In generale

$$\begin{cases} \dfrac{dS_t}{S_t} = \mu(t)\,dt + \sqrt{V_t}\,dW_t^{(1)} \\[2ex] dV_t = \alpha(t,S,V)\,dt + \beta(t,S,V)\,dW_t^{(2)} \end{cases}$$

con $\langle dW_t^{(1)}, dW_t^{(2)} \rangle = \rho\,dt$

OSS:

In BS un solo MB $\Rightarrow$ COMPLETEZZA

$\Rightarrow$ Δ-hedging

In SV: 2 MB correlati

$\Rightarrow$ come si costruisce il ptf di copertura risk-free?

$$\underbrace{\Pi}_{\text{ptf}} = \underbrace{F}_{\text{Derivato 1}} - \underbrace{\Delta}_{\text{quote di underlying}} \cdot \underbrace{S}_{\text{underlying}} - \underbrace{\Delta_1}_{\text{quote di derivato 2}} \underbrace{F_1}_{\text{derivato 2}}$$

NB: $F = F(t,S,V)$, $F_1 = F_1(t,S,V)$

$$\Rightarrow d\Pi = dF - \Delta dS - \Delta_1 dF_1$$

RECALL:

$$dF \overset{(ITO)}{=} \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S}dS + \frac{\partial F}{\partial V}dV + $$
$$+ \frac{1}{2}\left(\frac{\partial^2 F}{\partial S^2}\langle dS, dS\rangle + 2\frac{\partial^2 F}{\partial S\partial V}\langle dS, dV\rangle + \right.$$
$$\left. + \frac{\partial^2 F}{\partial V^2}\langle dV, dV\rangle\right)$$

$$\Rightarrow d\Pi = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S}\left[S\cdot\mu(t)dt + S\sqrt{V_t}\,dW_t^{(1)}\right] +$$
$$+ \frac{\partial F}{\partial V}\left[\alpha\, dt + \beta\, dW_t^{(2)}\right] +$$
$$\underbrace{+ \frac{1}{2}\left(\frac{\partial^2 F}{\partial S^2}S^2 V_t + 2\frac{\partial^2 F}{\partial S\partial V}S\beta\sqrt{V}\rho + \frac{\partial^2 F}{\partial V^2}\beta^2\right)dt}_{A(F)} +$$
$$- \Delta\left[S\cdot\mu(t)dt + S\sqrt{V_t}\,dW_t^{(1)}\right] +$$
$$- \Delta_1\left[\frac{\partial F_1}{\partial t}dt + \frac{\partial F_1}{\partial S}\left[S\cdot\mu(t)dt + S\cdot\sqrt{V_t}\,dW_t^{(1)}\right] + \right.$$
$$+ \frac{\partial F_1}{\partial V}\left[\alpha\, dt + \beta\, dW_t^{(2)}\right] +$$
$$\left. \underbrace{+ \frac{1}{2}\left(\frac{\partial^2 F_1}{\partial S^2}S^2 V_t + 2\frac{\partial^2 F_1}{\partial S\partial V}S\beta\sqrt{V_t}\rho + \frac{\partial^2 F_1}{\partial V^2}\beta^2\right)dt}_{A(F_1)}\right]$$

[Ta]

$$\Rightarrow d\Pi_t = \left[\frac{\partial F}{\partial t} + S\mu \frac{\partial F}{\partial S} + \alpha \frac{\partial F}{\partial V} + A(F) - \Delta S\mu + \right.$$

$$\left. - \Delta_1 \left(\frac{\partial F_1}{\partial t} + S\mu \frac{\partial F_1}{\partial S} + \alpha \frac{\partial F_1}{\partial V} + A(F_1) \right) \right] dt +$$

$$+ \left[\frac{\partial F}{\partial S} S\sqrt{V_t} - \Delta S\sqrt{V_t} - \Delta_1 S\sqrt{V_t} \frac{\partial F_1}{\partial S} \right] dW_t^{(1)} +$$

$$+ \left[\beta \frac{\partial F}{\partial V} - \Delta_1 \beta \frac{\partial F_1}{\partial V} \right] dW_t^{(2)} \qquad (\bullet)$$

