2) MODELLO CIR

$$dr_t = a(\gamma - r_t)dt + \sigma\sqrt{r_t}\,dW_t \quad (\text{sotto } \mathbb{Q} \sim \mathbb{P}),$$

con $a, \gamma, \sigma \in \mathbb{R}^+$.

Si usa la stessa tecnica del modello di Vasicek:

$$dr_t = a\gamma dt - a r_t dt + \sigma\sqrt{r_t}\,dW_t$$

$$\Rightarrow dr_t + a r_t dt = a\gamma dt + \sigma\sqrt{r_t}\,dW_t$$

$$\Rightarrow e^{at}\left(dr_t + a r_t dt\right) = e^{at} a\gamma dt + e^{at}\sigma\sqrt{r_t}\,dW_t$$

$$\Rightarrow e^{at} dr_t + a e^{at} r_t dt = e^{at} a\gamma dt + e^{at}\sigma\sqrt{r_t}\,dW_t$$

$$\Rightarrow d\left(r_t e^{at}\right) = e^{at} a\gamma dt + e^{at}\sigma\sqrt{r_t}\,dW_t$$

$$\Rightarrow \int_t^s d\left(r_u e^{au}\right) = \int_t^s e^{au} a\gamma du + \int_t^s e^{au}\sigma\sqrt{r_u}\,dW_u$$

$$\Rightarrow r_s e^{as} - r_t e^{at} = \gamma\left[e^{a(s\cancel{-t})} - e^{at}\right] + \sigma\int_t^s e^{au}\sqrt{r_u}\,dW_u$$

$$\Rightarrow r_s = r_t e^{a(t-s)} + \gamma e^{-as}\left[e^{as} - e^{at}\right] + \sigma e^{-as}\int_t^s e^{au}\sqrt{r_u}\,dW_u$$

$$\Rightarrow r_s = r_t e^{-a(s-t)} + \gamma\left[1 - e^{-a(s-t)}\right] + \sigma\int_t^s e^{-a(s-u)}\sqrt{r_u}\,dW_u$$

OSS: la variabile compare anche al secondo membro

$\Rightarrow$ non possiamo ragionare come in Vasicek!

$\Rightarrow$ che distribuzione ha il tasso r_t?

$r_t \sim \chi^2(k, \lambda)$ chi-squared non centrato

Calcoliamone i momenti:

$$\mathbb{E}[r_s] = \mathbb{E}\left[r_t e^{-a(s-t)} + \gamma\left[1 - e^{-a(s-t)}\right] + \sigma \int_t^s e^{-a(s-u)} \sqrt{r_u}\, dW_u\right]$$

$$= \mathbb{E}\left[r_t e^{-a(s-t)} + \gamma\left[1 - e^{-a(s-t)}\right]\right] + \underbrace{\mathbb{E}\left[\sigma \int_t^s e^{-a(s-u)} \sqrt{r_u}\, dW_u\right]}_{=0}$$

In generale: se f è adattato, è misurabile e quadrato-integrabile
$\Rightarrow \mathbb{E}\left[\int_t^T f(\omega,t)\, dW_t\right] = 0.$

$$= r_t e^{-a(s-t)} + \gamma\left(1 - e^{-a(s-t)}\right)$$

$$= e^{-a(s-t)}\left[r_t - \gamma\right] + \gamma$$

$$\mathbb{E}[r_s^2] = \mathbb{E}\left[\left(r_t e^{-a(s-t)} + \gamma\left(1 - e^{-a(s-t)}\right) + \sigma \int_t^s e^{-a(s-u)} \sqrt{r_u}\, dW_u\right)^2\right]$$

$$= \mathbb{E}\left[\left(r_t e^{-a(s-t)} + \gamma\left(1 - e^{-a(s-t)}\right)\right)^2\right] + \sigma^2\, \mathbb{E}\left[\left(\int_t^s e^{-a(s-u)} \sqrt{r_u}\, dW_u\right)^2\right]$$

$$+ 2\sigma \underbrace{\mathbb{E}\left[\left\{r_t e^{-a(s-t)} + \gamma\left(1 - e^{-a(s-t)}\right)\right\}\left(\int_t^s e^{-a(s-u)} \sqrt{r_u}\, dW_u\right)\right]}_{=0}$$

