DEF. X ps è PROGRESSIVAMENTE MISURABILE w.r.t. $(\mathcal{F}_t)$ se, $\forall\, t$,

$$\{(s,\omega) \in [0,t] \times \Omega \mid X_s(\omega) \in H\} \in \mathcal{B}([0,t]) \otimes \mathcal{F}_t, \quad \forall\, H \in \mathcal{B}$$

DEF. Data $g : [0,t] \to \mathbb{R}^N$ ed una partizione $\xi = \{t_0, \dots, t_N\}$ di $[0,t]$, la VARIAZIONE QUADRATICA di g relativa a ξ è

$$V_t^{(2)}(g,\xi) := \sum_{k=1}^{N} |g(t_k) - g(t_{k-1})|^2$$

THM Se W è BM, allora

$$\lim_{|\xi| \to 0} V_t^{(2)}(g,\xi) = t \quad \text{in } L^2(\Omega, \mathbb{P})$$

Diciamo che $f(t)=t$ è la VQ di W e scriviamo

$$\langle W \rangle_t = t, \quad t \geq 0.$$

PROOF TH: $\lim_{|\xi| \to 0} \mathbb{E}\left[\left(V_t^{(2)}(W,\xi) - t\right)^2\right] = 0.$

$$\Delta_k := W_{t_k} - W_{t_{k-1}}, \quad \delta_k := t_k - t_{k-1}, \quad k = 1, \dots, N.$$

$$\Rightarrow \mathbb{E}[\Delta_k^2] = t_k - t_{k-1} = \delta_k.$$

Inoltre,

$$\mathbb{E}[\Delta_k^4] = \int_{\mathbb{R}} y^4\, \Gamma(y, \delta_k)\, dy, \quad \text{dove } \Gamma(y,t) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}}$$

$$\Rightarrow \mathbb{E}[\Delta_k^4] = 3\delta_k^2 \quad (\text{verifica x es.})$$

$$\Rightarrow \mathbb{E}\left[\left(V_t^2(W,\xi)-t\right)^2\right] = \mathbb{E}\left[\left(\sum_{k=1}^{N}|W_{t_k}-W_{t_{k-1}}|^2-t\right)^2\right]$$

$$= \mathbb{E}\left[\left(\sum_{k=1}^{N}|\Delta_k|^2-t\right)^2\right] = \mathbb{E}\left[\left(\sum_{k=1}^{N}\left(|\Delta_k|^2-\delta_k\right)\right)^2\right]$$

$$\left(\sum \delta_k = t\right) \qquad \sum_{k=1}^{N}\delta_k = \delta_1+\delta_2+\delta_3+\dots+\delta_N = (t_1-t_0)+(t_2-t_1)+\dots+(t_N-t_{N-1}) = -t_0+t_N = 0+t = t.$$

$$\overset{*}{=} \sum_{k=1}^{N}\mathbb{E}\left[\left(\Delta_k^2-\delta_k\right)^2\right] + 2\sum_{h<k}\mathbb{E}\left[\left(\Delta_k^2-\delta_k\right)\left(\Delta_h^2-\delta_h\right)\right]$$

$$\overset{(\text{ind})}{=} \sum_{k=1}^{N}\mathbb{E}\left[\left(\Delta_k^2-\delta_k\right)^2\right] + 2\sum_{h<k}\underbrace{\mathbb{E}\left[\Delta_k^2-\delta_k\right]}_{=0}\underbrace{\mathbb{E}\left[\Delta_h^2-\delta_h\right]}_{=0}$$

$$= \sum_{k=1}^{N}\left(\mathbb{E}\left[\Delta_k^4\right]+\delta_k^2-2\delta_k\mathbb{E}\left[\Delta_k^2\right]\right)\cancel{+2\mathbb{E}\left[\Delta_k^2-\delta_k\right]\sum_{h<k}\mathbb{E}\left[\Delta_h^2-\delta_h\right]}$$

$$= \sum_{k=1}^{N}\left(3\delta_k^2+\delta_k^2-2\delta_k^2\right) = 2\sum_{k=1}^{N}\delta_k^2 \leq 2t|\xi|$$

$$2\sum_k \delta_k^2 = 2\sum_k\left(\delta_k\cdot\underbrace{\delta_k}_{\leq|\xi|}\right) \leq 2|\xi|\sum_k\delta_k = 2t|\xi|$$

$$\Rightarrow \lim_{|\xi|\to 0}\mathbb{E}\left[\left(V_t^2(W,t)-t\right)^2\right] = 0$$

perché un intervallo è sempre + piccolo della lunghezza dell'intera partizione

$$\Rightarrow V_t^2(W,t) = t \quad \text{in } L^2(\Omega,\mathbb{P}) \quad \blacksquare$$

$*$ Es. per $N=3$.

$$\mathbb{E}\left[\left(\sum_{k=1}^{3}(a_k+b_k)\right)^2\right] = \mathbb{E}\left[\left((a_1+b_1)+(a_2+b_2)+(a_3+b_3)\right)^2\right] =$$

$$= \mathbb{E}\left[(a_1+b_1)^2+(a_2+b_2)^2+(a_3+b_3)^2+2(a_1+b_1)(a_2+b_2)+2(a_1+b_1)(a_3+b_3)+2(a_2+b_2)(a_3+b_3)\right]$$

$$= \sum_{k=1}^{3}\mathbb{E}\left[(a_k+b_k)^2\right] + 2\sum_{\substack{h,k=1\\h<k}}^{3}(a_k+b_k)(a_h+b_h)$$

