2) MODELLO DI HESTON

$$\begin{cases} \dfrac{dS_t}{S_t} = \mu\, dt + \sqrt{v_t}\, dW_t^{(1)} & (\text{under } \mathbb{P}) \\ dv_t = k(\theta - v_t)\, dt + \sigma \sqrt{v_t}\, dW_t^{(2)} \\ \langle dW_t^{(1)}, dW_t^{(2)} \rangle = \rho\, dt \end{cases}$$

COEFF:

- μ = drift del processo per lo stock
- k = mean-reversion speed per la varianza
- θ = mean-reversion level per la varianza
- σ = vol-of-vol
- v_0 = varianza istantanea iniziale
- ρ = coeff. di correlazione tra MB

OSS: Non si modellizza la volatilità, ma la varianza

$\Rightarrow h_t := \sqrt{v_t}$

$\Rightarrow dh_t = ?$ CALCOLARE PER ESERCIZIO

Cosa succede sotto $\mathbb{Q}$!

(GIRSANOV)

$$\Rightarrow \begin{cases} \widetilde{W}_t^1 = W_t^1 + \dfrac{\mu - r}{\sqrt{v_t}}\, t \\[2ex] \widetilde{W}_t^2 = W_t^2 + \dfrac{\lambda(t,S,v)}{\sigma\sqrt{v_t}}\, t, \quad \text{con } \underline{\lambda(t,S,v) := \lambda v} \end{cases}$$

↳ ASSUNZIONE FATTA DA HESTON!

$$\mu\, dt + \sqrt{v_t}\left(d\widetilde{W}_t^{(1)} - \frac{\mu - r}{\sqrt{v_t}}\, dt\right)$$

$$\Rightarrow \begin{cases} \dfrac{dS_t}{S_t} = r\, dt + \sqrt{v_t}\, d\widetilde{W}_t^{(1)} \, * \\[2ex] dv_t = (k\theta - k v_t)\, dt + \sigma\sqrt{v_t}\left(-\dfrac{\lambda v_t}{\sigma\sqrt{v_t}}\, dt + d\widetilde{W}_t^2\right) \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{dS_t}{S_t} = r\, dt + \sqrt{v_t}\, d\widetilde{W}_t^{(1)} \\[2ex] dv_t = (k\theta - k v_t - \lambda v_t)\, dt + \sigma\sqrt{v_t}\, d\widetilde{W}_t^{(2)} \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{dS_t}{S_t} = r\, dt + \sqrt{v_t}\, d\widetilde{W}_t^{(1)} \\[2ex] dv_t = \left[k\theta - (k+\lambda) v_t\right] dt + \sigma\sqrt{v_t}\, d\widetilde{W}_t^{(2)} \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{dS_t}{S_t} = r\, dt + \sqrt{v_t}\, d\widetilde{W}_t^{(1)} \\[2ex] dv_t = (k+\lambda)\left[\dfrac{k\theta}{k+\lambda} - v_t\right] dt + \sigma\sqrt{v_t}\, d\widetilde{W}_t^{(2)} \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{dS_t}{S_t} = r\, dt + \sqrt{v_t}\, d\widetilde{W}_t^{(1)} \\[2ex] dv_t = k^*\left(\theta^* - v_t\right) dt + \sigma\sqrt{v_t}\, d\widetilde{W}_t^{(2)} \end{cases}$$

con $k^* = k + \lambda$, $\theta^* = \dfrac{k\theta}{k+\lambda}$

9c

GOAL: pricing ~~?~~ derivati europei in SVM

$$C_t = C(S_0, K, T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u \, du} (S_T - K)^+ \,\middle|\, \mathcal{F}_t \right]$$

$$\Rightarrow C_0 = \mathbb{E}^{\mathbb{Q}}\left[(S_T - K)^+ \right] e^{-rT} = e^{-rT}\, \mathbb{E}^{\mathbb{Q}}\left[(S_T - K) \mathbb{1}_{\{S_T \geq K\}} \right]$$

$$= \underbrace{e^{-rT}\, \mathbb{E}^{\mathbb{Q}}\left[S_T \mathbb{1}_{\{S_T \geq K\}} \right]}_{A} - K \underbrace{e^{-rT}\, \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{S_T \geq K\}} \right]}_{B} = ?$$