(ISOMETRIA DI ITO)

$$= \cancel{\mathbb{E}}\left\{e^{-a(s-t)}\left(r_t - \gamma\right) + \gamma\right\}^2 + \sigma^2\, \mathbb{E}\left[\int_t^s e^{-2a(s-u)} r_u\, du\right]$$

$$\Rightarrow \mathrm{Var}[r_s] = \mathbb{E}[r_s^2] - \left(\mathbb{E}[r_s]\right)^2 = \cancel{\left[e^{-a(s-t)}(r_t - \gamma) + \gamma\right]^2} + \sigma^2\, \mathbb{E}\left[\int_t^s e^{-2a(s-u)} r_u\, du\right] +$$

$$- \cancel{\left[e^{-a(s-t)}(r_t - \gamma) + \gamma\right]^2} =$$

$$\overset{(FLTC)}{=} \sigma^2 \mathbb{E}\left[\int_t^s e^{-2a(s-u)} r_u \, du\right] = \sigma^2 \int_t^s e^{-2a(s-u)} \mathbb{E}[r_u] \, du$$

$$\overset{(MEDIA)}{=} \sigma^2 \int_t^s e^{-2a(s-u)} \left[r + e^{-a(u-t)} (r_t - r) \right] du$$

$$= \sigma^2 \int_t^s \left[r e^{-2a(s-u)} + (r_t - r) e^{-2a(s-u) - a(u-t)} \right] du$$

$$= \sigma^2 r \int_t^s e^{-2a(s-u)} \, du + \sigma^2 (r_t - r) \int_t^s e^{-2as + 2au - au + at} \, du$$

$$= \sigma^2 r \int_t^s e^{-2a(s-u)} \, du + \sigma^2 (r_t - r) \int_t^s e^{-2as + au + at} \, du$$

$$= \sigma^2 r \int_t^s e^{-2a(s-u)} \, du + \sigma^2 (r_t - r) e^{-2as + at} \int_t^s e^{au} \, du$$

$$=: I_1 + I_2$$

Per I_1 abbiamo:

$s - u = y \Rightarrow u = s - y \Rightarrow du = -dy$

$u = t \Rightarrow y = s - t \,; \quad u = s \Rightarrow y = 0$

$$\Rightarrow I_1 = \sigma^2 r \int_{s-t}^{0} - e^{-2ay} \, dy = \frac{\sigma^2 r}{-2a} \int_0^{s-t} (-2a) e^{-2ay} \, dy$$

$$= -\frac{\sigma^2 r}{2a} \left(e^{-2a(s-t)} - 1 \right) = \frac{\sigma^2 r}{2a} \left(1 - e^{-2a(s-t)} \right)$$

Per I_2 abbiamo:

$$I_2 = \sigma^2 (r_t - \gamma) \frac{e^{-2as+at}}{a} \int_t^s a e^{au} du$$

$$= \sigma^2 (r_t - \gamma) \frac{e^{-2as+at}}{a} \left[e^{as} - e^{at} \right]$$

$$= \frac{\sigma^2 (r_t - \gamma)}{a} \left[e^{-2as+at+as} - e^{-2as+at+at} \right]$$

$$= \frac{\sigma^2 (r_t - \gamma)}{a} \left[e^{-a(s-t)} - e^{-2a(s-t)} \right]$$

$$\Rightarrow Var[r_s] = \frac{\sigma^2 \gamma}{2a} \left(1 - e^{-2a(s-t)}\right) + \frac{\sigma^2}{a} (r_t - \gamma) \left(e^{-a(s-t)} - e^{-2a(s-t)} \right)$$