PTF RISK-FREE:

$$\begin{cases} \frac{\partial F}{\partial S} \cancel{S\sqrt{V_t}} - \Delta \cancel{S\sqrt{V_t}} - \Delta_1 \cancel{S\sqrt{V_t}} \frac{\partial F_1}{\partial S} = 0 \\ \cancel{\beta} \frac{\partial F}{\partial V} - \Delta_1 \cancel{\beta} \frac{\partial F_1}{\partial V} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{\partial F}{\partial S} - \Delta - \Delta_1 \frac{\partial F_1}{\partial S} = 0 \\ \frac{\partial F}{\partial V} - \Delta_1 \frac{\partial F_1}{\partial V} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \Delta = \frac{\partial F}{\partial S} - \frac{\frac{\partial F_1}{\partial S} \frac{\partial F}{\partial V}}{\frac{\partial F_1}{\partial V}} \\ \Delta_1 = \frac{\frac{\partial F}{\partial V}}{\frac{\partial F_1}{\partial V}} \end{cases}$$

$$\Rightarrow d\Pi = \left[\frac{\partial F}{\partial t} + S\mu\frac{\partial F}{\partial S} + \alpha\frac{\partial F}{\partial V} + A(F) + \right.$$

$$\left. - \Delta S\mu - \Delta_1\left(\frac{\partial F_1}{\partial t} + S\mu\frac{\partial F_1}{\partial S} + \alpha\frac{\partial F_1}{\partial V} + A(F_1)\right)\right] dt$$

D' altra parte,

$$d\Pi_t = r_t \Pi_t \, dt \quad (\text{Risk-Free})$$

$$\Rightarrow r_t\left(F - \Delta S - \Delta_1 F_1\right)\cancel{dt} = \left[\frac{\partial F}{\partial t} + S\mu\frac{\partial F}{\partial S} + \alpha\frac{\partial F}{\partial V} + A(F) + \right.$$

$$\left. - \Delta S\mu - \Delta_1\left(\frac{\partial F_1}{\partial t} + S\mu\frac{\partial F_1}{\partial S} + \alpha\frac{\partial F_1}{\partial V} + A(F_1)\right)\right]\cancel{dt}$$

$$\Rightarrow rF - Sr\left(\frac{\partial F}{\partial S} - \frac{\frac{\partial F_1}{\partial S}\frac{\partial F}{\partial V}}{\frac{\partial F_1}{\partial V}}\right) - rF_1\frac{\frac{\partial F}{\partial V}}{\frac{\partial F_1}{\partial V_1}} =$$

$$= \frac{\partial F}{\partial t} + S\mu\frac{\partial F}{\partial S} + \alpha\frac{\partial F}{\partial V} + A(F) - S\mu\left(\frac{\partial F}{\partial S} - \frac{\frac{\partial F_1}{\partial S}\frac{\partial F}{\partial V}}{\frac{\partial F_1}{\partial V}}\right) +$$

$$- \frac{\frac{\partial F}{\partial V}}{\frac{\partial F_1}{\partial V}}\left(\frac{\partial F_1}{\partial t} + S\mu\frac{\partial F_1}{\partial S} + \alpha\frac{\partial F_1}{\partial V} + A(F_1)\right)$$

$$\cancel{\Rightarrow \frac{\frac{\partial F}{\partial V}}{\frac{\partial F_1}{\partial V}}\left(\frac{\partial F_1}{\partial t} + S\mu\frac{\partial F_1}{\partial S} + \alpha\frac{\partial F_1}{\partial V} + A(F_1)\right) - rF_1\frac{\frac{\partial F}{\partial V}}{\frac{\partial F}{\partial V_1}} =}$$