(B) = prob. ~~chiusura~~ della call ITM sotto $\mathbb{Q}$ che rende $W_t^{(1)}$ e $W_t^{(2)}$ del modello dei MB (risk-neutral)

$$\Rightarrow \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{S_T \geq K\}} \right] = \mathbb{Q}(S_T \geq K) = \mathbb{Q}\left(\ln(S_T) \geq \ln(K) \right) =: P_2$$

$$\ln\left(\frac{B_t}{B_0}\right) = \ln(B_t) - \ln(B_0) = \int_0^t r \, ds \Leftarrow \frac{dB_t}{B_t} = r\,dt \Rightarrow B_t = e^{rt} \Rightarrow \frac{B_t}{B_0} = e^{rt}$$

$$\left(B_t = \exp\left\{ \int_0^t r \, ds \right\} = e^{rt} \right)$$

(A) = cambio di numerario!

$$\Rightarrow \frac{d\mathbb{Q}}{d\mathbb{Q}^S} = \frac{B_T S_0}{B_0 S_T} = \frac{B_T / B_0}{S_T / S_0} =: \frac{\mathbb{E}^{\mathbb{Q}}[e^{rT}]}{e^{rT}}$$

$$\Rightarrow \frac{B_T}{B_0} = e^{+rT} \quad (*)$$

Inoltre, poiché sotto $\mathbb{Q}$ i titoli rischiosi crescono al tasso free-risk r

allora $S_0 e^{rT} = \mathbb{E}^{\mathbb{Q}}[e^{rT}]$ + (*)

$$\frac{B_0}{B_T} \mathbb{E}^{\mathbb{Q}}\left[\frac{S_T S_0}{S_0} \mathbb{1}_{\{S_T \geq K\}} \right] = S_0 \mathbb{E}^{\mathbb{Q}}\left[\frac{B_0 S_T}{B_T S_0} \right]$$ (molt. e div. per S_0)

$$\Rightarrow e^{-rT}\, \mathbb{E}^{\mathbb{Q}}\left[S_T \mathbb{1}_{\{S_T \geq K\}} \right] = S_0\, \mathbb{E}^{\mathbb{Q}}\left[\frac{S_T / S_0}{B_T / B_0} \mathbb{1}_{\{S_T \geq K\}} \right]$$

$$\mathbb{E}^{\mathbb{Q}}[X] = \mathbb{E}^{\mathbb{Q}^S}[X \cdot L]$$

$$= S_0\, \mathbb{E}^{\mathbb{Q}^S}\left[\frac{S_T / S_0}{B_T / B_0} \mathbb{1}_{\{S_T \geq K\}} \underbrace{\frac{d\mathbb{Q}}{d\mathbb{Q}^S}}_{\frac{B_T/B_0}{S_T/S_0}} \right] = S_0\, \mathbb{E}^{\mathbb{Q}^S}\left[\mathbb{1}_{\{S_T \geq K\}} \right]$$

$$= S_0\, \mathbb{Q}^S(S_T \geq K) =: S_0 P_1$$

$$\Rightarrow C_0 = S_0 \cdot P_1 - K e^{-rT} P_2$$

$$\Rightarrow C_t = S_t \, \mathbb{Q}^S(S_T \geq K) - K e^{-r(T-t)} \, \mathbb{Q}(S_T \geq K)$$

$$= S_t P_1 - K e^{-r(T-t)} P_2$$

DOMANDA!

Come si calcola praticamente P_j, $\forall j = 1,2$?

$\Rightarrow$ Tramite il seguente

TEOR (GIL-PELAEZ)

$$P_j = \frac{1}{2} + \frac{1}{2\pi} \int_0^{+\infty} \mathrm{Re}\left[\frac{e^{-iu\ln(K)} f_j(u; x, v)}{iu} \right] du, \quad \forall j = 1,2$$

dove $f_j := f_j(u; x, v)$ è la FUNZIONE CARATTERISTICA della v.a. $x = \ln(S)$ ~~condizionata~~

Usiamo la TRASFORMATA DI FOURIER! $f: \mathbb{R} \to \mathbb{C}$, $u \in \mathbb{R}$

$$\varphi(u) = \int_{-\infty}^{+\infty} e^{iux} f(x)\, dx \iff f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iux} \varphi(u)\, du$$

OSS: Se f è pdf di X, allora

$\varphi(u) = \mathbb{E}\left[e^{iuX}\right]$ è la funzione caratteristica di X.

