1/3/2019 $\boxed{5}$

DEF. $u \in \mathbb{L}^2$ se

1) u prog. mis. w.r.t. $\mathcal{F}_t$

2) $u \in \underbrace{L^2([0,T] \times \Omega)}_{}$ $\rightsquigarrow$ $\mathbb{E}\left[\int_0^T [u(t)^2]\, dt\right] < \infty$ (Quadrato-integrabile)

↳ CONDIZIONE DI INTEGRABILITÀ

DEF. $u \in \mathbb{L}^2$ è SEMPLICE se

(***) $u_t = \sum_{k=1}^{N} e_k \mathbb{1}_{\{(t_{k-1}, t_k]\}}(t)$, $\forall t \in [0,T]$,

⇒ e_k è $\mathcal{F}_{t_{k-1}}$-misurabile

$0 \leq t_0 < t_1 < \dots < t_N \leq T$ e e_k v. a. su $(\Omega, \mathcal{F}, \mathbb{P})$, $\forall k = 1, \dots, N$

DEF. $\boxed{\text{INTEGRALE DI ITO}}$

$$\Rightarrow \underbrace{\int u_t \, dW_t}_{\text{INTEGRALE DI ITO}} := \sum_{k=1}^{N} e_k \underbrace{(W_{t_k} - W_{t_{k-1}})}_{\text{indipendenti}} \quad (\text{ITO})$$

INTEGRALE DI ITO ⇒ come in integrazione di Riemann (Rettangoloidi)

$$\Rightarrow \int_a^b u_t \, dW_t = \int u_t \mathbb{1}_{\{(a,b]\}}(t)\, dW_t \quad (**)$$

EX: VERIFICARE che $W_t = \int_0^t dW_s$

(Hint: usare $u_t := \mathbb{1}_{\{(0,t]\}}(t)$)

THM: $\forall u, v \in \mathbb{L}^2$ prox semplici, $\forall \alpha, \beta \in \mathbb{R}$, $\forall 0 \leq a < b < c \leq T$

1) $\int (\alpha u_t + \beta v_t)\, dW_t = \alpha \int u_t \, dW_t + \beta \int v_t \, dW_t$ (linearità)

2) $\int_a^b u_t \, dW_t + \int_b^c u_t \, dW_t = \int_a^c u_t \, dW_t$ (additività)

3) $\mathbb{E}\left[\int_a^b u_t \, dW_t \,\middle|\, \mathcal{F}_a\right] = 0$ ⇒ $\mathbb{E}\left[\int_0^T u_t \, dW_t\right] = 0$

4) $\mathbb{E}\left[\int_a^b u_t \, dW_t \int_a^b v_t \, dW_t \,\middle|\, \mathcal{F}_a\right] = \mathbb{E}\left[\int_a^b u_t v_t \, dt \,\middle|\, \mathcal{F}_a\right]$

⇒ $\mathbb{E}\left[\left(\int_a^b u_t \, dW_t\right)^2\right] = \mathbb{E}\left[\int_a^b u_t^2 \, dt \,\middle|\, \mathcal{F}_a\right]$ (ISOMETRIA DI ITO)

PROOF:

1), 2) $\Rightarrow$ VERIFICA PER ESERCIZIO

det. prox. semplice

3) $$\mathbb{E}\left[\int_a^b u_t\, dW_t \,\middle|\, \mathcal{F}_a\right] \overset{\uparrow}{=} \mathbb{E}\left[\sum_{k=1}^{N} e_k \left(W_{t_k}-W_{t_{k-1}}\right) \,\middle|\, \mathcal{F}_a\right]$$

$$= \sum_{k=1}^{N} \mathbb{E}\left[e_k \mid \mathcal{F}_a\right]\cdot \underbrace{\mathbb{E}\left[W_{t_k}-W_{t_{k-1}}\right]}_{=0} = 0$$

e_k è $\mathcal{F}_{t_{k-1}}$-mis, $\forall k$ $\Rightarrow$ e_k è $\mathcal{F}_a$-mis ~~[illegible]~~

