PDE per modelli a volatilità stocastica:

$$\frac{\partial F}{\partial t} + rS\frac{\partial F}{\partial S} + \alpha\frac{\partial F}{\partial V} + \frac{\sigma^2}{2}V\frac{\partial^2 F}{\partial S^2} + S\beta\rho\sqrt{V}\frac{\partial^2 F}{\partial S\partial V} + \frac{\partial^2 F}{\partial V^2}\frac{\beta^2}{2} - rF = \lambda(t,S,V)\frac{\partial F}{\partial V}$$

(HESTON) $\Rightarrow$

$$\frac{\partial F}{\partial t} + rS\frac{\partial F}{\partial S} + k(\theta - V)\frac{\partial F}{\partial V} + \frac{S^2}{2}V\frac{\partial^2 F}{\partial S^2} + S\rho\sqrt{V}\,\sigma\sqrt{V}\frac{\partial^2 F}{\partial S\partial V} + \frac{\sigma^2 V}{2}\frac{\partial^2 F}{\partial V^2} - rF = \lambda V\frac{\partial F}{\partial V}$$

$$\Rightarrow \frac{\partial F}{\partial t} + rS\frac{\partial F}{\partial S} + \left[k(\theta - V) - \lambda V\right]\frac{\partial F}{\partial V} + \frac{S^2}{2}V\frac{\partial^2 F}{\partial S^2} + S\rho\sigma V\frac{\partial^2 F}{\partial S\partial V} + \frac{\sigma^2 V}{2}\frac{\partial^2 F}{\partial V^2} - rF = 0$$

con le seguenti boundary conditions:

$$\begin{cases} F(T,S,V) = \phi(S_T) \quad (\text{e.g. call option} \rightarrow \max\{S_T - K, 0\}) \\ F(t,0,V) = 0 \\ \dfrac{\partial F(t,\infty,V)}{\partial S} = 1 \\ F(t,S,+\infty) = S \end{cases}$$

Per comodità, poniamo $x = \ln(S)$ e riscriviamo la PDE:

$$\frac{\partial F}{\partial S} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial S} = \frac{\partial F}{\partial x}\frac{1}{S}$$

$$\frac{\partial^2 F}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{\partial F}{\partial S}\right) = \frac{\partial}{\partial S}\left(\frac{\partial F}{\partial x}\cdot\frac{1}{S}\right) = \frac{\partial}{\partial S}\left(\frac{\partial F}{\partial x}\right)\cdot\frac{1}{S} +$$

$$+ \frac{\partial F}{\partial x}\left(-\frac{1}{S^2}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial S}\right)\cdot\frac{1}{S} - \frac{1}{S^2}\frac{\partial F}{\partial x}$$

$$= \frac{1}{S}\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\cdot\frac{1}{S}\right) - \frac{1}{S^2}\frac{\partial F}{\partial x}$$

$$= \frac{1}{S^2}\frac{\partial^2 F}{\partial x^2} - \frac{1}{S^2}\frac{\partial F}{\partial x} = \frac{1}{S^2}\left(\frac{\partial^2 F}{\partial x^2} - \frac{\partial F}{\partial x}\right)$$

$$\frac{\partial^2 F}{\partial S \partial v} = \frac{\partial}{\partial v}\left(\frac{\partial F}{\partial S}\right) = \frac{\partial}{\partial v}\left(\frac{\partial F}{\partial x}\cdot\frac{1}{S}\right) = \frac{\partial^2 F}{\partial x \partial v}\cdot\frac{1}{S}$$

$\Rightarrow$ Riscriviamo la PDE:

$$\frac{\partial F}{\partial t} + r\cancel{S}\frac{\partial F}{\partial x}\cdot\frac{1}{\cancel{S}} + \left[k(\theta - v) - \lambda v\right]\frac{\partial F}{\partial v} + \frac{\cancel{S^2}v}{2}\frac{1}{\cancel{S^2}}\left(\frac{\partial^2 F}{\partial x^2} - \frac{\partial F}{\partial x}\right) +$$

$$+ S\rho\sigma v\frac{1}{\cancel{S}}\frac{\partial^2 F}{\partial x \partial v} + \frac{\sigma^2 v}{2}\frac{\partial^2 F}{\partial v^2} - rF = 0$$

$$\Rightarrow \frac{\partial F}{\partial t} + r\frac{\partial F}{\partial x} + \left[k(\theta - v) - \lambda v\right]\frac{\partial F}{\partial v} + \frac{v}{2}\left(\frac{\partial^2 F}{\partial x^2} - \frac{\partial F}{\partial x}\right) +$$

