PROOF:

Sia $\pi = \{t_0 = 0, t_1, \dots, t_{N-1}, t_N = t\}$ una partizione di $[0,t]$.

Definiamo

$$f_k := f(W_{t_k}), \qquad \Delta_k := W_{t_k} - W_{t_{k-1}}, \qquad \forall k = 0, \dots, N$$

$$\Rightarrow f(W_t) - f(W_0) = f_N - f_0 = (f_1 - f_0) + (f_2 - f_1) + \dots + \dots + (f_N - f_{N-1}) = \sum_{k=1}^{N} (f_k - f_{k-1})$$

$$= \sum_{k=1}^{N} f'_{k-1} \Delta_k + \frac{1}{2} \sum_{k=1}^{N} f''_{k-1} \Delta_k^2 + \frac{1}{2} \sum_{k=1}^{N} \left[f''(W_{t_k^*}) - f''_{k-1} \right] \Delta_k^2,$$

sviluppo in serie di Taylor con resto di Lagrange

dove $t_k^* \in [t_{k-1}, t_k]$

$$=: I_1^{(\pi)} + I_2^{(\pi)} + I_3(\pi).$$

OSS:

$$\lim_{|\pi| \to 0^+} I_1(\pi) = \int_0^t f'(W_s)\, dW_s \quad \text{in } L^2$$

→ in quanto f' limitata per ipotesi e sfruttando la def. di integrale di Itô

$$\lim_{|\pi| \to 0^+} I_2(\pi) = \int_0^t f''(W_s)\, ds \quad \text{in } L^2$$

→ come nella dim. della variazione quadratica di W, usando il fatto che $\langle W_s \rangle = s$ (integrazione "deterministica")

Resta da provare che

$$\lim_{|\pi| \to 0^+} I_3(\pi) = 0 \quad \text{in } L^2 \longrightarrow \lim_{|\pi| \to 0} \mathbb{E}\left[(I_3(\pi) - 0)^2 \right] = 0$$

(convergenza in media quadratica)

$|+c|$

Allora:

$$\mathbb{E}\left[\left(I_3(\pi)\right)^2\right] = \mathbb{E}\left[\left(\sum_{k=1}^{N}\left(f''(W_{t_k^*}) - f''_{k-1}\right)\Delta_k^2\right)^2\right]$$

$$= \mathbb{E}\left[\sum_{k=1}^{N}\left(f''(W_{t_k^*}) - f''_{k-1}\right)^2\Delta_k^4\right] +$$

NOTA pag. 6b (proprietà integrale di Itô)

$$+ \cancel{2\,\mathbb{E}}\left[\sum_{\substack{h,k=1\\ h<k}}^{N}\left(f''(W_{t_k^*}) - f''_{k-1}\right)\Delta_k^2\cdot\left(f''(W_{t_h^*}) - f''_{h-1}\right)\Delta_h^2\right]$$

$$= \sum_{k=1}^{N}\mathbb{E}\left[\left(f''(W_{t_k^*}) - f''_{k-1}\right)^2\Delta_k^4\right] +$$

$$+ 2\sum_{\substack{h,k=1\\ h<k}}^{N}\mathbb{E}\left[\left(f''(W_{t_k^*}) - f''_{k-1}\right)\left(f(W_{t_h^*}) - f''_{h-1}\right)\Delta_k^2\Delta_h^2\right]$$

$$=: L_1(\pi) + 2\cdot L_2(\pi).$$

$f \in C_b^2$, $\mathbb{E}[\Delta_k^4] = 3\,(t_k - t_{k-1})^2$

Per quel che riguarda $L_1(\pi)$: $\left|f''(W_{t_k^*}) - f''_{k-1}\right|^2 \leq c_1 \sup|f''|^2$, c_1 costante $\forall\, t_k^* \in (t_{k-1}, t_k]$,

$$\Rightarrow L_1(\pi) \leq \sum_{k=1}^{N}\mathbb{E}\left[c_1\sup|f''|^2\cdot\Delta_k^4\right] = c_1\sup|f''|^2\sum_{k=1}^{N}\mathbb{E}\left[\Delta_k^4\right]$$

$$= 3\cdot c_1\cdot\underbrace{\sup|f''|^2}_{<\infty,\ \text{poiché } f\in C_b^2 \text{ (derivate limitate)}}\cdot\sum_{k=1}^{N}(t_k - t_{k-1})^2 \leq \bar{c}\cdot\sum_{k=1}^{N}(t_k - t_{k-1})^2$$