$W_{t_k}-W_{t_{k-1}}$ è indip. da $\mathcal{F}_a$ e da e_k, $\forall k$

$\Rightarrow$ PROPRIETÀ MEDIA CONDIZIONATA

4) $$\mathbb{E}\left[\int_a^b u_t\, dW_t \int_a^b v_t\, dW_t \,\middle|\, \mathcal{F}_a\right] \underset{\text{linearità dell'integrale}}{=} \mathbb{E}\left[\sum_{k=1}^{N}\int_{t_{k-1}}^{t_k} u_t\, dW_t \sum_{h=1}^{N}\int_{t_{h-1}}^{t_h} v_t\, dW_t \,\middle|\, \mathcal{F}_a\right]$$

$$= \mathbb{E}\left[\sum_{k=1}^{N}\int_{t_{k-1}}^{t_k} e_k \underbrace{\mathbb{1}_{\{(t_{k-1},t_k]\}}(t)}_{=1}\, dW_t \cdot \sum_{h=1}^{N}\int_{t_{h-1}}^{t_h} d_h \underbrace{\mathbb{1}_{\{(t_{h-1},t_h]\}}(t)}_{=1}\, dW_t \,\middle|\, \mathcal{F}_a\right]$$

det processo semplice:
$u_t = \sum e_k \mathbb{1}_{\{\ldots\}}$

$$= \mathbb{E}\left[\left(\sum_{k=1}^{N}\int_{t_{k-1}}^{t_k} e_k\, dW_t\right)\cdot\left(\sum_{h=1}^{N}\int_{t_{h-1}}^{t_h} d_h\, dW_t\right) \,\middle|\, \mathcal{F}_a\right]$$

NOTA: (Esempio)

$$\sum_{k=1}^{3} a_k \cdot \sum_{h=1}^{3} b_h = (a_1+a_2+a_3)\cdot(b_1+b_2+b_3)$$

$$= a_1b_1 + a_1b_2 + a_1b_3 + a_2b_1 + a_2b_2 + a_2b_3 + a_3b_1 + a_3b_2 + a_3b_3$$

$$= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1b_2 + a_1b_3 + a_2b_3) + (a_2b_1 + a_3b_1 + a_3b_2)$$

(a, b intercambiabili)

$$= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1b_2 + a_1b_3 + a_2b_3) + (a_1b_2 + a_1b_3 + a_2b_3)$$

$$= \sum_{k=1}^{3} a_kb_k + 2\sum_{\substack{h,k=1\\ h<k}}^{3} a_kb_h$$

5c

Allora:

$$\mathbb{E}\left[\int_a^b u_t\,dW_t \int_a^b v_t\,dW_t \,\Big|\, \mathcal{F}_a\right] \overset{(\text{NOTA})}{=}$$

$$= \mathbb{E}\left[\sum_{k=1}^N \int_{t_{k-1}}^{t_k} e_k\,dW_t \cdot \int_{t_{k-1}}^{t_k} d_k\,dW_t \,\Big|\, \mathcal{F}_a\right] +$$

$$\underbrace{+\,2\,\mathbb{E}\left[\sum_{\substack{h,k=1\\h<k}}^N \int_{t_{k-1}}^{t_k} e_k\,dW_t \int_{t_{h-1}}^{t_h} d_h\,dW_t \,\Big|\, \mathcal{F}_a\right]}_{=\,0 \text{ (applicazione punto 3))}}$$

$$\overset{(\text{ITO})}{=} \mathbb{E}\left[\sum_{k=1}^N e_k\left(W_{t_k}-W_{t_{k-1}}\right)\cdot d_k\left(W_{t_k}-W_{t_{k-1}}\right) \,\Big|\, \mathcal{F}_a\right]$$

$$= \sum_{k=1}^N \mathbb{E}\left[e_k d_k \left(W_{t_k}-W_{t_{k-1}}\right)^2 \,\Big|\, \mathcal{F}_a\right]$$

$$= \sum_{k=1}^N \mathbb{E}\left[e_k d_k \,|\, \mathcal{F}_a\right]\cdot \underbrace{\mathbb{E}\left[\left(W_{t_k}-W_{t_{k-1}}\right)^2\right]}_{=(t_k - t_{k-1})}$$