$$+ \rho\sigma v\frac{\partial^2 F}{\partial x \partial v} + \frac{\sigma^2 v}{2}\frac{\partial^2 F}{\partial v^2} - rF = 0$$

$$\Rightarrow \frac{\partial F}{\partial t} + \left(r - \frac{v}{2}\right)\frac{\partial F}{\partial x} + \left[k(\theta - v) - \lambda v\right]\frac{\partial F}{\partial v} + \frac{v}{2}\frac{\partial^2 F}{\partial x^2} +$$

$$+ \frac{\sigma^2 v}{2}\frac{\partial^2 F}{\partial v^2} + \rho\sigma v\frac{\partial^2 F}{\partial x \partial v} - rF = 0, \quad \forall F \quad \text{(F\_PDE)}$$

Questa PDE vale anche per un'opzione call, per la quale si ha

$$C_t := S_t P_1 - Re^{-r(T-t)}P_2 = e^x P_1 - Re^{-r(T-t)}P_2$$

$\hookrightarrow$ (STRIKE PRICE)

$\Rightarrow$ Allora, si ha:

$$\frac{\partial C}{\partial t} = e^x \frac{\partial P_1}{\partial t} - Re^{-r(T-t)}\frac{\partial P_2}{\partial t} - Rre^{-r(T-t)}P_2$$

$$= e^x \frac{\partial P_1}{\partial t} - Re^{-r(T-t)}\left[rP_2 + \frac{\partial P_2}{\partial t}\right]$$

$$\frac{\partial C}{\partial x} = e^x P_1 + e^x \frac{\partial P_1}{\partial x} - Re^{-r(T-t)}\frac{\partial P_2}{\partial x}$$

$$\frac{\partial^2 C}{\partial x^2} = e^x P_1 + e^x \frac{\partial P_1}{\partial x} + e^x \frac{\partial P_1}{\partial x} + e^x \frac{\partial^2 P_1}{\partial x^2} - Re^{-r(T-t)}\frac{\partial^2 P_2}{\partial x^2}$$

$$= e^x\left[P_1 + 2\frac{\partial P_1}{\partial x} + \frac{\partial^2 P_1}{\partial x^2}\right] - Re^{-r(T-t)}\frac{\partial^2 P_2}{\partial x^2}$$

$$\frac{\partial C}{\partial v} = e^x \frac{\partial P_1}{\partial v} - Re^{-r(T-t)}\frac{\partial P_2}{\partial v}$$

$$\frac{\partial^2 C}{\partial v^2} = e^x \frac{\partial^2 P_1}{\partial v^2} - K e^{-r(T-t)} \frac{\partial^2 P_2}{\partial v^2}$$

$$\frac{\partial^2 C}{\partial x \partial v} = e^x \frac{\partial P_1}{\partial v} + e^x \frac{\partial^2 P_1}{\partial x \partial v} - K e^{-r(T-t)} \frac{\partial^2 P_2}{\partial x \partial v}$$

Sostituendo in (F-PDE), si ha

$$e^x \frac{\partial P_1}{\partial t} - K e^{-r(T-t)} \left[\cancel{r P_2} + \frac{\partial P_2}{\partial t} \right] + \left(r - \frac{v}{2} \right) \left[\cancel{e^x P_1} + e^x \frac{\partial P_1}{\partial x} + K e^{-r(T-t)} \frac{\partial}{\partial x} \right.$$

$$+ \left[k(\theta - v) - \lambda v \right] \left[e^x \frac{\partial P_1}{\partial v} - K e^{-r(T-t)} \frac{\partial P_2}{\partial v} \right] +$$

$$+ \frac{v}{2} \left[e^x \left(\cancel{P_1} + 2 \frac{\partial P_1}{\partial x} + \frac{\partial^2 P_1}{\partial x^2} \right) - K e^{-r(T-t)} \frac{\partial^2 P_2}{\partial x^2} \right] +$$

$$+ \frac{\sigma^2 v}{2} \left[e^x \frac{\partial^2 P_1}{\partial v^2} - K e^{-r(T-t)} \frac{\partial^2 P_2}{\partial v^2} \right] +$$

$$+ \rho \sigma v \left[e^x \frac{\partial P_1}{\partial v} + e^x \frac{\partial^2 P_1}{\partial x \partial v} - K e^{-r(T-t)} \frac{\partial^2 P_2}{\partial x \partial v} \right] +$$