$$= \frac{\sigma^2 \gamma}{2a} \left(1 - e^{-2a(s-t)}\right) + \frac{\sigma^2}{a} r_t \left(e^{-a(s-t)} - e^{-2a(s-t)} \right) +$$
$$- \frac{2\sigma^2 \gamma}{2a} \left(e^{-a(s-t)} - e^{-2a(s-t)} \right)$$

$$= \frac{\sigma^2}{2a} \left[\gamma \left(1 - e^{-2a(s-t)} - 2 e^{-a(s-t)} + 2 e^{-2a(s-t)} \right) \right] + \frac{\sigma^2}{a} r_t \left(e^{-a(s-t)} - e^{-2a(s-t)} \right)$$

$$= \frac{\sigma^2}{2a} \left[\gamma \left(1 - 2 e^{-a(s-t)} + e^{-2a(s-t)} \right) \right] + \frac{\sigma^2}{a} r_t \left(e^{-a(s-t)} - e^{-2a(s-t)} \right)$$

$$= \frac{\sigma^2}{2a} \left[\gamma \left(1 - e^{-a(s-t)} \right)^2 + 2 r_t \left(e^{-a(s-t)} - e^{-2a(s-t)} \right) \right]$$

<u>NOTA</u>: la varianza non condizionata sarà:

$$\lim_{s \to \infty} Var[r_s] = \lim_{s \to \infty} \left[\frac{\sigma^2}{2a} \left(\gamma \left(1 - e^{-a(s-t)}\right)^2 + 2 r_t \left(e^{-a(s-t)} - e^{-2a(s-t)} \right) \right) \right]$$

$$= \frac{\sigma^2 \gamma}{2a}$$

DOMANDA: Quanto vale il prezzo di ZCB in questo modello?

17e

$$p(t,T) = F^T(t, r_t) \quad t.c.$$

$$\begin{cases} \dfrac{\partial F^T}{\partial t} + a(\gamma - r_t)\dfrac{\partial F^T}{\partial r} + \dfrac{1}{2}\sigma^2 r_t \dfrac{\partial^2 F^T}{\partial r^2} - r F^T = 0 \\ F(T, r_T) = 1 \end{cases}$$

(PROP. in HO-LEE) $\Rightarrow$ $p(t,T) = \exp\{A(t,T) - B(t,T)\, r\}$,

con $\alpha_t = -a$, $\beta_t = a\gamma$, $\gamma_t = \sigma^2$, $\delta_t = 0$

$$\Rightarrow \begin{cases} \dot{A}(t,T) - a\gamma B(t,T) = 0, & A(T,T) = 0 \\ \dot{B}(t,T) - aB(t,T) - \frac{1}{2}\sigma^2 (B(t,T))^2 + 1 = 0, & B(T,T) = 0 \end{cases}$$

↳ ODE di Riccati (Eq. diff. ordinaria, del primo ordine, quadratica)

$$\tau := T - t \Rightarrow \dot{B}(\tau) = \frac{1}{2}\sigma^2 B^2(\tau) + aB(\tau) - 1, \quad B(0) = 0$$

$$\Rightarrow \frac{\sigma^2}{2} B^2 - \dot{B} = -aB + 1$$

GUESS AND VERIFY APPROACH: $B(\tau) = -\dfrac{2}{\sigma^2}\dfrac{y'(\tau)}{y(\tau)}$, $y(\tau) \neq 0$ (*)

$$\Rightarrow \dot{B}(\tau) = -\frac{2}{\sigma^2}\frac{y''\cdot y - (y')^2}{y^2} = -\frac{2}{\sigma^2}\frac{y''}{y} + \frac{2}{\sigma^2}\frac{(y')^2}{y^2}$$

