

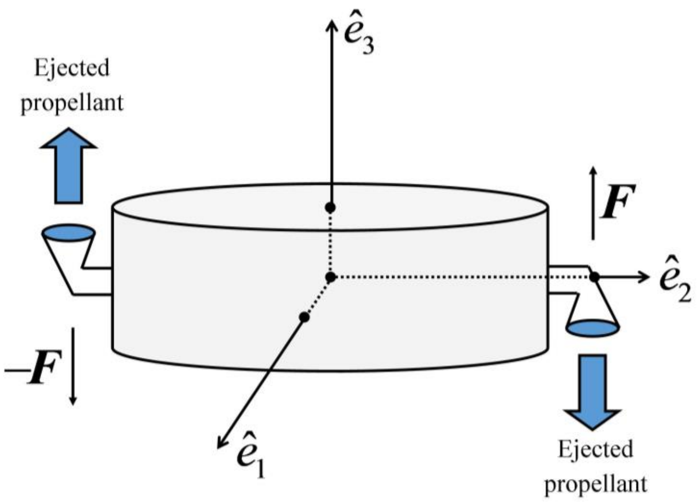
and is subject to no external torque. Axis $\hat{E}_3$ of the inertial reference frame is aligned with the angular momentum $\boldsymbol{H}_C\left(t_0^-\right)$; the spacecraft rotates with a constant nutation angle θ of 60 deg and a transverse velocity component $\omega_{12} = \sqrt{\omega_1^2 + \omega_2^2} = 0.3 \text{ sec}^{-1}$. At t_0^- $\omega_1\left(t_0^-\right) = -\omega_{12}$.

(a) Determine the angular velocity component ω_3.

(b) The spacecraft is equipped with two thrusters (see figure below), which are ignited for 1 sec at t_0 and provide a propulsive thrust $\boldsymbol{F}$, whose magnitude F equals 1 newton. Under the impulsive torque assumption, obtain the components (along the body axes) of the angular momentum $\boldsymbol{H}_C\left(t_0^+\right)$ after ignition of the two thrusters.

(c) A new inertial reference frame is defined, with axis $\hat{E}_3'$ aligned with $\boldsymbol{H}_C\left(t_0^+\right)$. Calculate the nutation angle at t_0^+, with respect to $\hat{E}_3'$.

(d) In the new inertial reference frame, the precession angle at t_0^+ is $\psi\left(t_0^+\right) = 0$. Determine the principal axis and angle associated with the instantaneous orientation of the spacecraft at time $t = t_0 + 60 \text{ sec}$.



4. An axisymmetric satellite, with symmetry axis $\hat{e}_3$ aligned with the satellite longitudinal axis, is not subject to external torques and no energy dissipation occurs. At a given time t_0, the satellite rotates with angular velocity components given by

$$\omega_{10} = -0.05 \text{ sec}^{-1} \qquad \omega_{20} = 0.02 \text{ sec}^{-1} \qquad \omega_{30} = 0.5 \text{ sec}^{-1}$$

whereas the principal inertia moments are

$$I_1 = I_2 = I_T = 100 \text{ kg}\cdot\text{m}^2 \qquad I_3 = 150 \text{ kg}\cdot\text{m}^2$$

The inertial axis $\hat{E}_3$ is aligned with the angular momentum $\boldsymbol{H}_C$, and the Euler angles (sequence 3-1-3) are employed to describe the instantaneous spacecraft orientation. The precession angle ψ at t_0 equals 30 deg.

(a) Calculate the angular momentum components along the principal axes of inertia at t_0

(b) Evaluate the nutation angle θ and the spin angle ϕ at t_0

(c) At t_0, obtain the rotation matrix $\underset{B\leftarrow I}{\mathbf{R}}$, where B $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ is the reference frame associated with the principal axes of inertia and I $(\hat{E}_1, \hat{E}_2, \hat{E}_3)$ is the inertial frame.