$$- r \left[\cancel{e^x P_1} - \cancel{K e^{-r(T-t)} P_2} \right] = 0$$

Raccogliamo a fattor comune:

$$e^{x}\left[\frac{\partial P_1}{\partial t}+\left(r-\frac{v}{2}\right)\frac{\partial P_1}{\partial x}+\left[\kappa(\theta-v)-\lambda v\right]\frac{\partial P_1}{\partial v}+v\frac{\partial P_1}{\partial x}+\frac{v}{2}\frac{\partial^2 P_1}{\partial x^2}+\frac{\sigma^2 v}{2}\frac{\partial^2 P_1}{\partial v^2}+\rho\sigma v\frac{\partial^2 P_1}{\partial x\partial v}+\rho\sigma v\frac{\partial P_1}{\partial v}\right]+$$

$$-Ke^{-r(T-t)}\left[\frac{\partial P_2}{\partial t}+\left(r-\frac{v}{2}\right)\frac{\partial P_2}{\partial x}+\left[\kappa(\theta-v)-\lambda v\right]\frac{\partial P_2}{\partial v}+\frac{v}{2}\frac{\partial^2 P_2}{\partial x^2}+\frac{\sigma^2 v}{2}\frac{\partial^2 P_2}{\partial v^2}+\rho\sigma v\frac{\partial^2 P_2}{\partial x\partial v}\right]=0$$

$$\Rightarrow\left\{\begin{aligned}&\frac{\partial P_1}{\partial t}+\left(r+\frac{v}{2}\right)\frac{\partial P_1}{\partial x}+\left[\kappa(\theta-v)-\lambda v+\rho\sigma v\right]\frac{\partial P_1}{\partial v}+\frac{v}{2}\frac{\partial^2 P_1}{\partial x^2}+\frac{\sigma^2 v}{2}\frac{\partial^2 P_1}{\partial v^2}+\rho\sigma v\frac{\partial^2 P_1}{\partial x\partial v}=0\\ &\qquad \left[\ldots\right]=+\kappa\theta-(\kappa+\lambda-\rho\sigma)v\\ &\frac{\partial P_2}{\partial t}+\left(r-\frac{v}{2}\right)\frac{\partial P_2}{\partial x}+\left[\kappa(\theta-v)-\lambda v\right]\frac{\partial P_2}{\partial v}+\frac{v}{2}\frac{\partial^2 P_2}{\partial x^2}+\frac{\sigma^2 v}{2}\frac{\partial^2 P_2}{\partial v^2}+\rho\sigma v\frac{\partial^2 P_2}{\partial x\partial v}=0\\ &\qquad \left[\ldots\right]=+\kappa\theta-(\kappa+\lambda)v\end{aligned}\right.$$

In forma compatta:

$$\left\{\begin{aligned}&\frac{\partial P_j}{\partial t}+(r+u_j v)\frac{\partial P_j}{\partial x}+(a-b_j v)\frac{\partial P_j}{\partial v}+\frac{v}{2}\frac{\partial^2 P_j}{\partial x^2}+\frac{\sigma^2 v}{2}\frac{\partial^2 P_j}{\partial v^2}+\rho\sigma v\frac{\partial^2 P_j}{\partial x\partial v}=0,\quad\forall j=1,2 \qquad (*)\\ &u_1=\frac{1}{2},\ u_2=-\frac{1}{2},\ a=\kappa\theta,\ b_1=\kappa+\lambda-\rho\sigma,\ b_2=\kappa+\lambda\end{aligned}\right.$$

8t

RECALL: $P_j = \frac{1}{2} + \frac{1}{2\pi} \int_0^{+\infty} \mathrm{Re}\left[\frac{e^{-iux} f_j(u;x,v)}{iu} \right] du, \quad j=1,2$

$\Rightarrow$ possiamo esplicitare la funzione caratteristica?

$\Rightarrow$ Sì, attraverso le PDE associate a P_j, $j=1,2$

GUESS & VERIFY APPROACH:

(**) $f_j(u;x,v) = \exp\{ C_j(\tau,u) + D_j(\tau,u) V_\tau + iux_\tau \}$, $\tau := T-t$

$\Rightarrow$ vogliamo determinare C_j, D_j, $j=1,2$.