$W_{t_k}-W_{t_{k-1}}$ indipendente da $\mathcal{F}_a$ e da e_k, d_k
e_k, d_k $\mathcal{F}_{t_{k-1}}$-misurabili, $\forall k$ $\Big\} \Rightarrow$ PROPRIETÀ MEDIA CONDIZIONAT.

$$= \sum_{k=1}^N \mathbb{E}\left[e_k d_k \,|\, \mathcal{F}_a\right]\cdot(t_k - t_{k-1})$$

$$= \mathbb{E}\left[\sum_{k=1}^N e_k d_k (t_k - t_{k-1}) \,\Big|\, \mathcal{F}_a\right]$$

Int. di Riemann

$$= \mathbb{E}\left[\int_a^b u_t v_t\,dt \,\Big|\, \mathcal{F}_a\right] \quad \boxed{3}$$

OSS: $u = v \Rightarrow \mathbb{E}\left[\left(\int_a^b u_t\,dW_t\right)^2 \Big|\, \mathcal{F}_a\right] = \mathbb{E}\left[\int_a^b u_t^2\,dt \,\Big|\, \mathcal{F}_a\right]$

(ISOMETRIA DI ITO)

5d

Siamo in grado di estendere la def. di integrale stocastico di processi in $\mathbb{L}^2$

$$I_t(u) := \int_0^t u_s \, dW_s \,, \quad u \in \mathbb{L}^2$$

DEF.

Dato un processo $u \in \mathbb{L}^2$, l'integrale stocastico è

$$I_t(u) = \lim_{n \to +\infty} \int_0^t u_s^{(n)} \, dW_s \,,$$

dove $\{u_s^{(n)}\}$ è una sequenza di processi semplici che approssimano u in $\mathbb{L}^2$.

THM

$\forall \, u, v \in \mathbb{L}^2$, $\alpha \in \mathbb{R}$, $0 \leq a < b < c$, si ha

1) $\displaystyle \int_0^a (\alpha u_t + \beta v_t) \, dW_t = \alpha \int_0^a u_t \, dW_t + \beta \int_0^a v_t \, dW_t$

2) $\displaystyle \int_a^c u_t \, dW_t = \int_a^b u_t \, dW_t + \int_b^c u_t \, dW_t$

3) $\displaystyle \mathbb{E}\left[\int_a^b u_t \, dW_t \,\Big|\, \mathcal{F}_a \right] = 0$

4) $\displaystyle \mathbb{E}\left[\int_a^b u_t \, dW_t \int_a^b v_t \, dW_t \,\Big|\, \mathcal{F}_a \right] = \mathbb{E}\left[\int_a^b u_t v_t \, dt \,\Big|\, \mathcal{F}_a \right]$

$\blacksquare$

PROCESSI DI ITO

<u>DEF.</u> Un PS $X=\{X_t\}_{t\geq 0}$ della forma

$$X_t = \underbrace{X_0 + \int_0^t \overset{\mu(s,X_s)}{\overset{\shortparallel}{\mu_s}}\, ds}_{\text{PROC. A VARIAZIONE LIMITATA}} + \underbrace{\int_0^t \overset{\sigma(s,X_s)}{\overset{\shortparallel}{\sigma_s}}\, dW_s}_{\text{MARTINGALA}}, \quad t\in[0,T], \ (**)$$

dove $W=\{W_t\}_{t\geq 0}$ è un MB su $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$,

si dice PROC. DI ITO.