$\bar{c} = c_1 c_2$

$$= \bar{c}\sum_{k=1}^{N}(t_k - t_{k-1})\underbrace{(t_k - t_{k-1})}_{\leq|\pi|} \leq \bar{c}\,|\pi|\sum_{k=1}^{N}(t_k - t_{k-1})$$

$$= \bar{c}\cdot|\pi|\cdot t \xrightarrow{|\pi|\to 0} 0$$

$$\Rightarrow L_1(\pi) \xrightarrow{|\pi|\to 0} 0.$$

Per quel che riguarda $L_2(\pi)$:

Ricordiamo la <u>DISUGUAGLIANZA DI HÖLDER</u>

$\forall\, p,q>0$ t.c. $\frac{1}{p}+\frac{1}{q}=1$, si ha

$$\mathbb{E}\left[|X\cdot Y|\right] \leq \left(\mathbb{E}\left[|X|^p\right]\right)^{\frac{1}{p}} \cdot \left(\mathbb{E}\left[|Y|^q\right]\right)^{\frac{1}{q}} \quad (DH)$$

$$\Rightarrow L_2(\pi) = \sum_{\substack{h,k=1\\ h<k}}^{N} \mathbb{E}\Big[\underbrace{\left(f''(W_{t_k^*})-f''_{k-1}\right)\left(f''(W_{t_h^*})-f''_{h-1}\right)}_{|X|} \underbrace{\Delta_k^2\Delta_h^2}_{|Y|}\Big]$$

$$\overset{(DH)\,(p=q=2)}{\leq} \sum_{\substack{h,k=1\\ h<k}}^{N} \underbrace{\left(\mathbb{E}\left[\left(f''(W_{t_k^*})-f''_{k-1}\right)^2\left(f''(W_{t_h^*})-f''_{h-1}\right)^2\right]\right)^{\frac{1}{p}}}_{f\in C_b^2 \Rightarrow \exists \text{ una costante che maggiora tutto } A=A(\pi)} \left(\mathbb{E}\left[\Delta_k^4\Delta_h^4\right]\right)^{\frac{1}{2}}$$

$$\leq A\cdot \sum_{\substack{k,h=1\\ h<k}}^{N}\left(\mathbb{E}\left[\Delta_h^4\Delta_k^4\right]\right)^{1/2} \leq A\cdot \sum_{\substack{h,k=1\\ h<k}}^{N}\left(\mathbb{E}\left[\Delta_h^4\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\Delta_k^4\right]\right)^{\frac{1}{2}}$$

$$= A\cdot \sum_{\substack{h,k=1\\ h<k}}^{N}\left(3\,(t_k-t_{k-1})^2\right)^{1/2}\left(3\,(t_h-t_{h-1})^2\right)^{1/2}$$

$$= 3\cdot A\cdot \sum_{\substack{h,k=1\\ h<k}}^{N}\underbrace{(t_k-t_{k-1})}_{\leq|\pi|}\underbrace{(t_h-t_{h-1})}_{|\pi|}$$

~~Quindi~~, <u>OSS</u>

$$t^2=\left[\sum_{k=1}^{N}(t_k-t_{k-1})\right]^2 = \underline{\sum_{k=1}^{N}(t_k-t_{k-1})^2} + 2\sum_{\substack{h,k=1\\ h<k}}^{N}(t_k-t_{k-1})(t_h-t_{h-1})$$

$$\Rightarrow \sum_{\substack{h,k=1\\ h<k}}^{N}(t_k-t_{k-1})(t_h-t_{h-1}) \leq \frac{t^2}{2}-\frac{t}{2} = \frac{t}{2}(t-1) \leq t \;\; |\pi|$$

??

~~tale~~

Quindi:

$$L_2(\pi) \leq 3 \cdot A|\pi|^{k} \xrightarrow{(|\pi| \to 0)} 0$$

$$\Rightarrow I_3(\pi) \xrightarrow{|\pi| \to 0} 0 \text{ in } L^2 \quad \blacksquare$$

DEF. X, Y processi continui, quadrato-integrabili. (martingale) $\boxed{**}$

$$\langle X, Y\rangle := \frac{1}{2}\left[\langle X+Y\rangle - \langle X-Y\rangle\right]$$

↳ COVARIAZIONE

PROPRIETÀ:

$\forall X, Y, Z$, si ha

1) $\langle X, Y\rangle = \langle Y, X\rangle$

2) $\langle \lambda X + \mu Y, Z\rangle = \lambda \langle X, Z\rangle + \mu \langle Y, Z\rangle, \ \forall \lambda, \mu \in \mathbb{R}$