(PDE in (*)) $\Rightarrow$

$$-\frac{\partial f_j}{\partial \tau} + (r + u_j v) \frac{\partial f_j}{\partial x} + (a - b_j v) \frac{\partial f_j}{\partial v} + \frac{1}{2} v \frac{\partial^2 f_j}{\partial x^2} + \frac{1}{2}\sigma^2 v \frac{\partial^2 f_j}{\partial v^2} + \rho\sigma v \frac{\partial^2 f_j}{\partial x \partial v} = 0$$

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial \tau}\frac{\partial \tau}{\partial t} = -\frac{\partial F}{\partial \tau}, \quad \tau = T-t \Leftrightarrow t = T-\tau \Rightarrow \frac{\partial t}{\partial \tau} = -1$$

Scriviamo le derivate, tenendo conto dell'ipotesi (**):

$$\frac{\partial f_j}{\partial \tau} = (\dot{C}_j + \dot{D}_j v)\,\underbrace{\exp\{C_j + D_j v + iux\}}_{f_j}$$

$$\frac{\partial f_j}{\partial x} = iu\,\underbrace{\exp\{C_j + D_j v + iux\}}_{f_j}$$

$$\frac{\partial^2 f_j}{\partial x^2} = -u^2\,\underbrace{\exp\{C_j + D_j v + iux\}}_{f_j}$$

$$\frac{\partial f_j}{\partial v} = D_j\,\underbrace{\exp\{C_j + D_j v + iux\}}_{f_j}$$

$$\frac{\partial^2 f_j}{\partial v^2} = D_j^2\,\underbrace{\exp\{C_j + D_j v + iux\}}_{f_j}$$

$$\frac{\partial^2 f_j}{\partial x\,\partial v} = iuD_j\,\underbrace{\exp\{C_j + D_j v + iux\}}_{f_j}$$

Dunque:

$$-(\dot{C}_j + \dot{D}_j v)\cancel{f_j} + (r + u_j v)\cancel{f_j}\,iu + (a - b_j v)D_j\cancel{f_j} + \frac{1}{2}v(-u^2)\cancel{f_j} + \frac{1}{2}\sigma^2 v D_j^2\cancel{f_j} + \rho\sigma v\,iuD_j\cancel{f_j} = 0, \quad \forall j = 1,2$$

$$\Rightarrow \left[-\dot{C}_j + iur + aD_j\right] + \left[-\dot{D}_j + u_j iu - b_j D_j - \frac{u^2}{2} + \frac{1}{2}\sigma^2 D_j^2 + \rho\sigma iuD_j\right]v = 0, \quad \forall j = 1,2$$

(8a)

$$\Rightarrow \begin{cases} -\dot{C}_j + iur + a D_j = 0 \\ -\dot{D}_j + iu \cdot u_j - b_j D_j - \frac{1}{2}u^2 + \frac{1}{2}\sigma^2 D_j^2 + \\ \quad + \rho\sigma iu D_j = 0 \end{cases}$$

con $C_j(0,u) = D_j(0,u) = 0, \ \forall j = 1,2$

Risolviamo le due ODEs.

<u>ODE per D_j:</u>

$$\dot{D}_j = \frac{\sigma^2}{2} D_j^2 + \underbrace{\left[\rho\sigma iu - b_j\right]}_{=:\alpha_j} D_j - \underbrace{\left[\frac{1}{2}u^2 - iu \cdot u_j\right]}_{=:\Psi_j}$$

$\hookrightarrow$ ODE DI RICCATI

$$D_j := -\frac{2}{\sigma^2}\frac{y'}{y}$$, dove $y = y(\tau)$ definita come nel modello CIR

$$\Rightarrow \dot{D}_j = -\frac{2}{\sigma^2}\frac{y'' \cdot y - (y')^2}{y^2}$$

$$\Rightarrow -\frac{2}{\sigma^2}\frac{y''}{y} + \frac{2}{\sigma^2}\frac{(y')^2}{y^2} = \frac{\cancel{\sigma^2}}{\cancel{2}} \cdot \frac{\cancel{4}^{2}}{\sigma^{\cancel{4}}{}_2}\frac{(y')^2}{y^2} + \alpha\left(-\frac{2}{\sigma^2}\frac{y'}{y}\right) + \\ -\Psi$$

$$\Rightarrow +\frac{2}{\sigma^2}\frac{y''}{y} = +\frac{2\alpha_j}{\sigma^2}\frac{y'}{y} + \Psi_j$$

