3) MODELLO DI HULL-WHITE (O VASICEK ESTESO)

$$dr_t = (\Theta(t) - a r_t)dt + \sigma dW_t, \quad (HW)$$

dove $a, \sigma \in \mathbb{R}^+$ e $\Theta(t)$ è funz. deterministica

PROBLEMA: $\Theta(t)$ non è esplicitamente definita!

$\Rightarrow$ vogliamo determinarla, attraverso la seguente procedura.

OSS: (HW) è un modello affine, infatti:

$$dr_t = (-a \cdot r_t + \Theta(t))dt + \sqrt{0 \cdot r_t + \sigma^2}\, dW_t$$

$$\Rightarrow \alpha_t = -a; \; \beta_t = \Theta(t); \; \gamma_t = 0; \; \delta_t = \sigma^2$$

$$\overset{(TEOR)}{\Rightarrow} p(t,T) = \exp\{A(t,T) - r_t \cdot B(t,T)\}, \text{ con}$$

$$\begin{cases} \dot{A}(t,T) - \beta_t \cdot B(t,T) + \frac{1}{2}\delta_t (B(t,T))^2 = 0, & A(T,T) = 0 \\ \dot{B}(t,T) + \alpha_t \cdot B(t,T) - \frac{1}{2}\gamma_t (B(t,T))^2 + 1 = 0, & B(T,T) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \dot{A}(t,T) - \Theta(t) B(t,T) + \frac{\sigma^2}{2}(B(t,T))^2 = 0, & A(T,T) = 0 \\ \dot{B}(t,T) - a \cdot B(t,T) + 1 = 0, & B(T,T) = 0 \end{cases}$$

Risolviamo la ODE per B (come per il modello di Vasicek): è della forma

$$\begin{cases} y'(x) + a(x) \cdot y(x) = g(x) \\ y(x_0) = y_0 \end{cases} \longrightarrow \text{ODE DEL PRIMO ORDINE, LINEARE, NON OMOGENEA}$$

$$\Rightarrow y(x) = e^{-G(x)} \left[y_0 + \int_{x_0}^{x} g p(s)\, e^{G(s)}\, ds \right]$$

$$G(x) = \int_{x_0}^{x} a_0(s)\, ds$$

Nel nostro caso:

$$\begin{cases} \dot{B}(t,T) - a \cdot B(t,T) = 1 \\ B(T,T) = 0 \end{cases}$$

$$\Rightarrow \quad G(t) = \int_T^t -a\, ds = -a(t-T) = a(T-t)$$

$$\Rightarrow \quad B(t,T) = e^{-a(T-t)} \int_T^t - e^{a(T-s)}\, ds$$

Risolviamo l'integrale per sostituzione:

$T - s = z \Rightarrow s = T - z \Rightarrow ds = -dz$

$s = T \rightsquigarrow z = 0$

$s = t \rightsquigarrow z = T - t$

$$\Rightarrow B(t,T) = e^{-a(T-t)} \int_0^{T-t} e^{az}\, dz = \frac{e^{-a(T-t)}}{a} \left(e^{a(T-t)} - 1 \right)$$

$$= \frac{1 - e^{-a(T-t)}}{a}$$

Risolviamo la ODE per A

$$\dot{A}(t,T) = -\frac{\sigma^2}{2}\left(B(t,T)\right)^2 + \theta(t)\,B(t,T)$$

$$\Rightarrow \int_t^T \dot{A}(s,T)\,ds = -\frac{\sigma^2}{2}\int_t^T \left(B(s,T)\right)^2 ds + \int_t^T \theta(s)\,B(s,T)\,ds$$

$$\Rightarrow \underbrace{A(T,T)}_{\shortparallel \atop 0} - A(t,T) = -\frac{\sigma^2}{2}\int_t^T \left[\frac{1-e^{-a(T-s)}}{a}\right]^2 ds + \int_t^T \theta(s)\cdot\frac{1-e^{-a(T-s)}}{a}\,ds$$

