

DEF. $u \in \mathbb{L}^2$ se

- 1) u prog. w.r.t. \mathcal{F}_t
- 2) $u \in L^2([0, T] \times \Omega)$ $\Leftrightarrow \mathbb{E} \left[\int_0^T |u(t)|^2 dt \right] < \infty$
 (CONDIZIONE DI INTEGRABILITÀ) (QUADRATO-INTEGRABILE)

DEF. $u \in \mathbb{L}^2$ è SEMIPRICE se

$$(***) M_t = \sum_{k=1}^N \rho_k M_{\{(t_{k-1}, t_k]\}}^{(t)}, \quad \forall t \in [0, T],$$

$$0 \leq t_0 < t_1 < \dots < t_N \leq T \quad \rho_k \bar{e} \mathcal{F}_{t_{k-1}} \text{-misurabile}$$

$$\rho_k \forall, \rho_k \text{ su } (\Omega, \mathcal{F}, \mathbb{P}), \quad \forall k=1, \dots, N$$

$$\Rightarrow \text{DEF. } \boxed{\text{INTEGRALE DI ITO}} := \sum_{k=1}^N \rho_k (M_{t_k} - M_{t_{k-1}}) \quad (ITO)$$

INTEGRALE DI ITO \Rightarrow come in integrazione di Riemann (Riemanniani)

$$\Rightarrow \int_a^b u_t dW_t = \int u_t M_{\{(0, b]\}}^{(t)} dW_t \quad (**)$$

EXI: VERIFICARE che $W_t = \int_0^t dW_s$

(HINT: usare $u_t := M_{\{(0, t]\}}^{(t)}$)

THM: $\forall u, v \in \mathbb{L}^2$, $\forall \alpha, \beta \in \mathbb{R}$, $\forall 0 \leq a < b < c \leq T$
 (prox. semplici)

$$1) \int (\alpha u_t + \beta v_t) dW_t = \alpha \int u_t dW_t + \beta \int v_t dW_t \quad (\text{lineare})$$

$$2) \int_a^b u_t dW_t + \int_b^c u_t dW_t = \int_a^c u_t dW_t \quad (\text{additività})$$

$$3) \mathbb{E} \left[\int_0^b u_t dW_t \mid \mathcal{F}_a \right] = 0 \Rightarrow \mathbb{E} \left[\int_0^T u_t dW_t \right] = 0$$

$$4) \mathbb{E} \left[\int_0^b u_t dW_t \int_0^b v_t dW_t \mid \mathcal{F}_a \right] = \mathbb{E} \left[\int_0^b u_t v_t dt \mid \mathcal{F}_a \right] \quad (\text{ISOMETRIA DI ITO})$$

$$\Rightarrow \mathbb{E} \left[\left(\int_0^b u_t dW_t \right)^2 \right] = \mathbb{E} \left[\int_0^b u_t^2 dt \mid \mathcal{F}_a \right]$$

Siemo in grado di intendere la def. di
integrale stocastico di processi in \mathbb{L}^2

$$I_t(u) := \int_0^t u_s dW_s, \quad u \in \mathbb{L}^2$$

DEF.

Dato un processo $u \in \mathbb{L}^2$, l'integrale stocastico è

$$I_t(u) = \lim_{n \rightarrow \infty} \int_0^t u_s^{(n)} dW_s,$$

dove $\{u_s^{(n)}\}$ è una sequenza di processi semplici
che approssimano u in \mathbb{L}^2 .

THM
 $\forall u, v \in \mathbb{L}^2$, $\alpha \in \mathbb{R}$, $0 \leq a < b < c$, si ha

$$1) \int_0^a (\alpha u_t + \beta v_t) dW_t = \alpha \int_0^a u_t dW_t + \beta \int_0^a v_t dW_t$$

$$2) \int_0^c u_t dW_t = \int_0^b u_t dW_t + \int_b^c u_t dW_t$$

$$3) \mathbb{E} \left[\int_0^b u_t dW_t \mid \mathcal{F}_a \right] = 0$$

$$4) \mathbb{E} \left[\int_0^b u_t dW_t \int_0^b v_t dW_t \mid \mathcal{F}_a \right] = \mathbb{E} \left[\int_0^b u_t v_t dt \mid \mathcal{F}_a \right]$$

□

DEF. Un BS $X = \{X_t\}_{t \geq 0}$ deve fornire una formula

$$X_t = X_0 + \underbrace{\int_0^t \mu_s^{(X)} ds}_{\text{PROX. A VARIAZIONE FINITA}} + \underbrace{\int_0^t \sigma_s^{(X)} dW_s}_{\text{MARTINGALA}}, \quad t \in [0, T], (**)$$

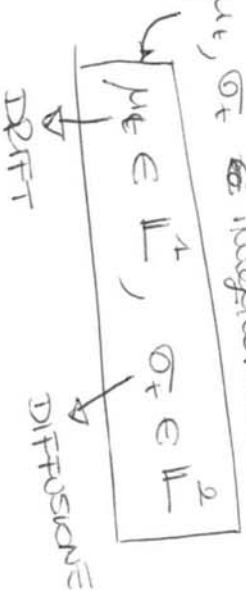
dove $W = \{W_t\}_{t \geq 0}$ è un MB su $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$,

si dice PROX. DI ITO.