SOLUTION OF EXERCISE 1

Point (a). The rotation matrix associated with the instantaneous orientation is found on the basis of the known values of quaternions,

$$\underset{B\leftarrow I}{\mathbf{R}} = \begin{bmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2+q_0q_3) & 2(q_1q_3-q_0q_2) \\ 2(q_1q_2-q_0q_3) & q_0^2+q_2^2-q_1^2-q_3^2 & 2(q_2q_3+q_0q_1) \\ 2(q_1q_3+q_0q_2) & 2(q_2q_3-q_0q_1) & q_0^2+q_3^2-q_2^2-q_1^2 \end{bmatrix} = \begin{bmatrix} 0.6250 & 0.6495 & -0.4330 \\ 0.6495 & -0.1250 & 0.7500 \\ 0.4330 & -0.7500 & -0.5000 \end{bmatrix}$$

Then, Bryant's angles are found from the general relations that yield them from $\underset{B\leftarrow I}{\mathbf{R}}$

$$\theta = -\arcsin r_{13} = 25.66 \text{ deg}$$

$$\sin\psi = \frac{r_{12}}{\cos\theta} = 0.7206 \quad \text{and} \quad \cos\psi = \frac{r_{11}}{\cos\theta} = 0.6934 \ \Rightarrow \ \psi = 2\arctan\frac{0.7206}{1+0.6934} = 46.10 \text{ deg}$$

$$\sin\phi = \frac{r_{23}}{\cos\theta} = 0.8321 \quad \text{and} \quad \cos\phi = \frac{r_{33}}{\cos\theta} = -0.5527 \ \Rightarrow \ \phi = 2\arctan\frac{0.8321}{1-0.5527} = 123.69 \text{ deg}$$

where $\{r_{ij}\}$ are the elements of matrix $\underset{B\leftarrow I}{\mathbf{R}}$.

Point (b). As a first step, using the relations that hold for a homogeneous cylinder, the principal moments of inertia are

$$I_3 = \frac{MR^2}{2} = 100 \text{ kg m}^2 \qquad I_1 = I_2 = \frac{M}{12}\left(3R^2+H^2\right) = 54.167 \text{ kg m}^2$$

Hence, the axis of maximal inertia is $\hat{e}_3$ and pure spin occurs about this axis. Due to conservation of the angular momentum,

$$I_3^-\omega_0 = \left[I_3^+ + 2\tilde{m}(l+R)^2\right]\omega_f \qquad \text{where}$$

$$I_3^- = I_3 = 100 \text{ kg m}^2 \qquad I_3^+ = \frac{M_f R^2}{2} = 95 \text{ kg m}^2 \quad \left(M_f = 190 \text{ kg}\right) \qquad \tilde{m} = 5 \text{ kg}$$

Hence, one obtains

$$(l+R)^2 = \frac{I_3^-\omega_0 - I_3^+\omega_f}{2\tilde{m}\omega_f} \qquad \Rightarrow \qquad l = 3.528 \text{ m}$$

SOLUTION OF EXERCISE 2

Point (a). The three components $(\hat{e}_1,\hat{e}_2,\hat{e}_3)$ of the angular momentum $\boldsymbol{H}_C$ are

$$H_1 = I_1\omega_{10} = 4 \text{ kg } \frac{\text{m}^2}{\text{sec}} \qquad H_2 = I_2\omega_{20} = -10 \text{ kg } \frac{\text{m}^2}{\text{sec}} \qquad H_3 = I_3\omega_{30} = 125 \text{ kg } \frac{\text{m}^2}{\text{sec}}$$

Point (b). Because $\hat{E}_3$ is aligned with the angular momentum $\boldsymbol{H}_C$, the spin angle ϕ_0 and the nutation angle θ_0 are given by

$$\theta_0 = \arccos\left[\frac{H_3}{H_C}\right] \qquad \left(H_C = \left|\boldsymbol{H}_C\right|\right) \qquad \Rightarrow \qquad \theta_0 = 4.9 \text{ deg}$$

$$\sin\phi_0 = \frac{H_1}{H_C \sin\theta_0} \qquad \text{and} \qquad \cos\phi_0 = \frac{H_2}{H_C \sin\theta_0} \qquad \Rightarrow \qquad \phi_0 = 158.2 \text{ deg}$$

Point (c). The rotation matrix $\underset{B\leftarrow I}{\mathbf{R}}$ is found as the result of three consecutive elementary rotations,

$$\underset{B\leftarrow I}{\mathbf{R}} = \mathbf{R}_3(\phi_0)\mathbf{R}_1(\theta_0)\mathbf{R}_3(\psi_0) = \begin{bmatrix} -0.7847 & -0.6191 & 0.0319 \\ 0.6154 & -0.7842 & -0.0797 \\ 0.0743 & -0.0429 & 0.9963 \end{bmatrix}$$

Point (d). An axisymmetric satellite rotates with angular velocity components (along $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$) given by

$$\omega_1 = \omega_{12}\cos(\omega_P t + \varphi) \qquad \omega_2 = \omega_{12}\sin(\omega_P t + \varphi) \qquad \omega_3 = \omega_{30}$$

where

$$\omega_{12} = \sqrt{\omega_{10}^2 + \omega_{20}^2} \qquad \omega_P = \omega_{30}\left(\frac{I_3}{I_T} - 1\right) \qquad \sin\varphi = \frac{\omega_{20}}{\omega_{12}} \qquad \cos\varphi = \frac{\omega_{10}}{\omega_{12}}$$