$$\Rightarrow A(t,T) = \frac{\sigma^2}{2a^2}\int_t^T \left(1+e^{-2a(T-s)}-2e^{-a(T-s)}\right)ds +$$

$$-\frac{1}{a}\int_t^T \theta(s)\left[1-e^{-a(T-s)}\right]ds = ?$$

Ricordiamo che

$$f(t,T) = -\frac{\partial}{\partial T}\left[\ln\left(p(t,T)\right)\right]$$

$$\overset{(t=0)}{\Rightarrow} f(0,T) = -\frac{\partial}{\partial T}\left[\ln\left(p(0,T)\right)\right]$$

$$\overset{\text{(modello affine)}}{=} -\frac{\partial}{\partial T}\left[\ln\left\{\exp\left(A(0,T) - r_0\cdot B(0,T)\right)\right\}\right]$$

$$= -\frac{\partial}{\partial T}\left[A(0,T) - r_0\cdot B(0,T)\right]$$

$$= -\frac{\partial A(0,T)}{\partial T} + r_0\,\frac{\partial B(0,T)}{\partial T}$$

Calcoli per I_1:

$T - s = z \Rightarrow s = T - z$

$\Rightarrow ds = -dz$

$s = 0 \rightarrow z = T$

$s = T \rightarrow z = 0$

$$\Rightarrow \frac{\partial}{\partial T}\left[\int_T^0 -\left(1 - e^{-az}\right)^2 dz\right] = \frac{\partial}{\partial T}\left[\int_0^T \left(1 - e^{-az}\right)^2 dz\right]$$

$$= \frac{\partial}{\partial T}\left[\int_0^T \left(1 + e^{-2az} - 2e^{-az}\right) dz\right]$$

$$= \frac{\partial}{\partial T}\left[T + \int_0^T \frac{-2a e^{-2az}}{-2a} dz + 2\int_0^T \frac{-a e^{-az}}{a} dz\right]$$

$$= \frac{\partial}{\partial T}\left[T - \frac{1}{2a}\left(e^{-2aT} - 1\right) + \frac{2}{a}\left(e^{-aT} - 1\right)\right]$$

$$= 1 + \frac{1}{\cancel{2a}}\left(+\cancel{2a}\right)e^{-2aT} + \frac{2}{\cancel{a}}\left(-\cancel{a}e^{-aT}\right)$$

$$= 1 + e^{-2aT} - 2e^{-aT} = \left(1 - e^{-aT}\right)^2$$

Calcoli per I_2:

$$\frac{\partial}{\partial T}\left[\int_0^T \Theta(s)\left(1 - e^{-a(T-s)}\right) ds\right]$$

$$= \int_0^T \frac{\partial}{\partial T}\left[\Theta(s)\left(1 - e^{-a(T-s)}\right) ds\right]$$

$$= \int_0^T \Theta(s) \cdot a\, e^{-a(T-s)} ds$$

$$= r_0 \frac{\partial}{\partial T}\left[\frac{1-e^{-aT}}{a}\right] - \frac{\partial}{\partial T}\left[\frac{\sigma^2}{2a^2}\int_0^T \left(1-e^{-a(T-s)}\right)^2 ds + \right.$$

$$\left. - \frac{1}{a}\int_0^T \theta(s)\left(1-e^{-a(T-s)}\right) ds\right]$$

$$= \frac{r_0}{\cancel{a}}\cdot \cancel{a}\, e^{-aT} - \frac{\sigma^2}{2a^2}\frac{\partial}{\partial T}\left[\int_0^T \left(1-e^{-a(T-s)}\right)^2 ds\right] +$$

$$+ \frac{1}{a}\frac{\partial}{\partial T}\left[\int_0^T \theta(s)\left(1-e^{-a(T-s)}\right) ds\right]$$

$$= r_0 e^{-aT} - \frac{\sigma^2}{2a^2}\left(1-e^{-aT}\right)^2 + \frac{1}{\cancel{a}}\cdot \cancel{a}\int_0^T \theta(s) e^{-a(T-s)} ds$$