NB: In maniera equivalente, (***) o può scrivere in forma differenziale

$$dX_t = \mu_t dt + \sigma_t dW_t \quad (\text{SDE})$$

ovvero μ_t, σ_t integrabili.



(SDE) rappresenta la dinamica del processo X

OSS: Se X è prox. Ito, allora

$$\langle X \rangle_t = \int_0^t \sigma_s^2 ds$$

PROOF: Esercizio.

DOMANDA! Sia W un BM e sia $f \in C^2 \left(\begin{matrix} \mathbb{R} \times [0, T] \\ \mathbb{R} \times [0, T] \end{matrix} \right)$ ~~65~~

\Rightarrow Cosa rappresenta $df(W, t)$? È quanto vale $f(W, t)$?

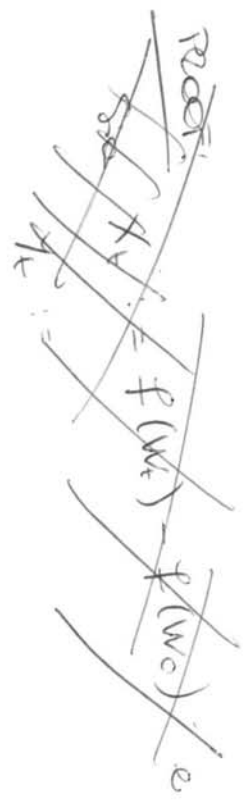
PROF. Sia $f \in C^2 \left(\begin{matrix} \mathbb{R} \times [0, T] \\ \mathbb{R} \times [0, T] \end{matrix} \right)$ con derivata limite e sia $\{W_t\}_{t \geq 0}$ un MB. Allora

$$df(W, t) = f(W, t) + \int_0^t \frac{\partial f(W, s)}{\partial W} dW_s +$$

$$+ \int_0^t \left[\frac{\partial^2 f(W, s)}{\partial W^2} \right] ds \quad (*)$$

TERMINI NUOVI!

\hookrightarrow dipende dal tempo, un $\langle W_t \rangle = t$



In forma differenziale:

$$df(W, t) = \frac{\partial f(W, t)}{\partial W} dW_t + \frac{\partial^2 f(W, t)}{\partial W^2} dt + \frac{1}{2} \frac{\partial^2 f(W, t)}{\partial W^2} dt \quad (**)$$

$$= \left(\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial W^2} \right) dt + \frac{\partial f}{\partial W} dW_t$$

NOTE! le relazioni (*) e (**) sono note in letteratura come BEFOMULE DI ITO PER I MOTI BROWNIANI.

ESEMPIO: $f(x) = x^2 \Rightarrow d(W_t^2) = 2W_t dW_t$ [X] 7

ESEMPIO

1) $f(x) = x^2 \Rightarrow d(W_t^2) = ?$

~~$d(W_t^2) = 2W_t dW_t$~~ $\frac{\partial f}{\partial t} = 0, \frac{\partial f}{\partial x} = 2x, \frac{\partial^2 f}{\partial x^2} = 2$

$\Rightarrow d(W_t^2) = 2W_t dW_t + \frac{1}{2} 2 dt$

$\Rightarrow \int_0^t W_s dW_s = \frac{W_t^2 - t}{2}$

2) Calcolare $\mathbb{E}[W_t^4]$.

$f(x) = x^4 \Rightarrow \frac{\partial f}{\partial t} = 0, \frac{\partial f}{\partial x} = 4x^3, \frac{\partial^2 f}{\partial x^2} = 12x^2$

$\Rightarrow dW_t^4 = 4W_t^3 dW_t + \frac{1}{2} 12W_t^2 dt$

$\Rightarrow W_t^4 = W_0^4 + \int_0^t 4W_s^3 dW_s + \int_0^t 6W_s^2 dt$

$\Rightarrow \mathbb{E}[W_t^4] = \mathbb{E}\left[\int_0^t 4W_s^3 dW_s\right] + \mathbb{E}\left[\int_0^t 6W_s^2 ds\right] = 0 + \int_0^t 6 \mathbb{E}[W_s^2] ds$

3) $X_t = e^{\sigma W_t} \Rightarrow dX_t = ?$ ESERCIZIO

$= 6 \int_0^t ds = 6 \frac{\Delta}{2} \Big|_0^t = 3$