<u>NB</u>: In maniera equivalente, $(**)$ si può scrivere in forma differenziale

$$dX_t = \mu_t\, dt + \sigma_t\, dW_t \quad (SDE)$$

assumendo μ_t, σ_t ~~e~~ integrabili!

$$\boxed{\underbrace{\mu_t}_{\text{DRIFT}} \in \mathbb{L}^1, \ \underbrace{\sigma_t}_{\text{DIFFUSIONE}} \in \mathbb{L}^2}$$

(SDE) rappresenta la dinamica del processo X

<u>OSS</u>: Se X è proc. Ito, allora

$$\langle X\rangle_t = \int_0^t \sigma_s^2\, ds$$

<u>PROOF</u>: Esercizio.

6b

DOMANDA! Sia W un BM e sia $f \in C^2(\mathbb{R} \times [0,T])$

$\Rightarrow$ Cosa rappresenta $df(W,t)$? E quanto vale $f(W,t)$?

PROP. Sia $f \in C^2(\mathbb{R} \times [0,T])$ con derivate limitate e sia $\{W_t\}_{t \geq 0}$ un MB. Allora

$$f(W_T, T) = f(W_0, 0) + \int_0^T \frac{\partial f(W_s, s)}{\partial W} \, dW_s +$$

$$+ \int_0^T \left\{ \cancel{\frac{\partial f(W_s, s)}{\partial t}} ds + \underbrace{\int_0^T \frac{1}{2} \frac{\partial^2 f(W_s, s)}{\partial W^2} \right\} ds}_{\text{TERMINE NUOVO!}} . \qquad (\bullet)$$

$\hookrightarrow$ dipende dal fatto che $\langle W_t \rangle = t$!

~~PROOF: $X_t := f(W_t) - f(W_0)$ e $Y_t :=$~~

In forma differenziale:

$$df(W,t) = \frac{\partial f(W,t)}{\partial W} dW_t + \cancel{\frac{\partial f(W,t)}{\partial t}} dt + \frac{1}{2} \frac{\partial^2 f(W,t)}{\partial W^2} dt \qquad (\bullet\bullet)$$

$$= \left(\cancel{\frac{\partial f}{\partial t}} + \frac{1}{2} \frac{\partial^2 f}{\partial W^2} \right) dt + \frac{\partial f}{\partial W} dW_t$$

NOTA! Le relazioni $(\bullet)$ e $(\bullet\bullet)$ sono note in letteratura come FORMULE DI ITO PER I MOTI BROWNIANI.

~~ESEMPIO: $f(x) = x^2 \Rightarrow d(W^2) = dt + 2W_t\, dW_t$~~

ESEMPI

1) $f(x) = x^2 \Rightarrow d(W_t^2) = ?$

$\cancel{\Rightarrow d(W_t^2)}$ $\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial x} = 2x, \quad \frac{\partial^2 f}{\partial x^2} = 2$

$$\Rightarrow d(W_t^2) = 2W_t\, dW_t + \frac{1}{\cancel{2}}\cancel{2}\, dt$$

$$\Rightarrow \int_0^t W_s\, dW_s = \frac{W_t^2 - t}{2}$$

2) Calcolare $\mathbb{E}[W_t^4]$.

$f(x) = x^4 \Rightarrow \frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial x} = 4x^3, \quad \frac{\partial^2 f}{\partial x^2} = 12x^2$

$$\Rightarrow dW_t^4 = 4W_t^3\, dW_t + \frac{1}{\cancel{2}}\cancel{12}^{6}\, W_t^2\, dt$$

$$\Rightarrow W_t^4 = \underbrace{W_0}_{0} + \int_0^t 4W_s^3\, dW_s + \int_0^t 6W_t^2\, dt$$

$$\Rightarrow \mathbb{E}[W_t^4] = \mathbb{E}\left[\int_0^t 4W_s^3\, dW_s\right] + \mathbb{E}\left[\int_0^t 6W_s^2\, ds\right] = \cancel{3t}$$

$$= 0 + \int_0^t 6\,\mathbb{E}[W_s^2]\, ds$$

$$= 6\int_0^t s\, ds = 6\,\frac{s^2}{2}\Big|_0^t = 3$$

3) $X_t = e^{\sigma W_t} \Rightarrow dX_t = ?$

ESERCIZIO