$$\Rightarrow \frac{y''}{y} = \alpha_j\frac{y'}{y} + \frac{\sigma^2}{2}\Psi_j$$

$$\Rightarrow y'' - \alpha_j y' - \frac{\sigma^2\Psi_j}{2} y = 0$$

Sol: $y = C_1 e^{x_1 \tau} + C_2 e^{x_2 \tau}$

$$\Rightarrow y' = C_1 x_1 e^{x_1 \tau} + C_2 x_2 e^{x_2 \tau}$$

Inoltre, $y = e^{x\tau} \Rightarrow x^2 - \alpha x - \frac{\sigma^2 \gamma}{2} = 0$

$$\Rightarrow x = \frac{\alpha_j \pm \sqrt{\alpha_j^2 + 2\sigma^2 \gamma_j}}{2} = \frac{\alpha_j \pm d_j}{2},$$

essendo $d_j = \sqrt{\alpha_j^2 + 2\sigma^2 \gamma_j}$

$$\Rightarrow d_j^2 = \alpha_j^2 + 2\sigma^2 \gamma_j$$

$$\Rightarrow d_j^2 - \alpha_j^2 = 2\sigma^2 \gamma_j$$

Allora, si ha:

$$D_j(\tau) = -\frac{2}{\sigma^2} \frac{C_1 x_1 e^{x_1 \tau} + C_2 x_2 e^{x_2 \tau}}{C_1 e^{x_1 \tau} + C_2 e^{x_2 \tau}}, \text{ con}$$

$$D_j(0) = -\frac{2}{\sigma^2} \frac{C_1 x_1 + C_2 x_2}{C_1 + C_2} = 0 \overset{(C_1 \neq -C_2)}{\Rightarrow} C_1 = -\frac{x_2}{x_1} C_2$$

$$\Rightarrow D_j = -\frac{2}{\sigma^2} \frac{-\frac{x_2}{x_1} C_2 x_1 e^{x_1 \tau} + C_2 x_2 e^{x_2 \tau}}{-\frac{x_2}{x_1} C_2 e^{x_1 \tau} + C_2 e^{x_2 \tau}}$$

$$= -\frac{2}{\sigma^2} \frac{-x_1 x_2 C_2 e^{x_1 \tau} + C_2 x_1 x_2 e^{x_2 \tau}}{-x_2 C_2 e^{x_1 \tau} + C_2 x_1 e^{x_2 \tau}}$$

$$= -\frac{2}{\sigma^2}\,\frac{+x_1x_2\cancel{c_2}\cancel{e}\,\left(1-e\right.}{+\cancel{c_2}\cancel{e^{x_1\tau}}\left(x_2-x_1\,e^{(x_2-x_1)\tau}\right)}$$

$$= -\frac{2}{\sigma^2}\,\frac{x_1x_2\left(1-e^{(x_2-x_1)\tau}\right)}{x_2-x_1\,e^{(x_2-x_1)\tau}}$$

$$= -\frac{\cancel{2}}{\sigma^2}\,\frac{\frac{\alpha_j^2-d_j^2}{\cancel{4}}\left(1-e^{d_j\tau}\right)}{\frac{\alpha_j+d_j}{\cancel{2}}-\frac{\alpha_j-d_j}{\cancel{2}}\,e^{d_j\tau}}$$

$$= +\frac{\cancel{2}}{\cancel{\sigma^2}}\,\frac{+\frac{2\cancel{\sigma^2}\gamma}{\cancel{4}}\left(1-e^{d\tau}\right)}{\frac{(\alpha+d)-(\alpha-d)\,e^{d\tau}}{\cancel{2}}} = \frac{2\gamma\left(1-e^{d\tau}\right)}{(\alpha-d)+2d-(\alpha-d)\,e^{d\tau}}$$

$$\Rightarrow D_j = \frac{2\gamma\left(1-e^{d\tau}\right)}{2d+(\alpha-d)\left(1-e^{d\tau}\right)}$$

$$\Rightarrow D_j = -\frac{\alpha_j^2-d_j^2}{\sigma^2}\,\frac{1-e^{d_j\tau}}{(\alpha_j+d_j)\left(1-\underbrace{\frac{\alpha_j-d_j}{\alpha_j+d_j}}_{=g_j}e^{d_j\tau}\right)}$$

$$\Rightarrow D_j = \frac{d_j-\alpha_j}{\sigma^2}\,\frac{1-e^{d_j\tau}}{1-g_j\,e^{d_j\tau}} = \frac{d_j+b_j-iu\rho\sigma}{\sigma^2}\,\frac{1-e^{d_j\tau}}{1-g_j\,e^{d_j\tau}}$$