PROOF (Gil - Pelaez):

Sia $\operatorname{sgn}(\alpha) := \begin{cases} \frac{\alpha}{|\alpha|}, & \alpha \neq 0 \\ 0, & \alpha = 0 \end{cases}$ $= \begin{cases} 1 & \alpha > 0 \\ -1 & \alpha < 0 \end{cases}$

→ FUNZIONE SEGNO

(NO PROOF) $\Rightarrow \operatorname{sgn}(\alpha) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\sin(\alpha x)}{x} dx$. (SGN) , $\alpha \neq 0$ → INTEGRALE DI DIRICHLET

Sia $f(x)$ la densità di $x = \ln(S)$, e sia $F(x)$ $= \int_{-\infty}^{x} f(z)dz$ la corrispondente distribuzione (f. di ripartizione)

$(\forall y \in \mathbb{R}$ fissato$) \Rightarrow \int_{-\infty}^{+\infty} \operatorname{sgn}(x-y) f(x) dx = \int_{y}^{+\infty} \underbrace{\operatorname{sgn}(x-y)}_{=+1} f(x)dx + \int_{-\infty}^{y} \underbrace{\operatorname{sgn}(x-y)}_{=-1} f(x) dx$

$= -\int_{-\infty}^{y} f(x)dx + \int_{y}^{+\infty} f(x)dx = -F(y) + \int_{-\infty}^{+\infty} f(x)dx - \int_{-\infty}^{y} f(x)dx$

$= [1 - F(y)] - F(y) = 1 - 2F(y)$ (**)

D'altra parte, $f(x) \overset{(TF)}{=} \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iux} \varphi(u) du$

$\Rightarrow \underset{P_2 =}{Q}(\underbrace{\ln(S)}_{x} > \underbrace{\ln(K)}_{k}) = Q(x > k) = \int_{k}^{+\infty} f(x)dx = \int_{k}^{+\infty} \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iux} \varphi(u) du\, dx$

(Fubini) $= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \varphi(u) \left[\int_{k}^{+\infty} e^{-iux} dx \right] du = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\varphi(u)}{-iu} \left[\int_{k}^{+\infty} -iu\, e^{-iux} dx \right] du$

$= -\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\varphi(u)}{iu} \left[e^{-iux} \right]_{k}^{+\infty} du = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\varphi(u)}{iu} e^{-iuk} du - \underbrace{\frac{1}{2\pi} \lim_{R \to +\infty} \int_{-\infty}^{+\infty} \frac{\varphi(u)}{iu} e^{-iuR} du}_{=: I}$,

$$I = \int_{-\infty}^{+\infty} \frac{\varphi(u)}{iu} e^{-iuR}\, du \overset{(FT)}{=} \int_{-\infty}^{+\infty} \frac{1}{iu} \left[\int_{-\infty}^{+\infty} e^{iux} f(x)\, dx \right] e^{-iuR}\, du$$

(Fubini)
$$= \int_{-\infty}^{+\infty} f(x) \left[\int_{-\infty}^{+\infty} \frac{e^{iu(x-R)}}{iu}\, du \right] dx$$

$\boxed{e^{i\alpha} = \cos(i\alpha) - i \sin(i\alpha)}$ con $\boxed{\alpha = u(x-R)}$

$$= \int_{-\infty}^{+\infty} f(x) \left[\int_{-\infty}^{+\infty} \Bigg(\frac{1}{i} \underbrace{\frac{\cos[u(x-R)]}{u}}_{\substack{\text{funzione pari} \\ \text{(si annulla su } \mathbb{R})}} + \underbrace{\frac{\sin[u(x-R)]}{u}}_{\text{funzione dispari}} \Bigg) du \right] dx$$

$$= \int_{-\infty}^{+\infty} f(x) \left[\int_{-\infty}^{+\infty} \frac{\sin[u(x-R)]}{u}\, du \right] dx \overset{(SGN)}{=} \int_{-\infty}^{+\infty} \pi \cdot \mathrm{sgn}(x-R)\, f(x)\, dx$$

$$\cancel{\Rightarrow P_2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\varphi(u)}{iu} e^{-iux}\, du - \frac{1}{2\pi} \lim}$$

$$\Rightarrow \frac{1}{2\pi} \lim_{R \to \infty} I = \frac{1}{2\pi} \lim_{R \to +\infty} \pi \int_{-\infty}^{+\infty} f(x)\, \mathrm{sgn}(x-R)\, dx$$

$$\overset{(**)}{=} \frac{1}{2} \lim_{R \to +\infty} [1 - 2\underbrace{F(R)}_{\substack{\to 1 \\ (R \to \infty)}}] = -\frac{1}{2}$$

$$\Rightarrow P_2 = \frac{1}{2} + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \varphi(u) \frac{e^{-iux}}{iu}\, du = \frac{1}{2} + \frac{1}{\pi} \int_{0}^{+\infty} \mathrm{Re}\left[\overset{f_j}{\varphi(u)} \frac{e^{-iux}}{iu} \right] du$$

perché $\mathrm{Im}[\dots]$ è funzione dispari, mentre $\mathrm{Re}[\dots]$ è funzione pari