$$\overset{(*)}{\Rightarrow} -\frac{2}{\sigma^2}\frac{y''}{y} + \frac{2}{\sigma^2}\frac{(y')^2}{y^2} = \frac{\sigma^2}{2}\left(-\frac{2}{\sigma^2}\frac{y'}{y}\right)^2 + a\left(-\frac{2}{\sigma^2}\frac{y'}{y}\right) - 1$$

$$\Rightarrow -\frac{2}{\sigma^2}\frac{y''}{y} + \frac{2}{\sigma^2}\frac{(y')^2}{y^2} = \frac{\cancel{\sigma^2}}{\cancel{2}}\,\frac{\cancel{4}\,2}{\cancel{\sigma^4}\,\sigma^2}\frac{(y')^2}{y^2} - \frac{2a}{\sigma^2}\frac{y'}{y} - 1$$

$$\Rightarrow -\frac{2}{\sigma^2}\frac{y''}{y} + \cancel{\frac{2}{\sigma^2}\frac{(y')^2}{y^2}} = \cancel{\frac{2}{\sigma^2}\frac{(y')^2}{y^2}} - \frac{2a}{\sigma^2}\frac{y'}{y} - 1$$

$$\Rightarrow -\frac{2}{\sigma^2}\frac{y''}{y} + \frac{2a}{\sigma^2}\frac{y'}{y} + 1 = 0$$

NB: $y(t) = \exp\left\{\dfrac{\sigma^2}{2}\displaystyle\int_t^T B(s,T)\,ds\right\}$

$$\Rightarrow B(s,T) = -\frac{2}{\sigma^2}\frac{y'(t)}{y(t)}$$

$$\Rightarrow B(\tau) = -\frac{2}{\sigma^2}\,\frac{x_2 \cancel{c_2}\left[e^{\dots} + e^{\dots}\right]}{\cancel{c_2}\left[-\frac{x_2}{x_1} e^{x_1\tau} + e^{x_2\tau}\right]} = -\frac{2}{\sigma^2}\,\frac{x_2\left(e^{\dots} - e^{\dots}\right)\dots}{\frac{e^{x_2\tau}x_1 - x_2 e^{x_1\tau}}{x_1}}$$

$$= -\frac{2}{\sigma^2}\,\frac{x_1 x_2\left(e^{x_2\tau} - e^{x_1\tau}\right)}{x_1 e^{x_2\tau} - x_2 e^{x_1\tau}} = -\frac{2}{\sigma^2}\,\frac{x_1 x_2 \cancel{e^{x_1\tau}}\left(e^{(x_2-x_1)\tau} - 1\right)}{\cancel{e^{x_1\tau}}\left(x_1 e^{(x_2-x_1)\tau} - x_2\right)}$$

$$\Rightarrow B(\tau) = -\frac{2}{\sigma^2}\,\frac{x_1 x_2\left(e^{(x_2-x_1)\tau} - 1\right)}{\left(x_1 e^{(x_2-x_1)\tau} - x_2\right)}$$

D'altra parte:

$$\Rightarrow x_1 \cdot x_2 = \frac{(a+b)(a-b)}{4} = \frac{1}{4}\left(a^2 - b^2\right)$$

$$x_2 - x_1 = $$

$$\Rightarrow B(\tau) = -\frac{2}{\sigma^2}\,\frac{\frac{1}{4}\left(a^2-b^2\right)\left(e^{b\tau}-1\right)}{\frac{(a+b)}{2}e^{b\tau} - \frac{(a-b)}{2}} = -\frac{2}{2\sigma^2}\,\frac{\frac{1}{2}\left(a^2-b^2\right)\left(e^{b\tau}-1\right)}{\frac{(a+b)e^{b\tau} - (a-b)}{2}}$$

$$= -\frac{2}{2\sigma^2}\,\frac{-\frac{\left(b^2-a^2\right)}{\cancel{2}}\left(e^{b\tau}-1\right)}{\frac{(a+b)e^{b\tau} - (a+b-b-b)}{\cancel{2}}}$$