$$\cancel{= \frac{\partial F}{\partial t} + S\mu\frac{\partial F}{\partial S} + \alpha\frac{\partial F}{\partial V} + A(F) - S\mu\frac{\partial F}{\partial S} + S\mu\frac{\frac{\partial F_1}{\partial S}\frac{\partial F}{\partial V}}{\frac{\partial F_1}{\partial V}} - rF}$$

$$\Rightarrow \frac{\frac{\partial F}{\partial V}}{\frac{\partial F_1}{\partial V}}\left(\frac{\partial F_1}{\partial t} + S\mu\frac{\partial F_1}{\partial S} + \alpha\frac{\partial F_1}{\partial V} + A(F_1)\right) - S\mu\frac{\partial F_1}{\partial S}\,\frac{\frac{\partial F}{\partial V}}{\frac{\partial F_1}{\partial V}} +$$

$$- rF_1\,\frac{\frac{\partial F}{\partial V}}{\frac{\partial F_1}{\partial V}} + Sr\frac{\partial F_1}{\partial S}\,\frac{\frac{\partial F}{\partial V}}{\frac{\partial F_1}{\partial V}} =$$

$$= \frac{\partial F}{\partial t} + \cancel{S\mu\frac{\partial F}{\partial S}} + \alpha\frac{\partial F}{\partial V} + A(F) - \cancel{S\mu\frac{\partial F}{\partial S}} + Sr\frac{\partial F}{\partial S} - rF$$

$$\Rightarrow \frac{\frac{\partial F}{\partial V}}{\frac{\partial F_1}{\partial V}}\left[\frac{\partial F_1}{\partial t} + \cancel{S\mu\frac{\partial F_1}{\partial S}} + \alpha\frac{\partial F_1}{\partial V} + A(F_1) - \cancel{S\mu\frac{\partial F_1}{\partial S}} + Sr\frac{\partial F_1}{\partial S} - rF_1\right] =$$

$$= \frac{\partial F}{\partial t} + Sr\frac{\partial F}{\partial S} + \alpha\frac{\partial F}{\partial V} + A(F) - rF$$

$$\Rightarrow \frac{\frac{\partial F_1}{\partial t} + \alpha\frac{\partial F_1}{\partial V} + Sr\frac{\partial F_1}{\partial S} + A(F_1) - rF_1}{\frac{\partial F_1}{\partial V}} =$$

$$= \frac{\frac{\partial F}{\partial t} + \alpha\frac{\partial F}{\partial V} + Sr\frac{\partial F}{\partial S} + A(F) - rF}{\frac{\partial F}{\partial V}}, \quad \forall F, F_1 \quad (\nexists \text{ arbitrag}$$

$$\Rightarrow \exists \lambda = \lambda(t, S, v) \text{ t.c. } \forall F = F(t, S, v)$$

$$\frac{\partial F}{\partial t} + Sr\frac{\partial F}{\partial S} + \alpha\frac{\partial F}{\partial V} + A(F) - rF = \lambda(t, S, v)\frac{\partial F}{\partial V} \qquad (\bullet\bullet)$$

(*) Se (di BS) $\Delta_1 = 0$ e $\Delta = \frac{\partial F}{\partial S}$ (vogliamo un solo derivato)

$$\Rightarrow d\Pi^{BS} = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S}dS + \frac{\partial F}{\partial V}dV +$$
$$+ \frac{1}{2}\left(\frac{\partial^2 F}{\partial S^2}\langle dS, dS\rangle + 2\frac{\partial^2 F}{\partial S \partial V}\langle dS, dV\rangle + \frac{\partial^2 F}{\partial V^2}\langle dV, dV\rangle\right)$$