After 60 sec, one obtains the following values for the angular velocity components (at $t_1 = t_0 + 60$ sec):

$$\omega_1(t_1) = 0.0538 \text{ sec}^{-1} \qquad \omega_2(t_1) = 0.0014 \text{ sec}^{-1} \qquad \omega_3(t_1) = 0.5 \text{ sec}^{-1}$$

and the corresponding components of the angular momentum are given by

$$H_1(t_1) = I_1\omega_1(t_1) \qquad H_2(t_1) = I_2\omega_2(t_1) \qquad H_3(t_1) = I_3\omega_3(t_1)$$

Right after the impulsive torque (at t_1^+), the satellite rotates with pure spin about axis 3, therefore the components of $\boldsymbol{H}_C$ are

$$H_1(t_1^+) = 0 \qquad H_2(t_1^+) = 0 \qquad H_3(t_1^+) = I_3\omega_3(t_1)$$

This means that

$$\boldsymbol{H}_C(t_1^+) - \boldsymbol{H}_C(t_1) = \boldsymbol{L}_C \Delta t \qquad (\Delta t = 1 \text{ sec})$$

and finally one obtains the three torque components

$$L_{C1} = \frac{H_1(t_1^+) - H_1(t_1)}{\Delta t} = -10.7665 \text{ N m}$$

$$L_{C2} = \frac{H_2(t_1^+) - H_2(t_1)}{\Delta t} = -0.2856 \text{ N m}$$

$$L_{C3} = \frac{H_3(t_1^+) - H_3(t_1)}{\Delta t} = 0 \text{ N m}$$

and the torque magnitude

$$L_C = 10.7703 \text{ N m}$$

SOLUTION OF EXERCISE 3

Point (a). For free-torque motion of an axisymmetric spacecraft, if $\hat{E}_3$ is aligned with the angular momentum, then the angular velocity component ω_3 and the transverse angular velocity ω_{12} fulfill the following relation:

$$\frac{I_T\omega_{12}}{I_3\omega_3}=\tan\theta \qquad \rightarrow \qquad \omega_3=\frac{I_T\omega_{12}}{I_3\tan\theta}=0.099\ \text{sec}^{-1}$$

Point (b). The two thrusters portrayed in the figure provide a torque $\boldsymbol{L}_C$ aligned with body axis $\hat{e}_1$, i.e. $\boldsymbol{L}_C=L_{C1}\hat{e}_1$. The torque component L_{C1} is given by

$$L_{C1}=2FR=2\ \text{N m}$$

This torque is applied for 1 sec, hence its action can be approximated as instantaneous (impulsive torque assumption). The angular momentum variation at t_0 is given by

$$\boldsymbol{H}_C\left(t_0^+\right)-\boldsymbol{H}_C\left(t_0^-\right)=\boldsymbol{L}_C\Delta t \qquad \left(\Delta t=1\ \text{sec}\right)$$

As a result, right after the impulse, the angular momentum is

$$\boldsymbol{H}_C\left(t_0^+\right)=\begin{bmatrix}-4 & 0 & 3.464\end{bmatrix}\begin{bmatrix}\hat{e}_1\\ \hat{e}_2\\ \hat{e}_3\end{bmatrix}\ \text{kg}\frac{\text{m}^2}{\text{sec}} \qquad \text{i.e.} \qquad \begin{cases}H_{C1}=-4\ \ \text{kg m}^2/\text{sec}\\ H_{C2}=0\\ H_{C3}=3.464\ \ \text{kg m}^2/\text{sec}\end{cases}$$

Point (c). At t_0^+, because $\hat{E}_3'$ is aligned with $\boldsymbol{H}_C\left(t_0^+\right)$, the following relations hold:

$$\cos\left[\theta\left(t_0^+\right)\right]=\frac{I_3\omega_3\left(t_0^+\right)}{H_C\left(t_0^+\right)} \qquad \rightarrow \qquad \theta\left(t_0^+\right)=49.1\ \text{deg}$$

$$\begin{cases}\sin\left[\phi\left(t_0^+\right)\right]=\dfrac{I_T\omega_1\left(t_0^+\right)}{H_C\left(t_0^+\right)\sin\left[\theta\left(t_0^+\right)\right]}\\[2ex] \cos\left[\phi\left(t_0^+\right)\right]=\dfrac{I_T\omega_2\left(t_0^+\right)}{H_C\left(t_0^+\right)\sin\left[\theta\left(t_0^+\right)\right]}\end{cases} \qquad \rightarrow \qquad \phi\left(t_0^+\right)=-90\ \text{deg}$$