ODE per C_j:

$$\dot{C}_j(\tau) = iur + a D_j(\tau)$$

$$\Rightarrow \int_0^{\tau} \dot{C}_j(s)\, ds = \int_0^{\tau} \left(iur + \frac{a\gamma}{\sigma^2} \frac{1 - e^{d_j s}}{1 - g_j e^{d_j s}} \right) ds$$

$$\Rightarrow C_j(\tau) - \cancel{C_j(0)} = iur\tau + \frac{a\gamma}{\sigma^2} \underbrace{\int_0^{\tau} \frac{1 - e^{d_j s}}{1 - g_j e^{d_j s}}\, ds}_{=: I}$$

Risolviamo I:

$$e^{d_j s} =: z \Rightarrow s = \frac{1}{d_j} \ln(z) \Rightarrow ds = \frac{1}{d_j} \frac{1}{z} dz$$

$$s = 0 \longrightarrow z = 1$$
$$s = \tau \longrightarrow z = e^{d_j \tau}$$

$$\Rightarrow I = \frac{1}{d_j} \int_1^{e^{d_j \tau}} \frac{1 - z}{(1 - g_j z) z}\, dz$$

NOTA: Cerchiamo F, G t.c. $\dfrac{1-z}{z(1-g_j z)} = \dfrac{F}{z} + \dfrac{G}{1 - g_j z}$

$$\Rightarrow 1 - z = F(1 - g_j z) + z \cdot G \Rightarrow 1 - z = F + (G - F \cdot g_j) z$$

$$\Rightarrow \begin{cases} F = 1 \\ G - F \cdot g_j = -1 \end{cases} \Rightarrow \begin{cases} F = 1 \\ G = g_j - 1 \end{cases}$$

$$\Rightarrow I = \frac{1}{d_j}\int_1^{\exp\{d_j\tau\}}\left(\frac{1}{z} - \frac{1-g_j}{1-g_j z}\right)dz$$

$$= \frac{1}{d_j}\ln(z)\Big|_1^{\exp\{d_j\tau\}} - \frac{1}{d_j}\int_1^{\exp\{d_j\tau\}}\frac{dz}{1-g_j z} + \frac{g_j}{d_j}\int_1^{\exp\{d_j\tau\}}\frac{dz}{1-g_j z}$$

$$= \frac{1}{d_j}\left[\ln\left(\exp\{d_j\tau\}\right) - \ln(1)\right] + \frac{1}{d_j g_j}\int_1^{\exp\{d_j\tau\}}\frac{-g_j\,dz}{1-g_j z} - \frac{1}{d_j}\int_1^{\exp\{d_j\tau\}}\frac{-g_j\,dz}{1-g_j z}$$

$$= \frac{1}{\cancel{d_j}}\cancel{d_j}\tau + \frac{1}{d_j g_j}\ln(1-g_j z)\Big|_1^{\exp\{d_j\tau\}} - \frac{1}{d_j}\ln(1-g_j z)\Big|_1^{\exp\{d_j\tau\}}$$

$$= \tau + \frac{1}{d_j g_j}\left[\ln\left(1-g_j e^{d_j\tau}\right) - \ln(1-g_j)\right] +$$

$$- \frac{1}{d_j}\ln\left(1-g_j e^{d_j\tau}\right) - \ln(1-g_j)\Big]$$

$$= \tau + \frac{1}{d_j g_j}\ln\left(\frac{1-g_j e^{d_j\tau}}{1-g_j}\right) - \frac{1}{d_j}\ln\left(\frac{1-g_j e^{d_j\tau}}{1-g_j}\right)$$

$$= \frac{1}{d_j}\left[d_j\tau + \left(\frac{1}{g_j}-1\right)\ln\left(\frac{1-g_j e^{d_j\tau}}{1-g_j}\right)\right]$$

$$\Rightarrow C_j = iru\tau + a\,\frac{d_j + \overbrace{b_j - iu\rho\sigma}^{-d_j}}{\sigma^2}\,\frac{1}{d_j}\left[d_j\tau + \overbrace{\frac{1-g_j}{g_j}}^{\frac{2d_j}{b_j-d_j}}\ln\left(\frac{1-g_j e^{d_j\tau}}{1-g_j}\right)\right]$$

$$= iru\tau + \frac{a}{\sigma^2}\left[(d_j+b_j-iu\rho\sigma)\tau + 2\ln\left(\frac{1-g_j e^{d_j\tau}}{1-g_j}\right)\right]$$