$$= +\frac{2}{\cancel{2\sigma^2}}\,\frac{\cancel{+2\sigma^2}\left(e^{b\tau}-1\right)}{(a+b)e^{b\tau} - (a+b) + 2b}$$

$$\Rightarrow B(t,T) = \frac{2\left(e^{b(T-t)} - 1\right)}{2b + (a+b)\left(e^{b(T-t)} - 1\right)}$$

$$\Rightarrow A(t,T) = -\int_t^T a\gamma\, B(s,T)\,ds = -\int_t^T a\gamma\,\frac{2\left[e^{b(T-s)} - 1\right]}{2b + (a+b)\left(e^{b(T-s)} - 1\right)}\,ds$$

$$= -\frac{2a\gamma}{\sigma^2}\ln\left[\frac{2b\, e^{\frac{a+b}{2}(T-t)}}{2b + (a+b)\left(e^{b(T-t)} - 1\right)}\right]$$

↓

$\boxed{\text{ESERCIZIO}}$

$$\Rightarrow -2y'' + 2ay' + \sigma^2 y = 0$$

$$\Rightarrow \underbrace{2y'' - 2ay' - \sigma^2 y = 0,}_{\text{ODE del secondo ordine, lineare}} \quad \text{(ODE-L)}$$

$\Rightarrow y = c_1 y_1 + c_2 y_2$, dove y_1, y_2 sono sol. particolari della eq. diff.

$\Rightarrow$ In particolare: $y_i = e^{x_i \tau}$, $\forall\, i = 1, 2$

$\Rightarrow y = c_1 e^{x_1\tau} + c_2 e^{x_2\tau}$

$$\Rightarrow y' = x e^{x\tau} = xy; \quad y'' = x^2 e^{x\tau} = x^2 y$$

(ODE-L)

$$\Rightarrow 2x^2 \cancel{y} - 2ax\cancel{y} - \sigma^2 \cancel{y} = 0$$

$$\Rightarrow \underbrace{2x^2 - 2ax - \sigma^2 = 0}_{\text{Eq. caratteristica}}$$

$$\Rightarrow x = \frac{a \pm \sqrt{a^2 + 2\sigma^2}}{2} = \frac{a \pm h}{2},$$

avendo posto $h := \sqrt{a^2 + 2\sigma^2}$

$$\Rightarrow h^2 = a^2 + 2\sigma^2$$
$$\Rightarrow 2\sigma^2 = h^2 - a^2$$

(*)

$$\Rightarrow y = c_1 e^{x_1\tau} + c_2 e^{x_2\tau}, \quad y' = c_1 x_1 e^{x_1\tau} + c_2 x_2 e^{x_2\tau}$$

NB: c_1, c_2 costanti da determinare con le cond. al contorno

$$\Rightarrow B(\tau) = -\frac{2}{\sigma^2} \frac{c_1 x_1 e^{x_1\tau} + c_2 x_2 e^{x_2\tau}}{c_1 e^{x_1\tau} + c_2 e^{x_2\tau}} \quad (**)$$

$$B(0) = 0 \Rightarrow -\frac{2}{\sigma^2} \frac{c_1 x_1 e^{x_1 0} + c_2 x_2 e^{x_2 0}}{c_1 e^{x_1 0} + c_2 e^{x_2 0}} = 0$$

$$\Rightarrow c_1 x_1 \cancel{e^{x_1 0}} + c_2 x_2 \cancel{e^{x_2 0}} = 0 \Rightarrow c_1 = -\frac{c_2 x_2}{x_1}$$

(**)

$$\Rightarrow B(\tau) = -\frac{2}{\sigma^2} \frac{-\frac{c_2 x_2}{\cancel{x_1}} e^{x_1\tau} \cancel{x_1} + c_2 x_2 e^{x_2\tau}}{-\frac{c_2 x_2}{x_1} e^{x_1\tau} + c_2 e^{x_2\tau}}$$