Dalla def. di MB si ricava che, $\forall\ t_1<t_2<\dots<t_N$, la v.a. $\overline{W} = (W_{t_1}, W_{t_2}, \dots, W_{t_N})$ è una gaussiana multivariata.

$$\Rightarrow Cov(W_{t_i}, W_{t_j}) = \min\{t_i, t_j\}, \quad \forall i,j = 1, \dots, N$$

PROOF:

Sia $i<j \Rightarrow t_i<t_j$

$$\Rightarrow Cov(W_{t_i}, W_{t_j}) \overset{(\text{def})}{=} \mathbb{E}[W_{t_i} \cdot W_{t_j}] - \mathbb{E}[W_{t_i}] \cdot \mathbb{E}[W_{t_j}]$$

$$= \mathbb{E}[W_{t_i} \cdot W_{t_j}] - \underbrace{0}_{\text{def. di MB}}$$

$$= \mathbb{E}[\underbrace{W_{t_i}}_{W_{t_i} - W_0}(W_{t_j} - W_{t_i} + W_{t_i})] =$$

$$= \mathbb{E}[W_{t_i}(W_{t_j} - W_{t_i}) + (W_{t_i})^2]$$

$$\overset{(\text{lin.})}{=} \mathbb{E}[\underbrace{W_{t_i}}\ \underbrace{(W_{t_j} - W_{t_i})}] + \mathbb{E}[(W_{t_i})^2]$$

indipendenti

$$= \underbrace{\underbrace{\mathbb{E}[W_{t_i}]}_{=0}\ \mathbb{E}[\underbrace{W_{t_j} - W_{t_i}}_{=0}]}_{\text{Def. di MB}} + t_i$$

$$\Rightarrow Cov(W_{t_i}, W_{t_j}) = t_i, \text{ se } t_i<t_j.$$

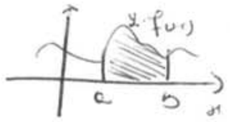
Analogamente, $Cov(W_{t_i}, W_{t_j}) = t_j$, se $t_i>t_j$ ∎

~~DEF: Una funzione $f: I \to \mathbb{R}$, $t \mapsto f(t)$ è NON ANTICIPATIVA~~

B

GOAL: definire $\int_0^T f(t)\, dW_t = ?$

Siamo abituati a $\int_a^b f(x)\, dx$ (Integrale di Reimann)



EX: $u \in C^1([0,1])$: $u(0) = u(1) = 0$

W BM

È v.a.? Momenti?

$$\int_0^1 u(t)\, dW_t := -\int_0^1 u'(t)\, \cancel{dW}\, W_t\, dt$$

$$\mathbb{E}\left[\int_0^1 u(t)\, dW_t\right] = ? \qquad \mathbb{E}\left[\left(\int_0^1 u(t)\, dW_t\right)^2\right] = ?$$

1) $$\mathbb{E}\left[\int_0^1 u(t)\, dW_t\right] = \mathbb{E}\left[-\int_0^1 u'(t) W_t\, dt\right]$$

$$\overset{\text{(Fubini-Tonelli)}}{=} -\int_0^1 u'(t)\, \mathbb{E}[W_t]\, dt = 0$$

2) $$\mathbb{E}\left[\left(\int_0^1 u(t)\, dW_t\right)^2\right] = \mathbb{E}\left[\left(\int_0^1 u'(t) W_t\, dt\right)\left(\int_0^1 u'(s) W_s\, ds\right)\right]$$

$$= \mathbb{E}\left[\int_0^1 \int_0^1 u'(t) u'(s) W_t W_s\, dt\, ds\right]$$

$$= \int_0^1 u'(t) \left(\int_0^1 u'(s) \underbrace{\mathbb{E}[W_t W_s]}_{\begin{cases} t, & t<s \\ s, & t>s \end{cases}} ds \right) dt$$

$\rightarrow$ prop. dimostrata prima, see pag (2b)

$$= \int_0^1 u'(t) \Big[\overbrace{\underbrace{\int_0^t s\, u'(s)\, ds}_{\text{(PARTI)}}}^{\text{quando } s<t} + t \overbrace{\int_t^1 u'(s)\, ds}^{\text{quando } t<s} \Big] dt$$

$$\cancel{= \int_0^1 u'(t) \Big[\int_0^t s\, u'(s)\, ds}$$

$$= \int_0^1 u'(t) \Big[s\,u(s)\Big|_0^t - \int_0^t u(s)\, ds + t\, u(s)\Big|_t^1 \Big] dt$$

$$= \int_0^1 u'(t) \Big[\cancel{t\,u(t)} - \int_0^t u(s)\, ds + \cancel{t\,u(1)} - \cancel{t\,u(t)} \Big] dt$$

$$= \int_0^1 u'(t) \left(- \int_0^t u(s)\, ds \right) dt$$

$$= u(t) \left(\cancel{- \int_0^t u(s)\, ds} \right) \Big|_0^1 + \int_0^1 u^2(t)\, dt \quad \blacksquare$$

Più in generale,

$\forall\, u \in L^2([0,1])$, (u_n) in $C_0^1([0,1])$

$\Rightarrow (u_n)$ è seq. di Cauchy in $L^2(\Omega, \mathcal{F}, \mathbb{P})$

$$\Rightarrow \int_0^1 u(t)\, dW_t = \lim_{n\to+\infty} \int_0^1 u_n(t)\, dW_t$$

$\triangle$ cosa succede se la funzione integranda è più complicata?