Point (d). The time evolution of the three Euler angles for an axisymmetric spacecraft is

$$\begin{aligned}&\psi(t)=\psi\left(t_0^+\right)+\frac{\omega_{12}}{\sin\theta}\left(t-t_0\right)\\ &\theta(t)=\theta\left(t_0^+\right)\\ &\phi(t)=\phi\left(t_0^+\right)+\omega_3\left(t_0^+\right)\left[1-\frac{I_3}{I_T}\right]\left(t-t_0\right)\end{aligned}$$

Right after the impulsive torque the angular velocity components are

$$\omega_1\left(t_0^+\right)=\frac{H_{C1}}{I_T}=-0.2\ \text{sec}^{-1} \qquad \omega_1\left(t_0^+\right)=\frac{H_{C2}}{I_T}=0 \qquad \omega_3\left(t_0^+\right)=\frac{H_{C3}}{I_3}=0.099\ \text{sec}^{-1}$$

Hence, at time $\bar{t}=t_0+60\ \text{sec}$, the three Euler angles equal

$$\psi\left(\bar{t}\right) = 189.5 \text{ deg} \qquad \theta\left(\bar{t}\right) = 49.1 \text{ deg} \qquad \phi\left(\bar{t}\right) = 14.8 \text{ deg}$$

At $\bar{t}$ the rotation matrix $\underset{B\leftarrow I}{\mathbf{R}}$ is the result of three subsequent elementary rotations 3-1-3 by angles $\psi\left(\bar{t}\right)$, $\theta\left(\bar{t}\right)$, and $\phi\left(\bar{t}\right)$. The general expression derived during the lectures can be used, to obtain

$$\underset{B\leftarrow I}{\mathbf{R}} = \begin{bmatrix} -0.9256 & -0.3253 & 0.1933 \\ 0.3571 & -0.5818 & 0.7308 \\ -0.1253 & 0.7455 & 0.6547 \end{bmatrix}$$

Once the rotation matrix is found, the principal axis and angle can be derived,

$$\Phi = \arccos\left\{\frac{1}{2}\left[r_{11} + r_{22} + r_{33} - 1\right]\right\} = 157.9 \text{ deg}$$

$$a_1 = \frac{r_{23} - r_{32}}{2\sin\Phi} = -0.0195$$

$$a_2 = \frac{r_{31} - r_{13}}{2\sin\Phi} = -0.4230$$

$$a_3 = \frac{r_{12} - r_{21}}{2\sin\Phi} = -0.9059$$

where $\left\{r_{ij}\right\}$ are the elements of matrix $\underset{B\leftarrow I}{\mathbf{R}}$.

SOLUTION OF EXERCISE 4

Point (a). The three components of the angular momentum (along the principal axes of inertia) are

$$H_1 = I_1\omega_{10} = -5 \text{ kg}\cdot\frac{\text{m}^2}{\text{sec}} \qquad H_2 = I_2\omega_{20} = 2 \text{ kg}\cdot\frac{\text{m}^2}{\text{sec}} \qquad H_3 = I_3\omega_{30} = 75 \text{ kg}\cdot\frac{\text{m}^2}{\text{sec}}$$

Point (b). When $\hat{E}_3$ is aligned with $\boldsymbol{H}_C$, the following relations hold:

$$\cos\theta_0 = \frac{I_3\omega_{30}}{H_C} \qquad \sin\phi_0 = \frac{I_1\omega_{10}}{H_C\sin\theta_0} \qquad \cos\phi_0 = \frac{I_2\omega_{20}}{H_C\sin\theta_0}$$

and lead to obtaining the desired values

$$\theta_0 = 4.1 \text{ deg} \quad \text{and} \quad \phi_0 = -68.2 \text{ deg}$$

Point (c). At t_0 the rotation matrix $\underset{B\leftarrow I}{\mathbf{R}}$ is the result of three subsequent elementary rotations 3-1-3 by angles ψ_0, θ_0, and ϕ_0 (the Euler angles). The general expression derived during the lectures can be used, to obtain

$$\underset{B\leftarrow I}{\mathbf{R}} = \begin{bmatrix} 0.7847 & -0.6163 & -0.0665 \\ 0.6189 & 0.7850 & 0.0266 \\ 0.0358 & -0.0620 & 0.9974 \end{bmatrix}$$