(RISK-FREE)

$$\overset{(\bullet)}{=} \left(\frac{\partial F}{\partial t} + \cancel{\mu S \frac{\partial F}{\partial S}} + \alpha\frac{\partial F}{\partial V} + A(F) - \cancel{\frac{\partial F}{\partial S} S\mu}\right)dt +$$
$$+ \left(\cancel{\frac{\partial F}{\partial S} S\sqrt{V}} - \cancel{S\sqrt{V}\frac{\partial F}{\partial S}}\right)dW_t^{(1)} + \beta\frac{\partial F}{\partial V}dW_t^{(2)}$$

costruzione del ptf di copertura

$$= \left(\frac{\partial F}{\partial t} + \alpha\frac{\partial F}{\partial V} + A(F)\right)dt + \beta\frac{\partial F}{\partial V}dW_t^{(2)}$$

$$\Rightarrow d\Pi^{BS} = \left(rF + \lambda(t,S,V)\frac{\partial F}{\partial V} - Sr\frac{\partial F}{\partial S}\right)dt + \beta\frac{\partial F}{\partial V}dW_t^{(2)}$$

$$\Rightarrow d\Pi^{BS} - r\Pi^{BS}dt = rF dt + \frac{\partial F}{\partial V}\left(\beta dW_t^{(2)} + \lambda(t,S,V)dt\right) +$$
$$- rS\frac{\partial F}{\partial S}dt - r\left(F - S\frac{\partial F}{\partial S}\right)dt$$

$$\Rightarrow d\Pi^{BS} - r\Pi^{BS}dt = \cancel{rF}dt + \frac{\partial F}{\partial V}\left(\lambda dt + \beta dW_t^{(2)}\right) +$$
$$- \cancel{rS\frac{\partial F}{\partial S}}dS - \cancel{rF}dt + \cancel{rS\frac{\partial F}{\partial S}}dt$$

$$\Rightarrow d\Pi_t^{BS} - r\Pi_t^{BS}dt = \frac{\partial F}{\partial V}\left(\lambda dt + \beta dW_t^{(2)}\right)$$

RENDIMENTO IN ECCESSO PER UNITÀ DI RISCHIO DI VOLATILITÀ DEL DERIVATO STESSO.

⇒ Fissato λ, si deve completare il mercato :

$$\begin{cases} \dfrac{\partial F}{\partial t} + rS\dfrac{\partial F}{\partial S} + \left[\alpha - \lambda(t,S,v)\right]\dfrac{\partial F}{\partial v} + \dfrac{1}{2} S^2 v \dfrac{\partial^2 F}{\partial S^2} + \\ \quad + S\beta\sqrt{v_t}\,\rho \dfrac{\partial^2 F}{\partial S \partial v} + \dfrac{1}{2}\beta^2 \dfrac{\partial^2 F}{\partial v^2} = rF, \\ \\ F(T,S,v) = \phi(S) \ \text{(noto)} \end{cases}$$

NOTA: $\lambda = \lambda(t,S,v)$ è detta MARKET PRICE OF VOLATILITY RISK

↳ ci dice quanto dei ritorni attesi è spiegato attraverso il rischio di v in APT. (*)

See pag. 74

OSS:

Nei mercati, $\lambda < 0$

⇒ investitori portati a pagare un prezzo più alto per i titoli che hanno payoff elevato, in corrispondenza degli stati di natura in cui la volatilità è più alta.

OSS: $F(t,S,v) = e^{-r(T-t)}\, \mathbb{E}^{\mathbb{Q}}\left[\phi(S_T^*)\right]$, con

$$\begin{cases} \dfrac{dS_t^*}{S_t^*} = r\,dt + \sqrt{v_t^*}\, dW_t^{1,*} \\ dv_t^* = \left[\alpha(t,S^*,v^*) - \lambda(t,S^*,v^*)\right]dt + \beta(t,S^*,v^*)\, dW_t^{2,*} \\ \langle dW_t^{1,*}, dW_t^{2,*}\rangle = \rho(t,S^*,v^*)\,dt \end{cases}$$

→ RISK-NEUTRAL DYNAMICS (basta a determinare i prezzi dei titoli derivati).