$\downarrow$

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$$\Rightarrow f(0,T) = r_0 e^{-aT} + \int_0^T \theta(s) e^{-a(T-s)} ds - \frac{\sigma^2}{2a^2}\left(1-e^{-aT}\right)^2 \quad (*)$$

Sia $g(t) := \dfrac{\sigma^2}{2a^2}\left(1-e^{-at}\right)^2$

$$\Rightarrow \dot{g}(t) = \frac{\sigma^2}{\cancel{2}a^{\cancel{2}}}\,\cancel{2}\left(1-e^{-at}\right)\left(+\cancel{a}\cdot e^{-at}\right) = \frac{\sigma^2}{a} e^{-at}\left(1-e^{-at}\right)$$

Sia inoltre $y(t)$ t.c. $\begin{cases} \dot{y}(t) + a\cdot y(t) = \theta(t) & (\#) \\ y(0) = r_0 \end{cases}$

$$\Rightarrow y(t) = e^{-G(t)}\left[r_0 + \int_0^t \theta(s)\, e^{G(s)} ds\right], \quad G(t) = \int_0^t a\, dt$$

ossia:

$$G(t) = a \cdot t \Rightarrow y(t) = e^{-at}\left[r_0 + \int_0^t \theta(s)\, e^{as}\, ds\right]$$

$$= r_0\, e^{-at} + \int_0^t \theta(s)\, e^{-a(t-s)}\, ds \quad (**)$$

Dal confronto tra (*) e (**) e per definizione di $g(t)$, si ricava

$$f(0,T) = y(T) + g(T)$$

$$\Rightarrow y(T) = f(0,T) - g(T)$$

Inoltre, per definizione,

$$\dot{y}(T) + a \cdot y(T) = \theta(T)$$

$$\Rightarrow \theta(T) = \frac{\partial}{\partial T}\left[f(0,T) - g(T)\right] + a \cdot \left[f(0,T) - g(T)\right]$$

$$= \underbrace{f_T(0,T)}_{\text{derivata rispetto a } T} - \dot{g}(T) + a \cdot f(0,T) - a \cdot g(T)$$

$$= \left(f_T(0,T) + a \cdot f(0,T)\right) - \frac{\sigma^2}{a} e^{-aT}\left(1 - e^{-aT}\right) +$$

$$- \cancel{a} \cdot \frac{\sigma^2}{2a^2}\left(1 - e^{-aT}\right)^2$$

$$\Rightarrow \theta(T) = \left[f_T(0,T) + a \cdot f(0,T)\right] - \frac{\sigma^2}{a} e^{-aT}\left(1 - e^{-aT}\right) +$$

$$- \frac{\sigma^2}{2a}\left(1 - e^{-aT}\right)^2$$

⟹ abbiamo dimostrato il seguente risultato:

PROP.

Sia $p^*(0,T)$, $\in C^2(\mathbb{R}^+)$, $T>0$ una curva bond osservata

Allora, $$\Theta(T) = \frac{\partial f(0,T)}{\partial T} + a \cdot f(0,T) + \frac{\sigma^2}{2a}\left(1-e^{-2aT}\right)$$

induce una struttura a termine affine tale che,

$\forall T \geq 0$, $p(0,T) = p^*(0,T)$.

OSS: Determinato Θ e fissati i parametri, abbiamo determinato la misura martingala Q.

⟹ possiamo calcolare il prezzo teorico del bond:

$$p(t,T) = \frac{p^*(0,T)}{p(0,T)} \cdot \exp\left\{ f^*(0,T) \cdot B(t,T) - \frac{\sigma^2}{4a} B^2(t,T)\left(1-e^{-2at}\right) - B(t,T)\, r_t \right\}.$$

ESERCIZIO!