3) $\left|\langle X, Y\rangle^2\right| \leq \langle X\rangle \cdot \langle Y\rangle$

LEMMA DI ITO ~~generalizzato~~ (VERSIONE MULTIVARIATA)

Sia X un processo di Itô ~~generalizzato~~ N-dimensionale:

$$dX_t^{(j)} = \mu_t^{(j)} dt + \sum_{k=1}^{d} \sigma_t^{j,k} dW_t^{(k)}, \ \forall j = 1, \dots, N$$

Sia $F = F(t, x) \in C^{1,2}(\mathbb{R} \times \mathbb{R}^N)$. Allora,

prodotto scalare

$$dF = \frac{\partial F}{\partial t} dt + \nabla F \cdot dX_t + \frac{1}{2} \sum_{i,j=1}^{N} \frac{\partial^2 F}{\partial X_t^{(i)} \partial X_t^{(j)}} d\langle X_t^{(i)}, X_t^{(j)}\rangle,$$

dove $\nabla F := \left(\frac{\partial F}{\partial X_t^{(1)}}, \dots, \frac{\partial F}{\partial X_t^{(N)}}\right)$

e $\nabla F \cdot dX_t = \sum_{i=1}^{N} \frac{\partial F}{\partial X_t^{(i)}} \cdot dX_t^{(i)}$

PROP (Lemma di Itô – versione generale)

Sia $\{X_t\}_{t\geq 0}$ un processo di Itô e sia $f = f(t,x) \in C^{1,2}(\mathbb{R}\times\mathbb{R})$. Allora, $Y_t := f(t, X_t)$ è un processo di Itô e si ha

$$dY_t = \frac{\partial f(t,X_t)}{\partial t}\,dt + \frac{\partial f(t,X_t)}{\partial X_t}\,dX_t + \frac{1}{2}\frac{\partial^2 f(t,X_t)}{\partial X_t^2}\langle dX_t, dX_t\rangle .$$

PROOF: Analoga al caso dei Moti Browniani ∎

NB: $dX_t = \mu(t,X_t)\,dt + \sigma(t,X_t)\,dW_t$

$$\Rightarrow dY_t = \frac{\partial f}{\partial t}\,dt + \frac{\partial f}{\partial x}\Big(\mu(t,X_t)\,dt + \sigma(t,X_t)\,dW_t\Big) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2(t,X_t)\,dt$$

$$= \left[\frac{\partial f}{\partial t} + \mu(t,X_t)\frac{\partial f}{\partial x} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2(t,X_t)\right]dt + \sigma(t,X_t)\frac{\partial f}{\partial x}\,dW$$

ESEMPIO: MOTO BROWNIANO GEOMETRICO.

$$\frac{dS_t}{S_t} = \mu\,dt + \sigma\,dW_t, \quad \mu\in\mathbb{R},\ \sigma\in\mathbb{R}^+$$

$\Rightarrow$ È possibile determinare una espressione esplicita di S_t? $\rightarrow$ Sì! (Usiamo Itô's Lemma)

|+4|

$$Y_t = f(t, S_t) = \ln(S_t)$$

$$\Rightarrow \frac{\partial f}{\partial t} = 0; \quad \frac{\partial f}{\partial S_t} = \frac{1}{S_t}; \quad \frac{\partial^2 f}{\partial S_t^2} = -\frac{1}{S_t^2}$$

$$\overset{(ITO)}{\Longrightarrow} dY_t = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S} dS_t + \frac{1}{2} \frac{\partial^2 f}{\partial S_t^2} \langle dS_t, dS_t \rangle$$

$$= \frac{1}{S_t}\left[\mu S_t dt + \sigma S_t dW_t\right] +$$

$$+ \frac{1}{2}\left(-\frac{1}{S_t^2}\right)\langle \mu S_t dt + \sigma S_t dW_t, \mu S_t dt + \sigma S_t dW_t \rangle$$

$$= \mu dt + \sigma dW_t - \frac{1}{2} \frac{1}{\cancel{S_t^2}} \sigma^2 \cancel{S_t^2} dt$$

$$\Rightarrow d\ln(S_t) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW_t$$

$$\Rightarrow \ln(S_t) - \ln(S_0) = \int_0^t \left(\mu - \frac{\sigma^2}{2}\right) ds + \int_0^t \sigma dW_s$$

$$\Rightarrow \ln(S_t) = \ln(S_0) + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma \underbrace{\int_0^t dW_s}_{= W_t}$$

VERIFICA PER ESERCIZIO!

$$\Rightarrow S_t = S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right\}$$