Ricapitolando:

t	T	S
vendo T-bond allo scoperto $\Downarrow$ incasso $p(t,T)$ + compro $\frac{p(t,T)}{p(t,S)}$	pago 1	Incasso $\frac{p(t,T)}{p(t,S)}$

In termini di tassi: sia $L(t,T,S)$ il tasso a capitalizzazione semplice, valutato in t, per il periodo $[T,S]$

$\Rightarrow$ in S realizzo $1 + L(t,T,S)(S-T)$

$\Rightarrow$ voglio che

$$1 + L(t,T,S)(S-T) = \frac{p(t,T)}{p(t,S)}$$

$$\Rightarrow L(t,T,S) = \frac{1}{(S-T)} \frac{p(t,T) - p(t,S)}{p(t,S)}$$

$$\Rightarrow \lim_{S \to T} L(t,T,S) = \lim_{S \to T} \frac{p(t,T) - p(t,S)}{(S-T)\,p(t,S)}$$

$$= -\frac{\partial [\ln(p(t,T))]}{\partial T} =: \underbrace{f(t,T)}_{\downarrow}$$

TASSO ISTANTANEO IN t PER UN INVESTIMENTO ISTANTANEO IN T.

$$\Rightarrow p(t,T) = \exp\left\{ -\int_t^T f(t,s)\,ds \right\}$$

OSS:

Sia $T=t \Rightarrow f(t,t) = r_t \rightsquigarrow$ TASSO SPOT

$\Rightarrow$ ci sono 3 grandezze (p, f, r) legate tra loro:

$$\begin{tikzcd}
p(t,T) \arrow[rr, leftrightarrow] \arrow[dr, bend right] & & f(t,T) \arrow[dl, bend left] \\
& r_t \arrow[ul, "?"'] \arrow[ur, "?"] &
\end{tikzcd}$$

DOMANDA: lo schema commuta?

$\Rightarrow$ vogliamo determinare una dinamica stocastica delle grandezze dello schema SENZA OPPORTUNITÀ DI ARBITRAGGIO

DINAMICA PER TASSO ISTANTANEO:

$$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\,dW_t$$ $\to$ specificheremo in seguito (vedi pag 13)

Sia $y(t,T) := -\int_t^T f(t,s)\,ds$

$$\Rightarrow p(t,T) = \exp\{ y(t,T) \}$$

DINAMICA PER PREZZO:

Inoltre: integrale deterministico

$$d y(t,T) = f(t,t)\,dt - \int_t^T d f(t,s)\,ds$$

NB: In questa dinamica abbiamo integrali doppi, poiché t compare nella funzione e come estremo di integrazione

$$\Rightarrow d y(t,T) = r_t\,dt - \int_t^T \left[\alpha(t,s)\,dt + \sigma(t,s)\,dW_t\right] ds$$

$$\overset{(FUBINI)}{=} r_t\,dt + \underbrace{\left[-\int_t^T \alpha(t,s)\,ds\right]}_{=: A(t,T)} dt + \underbrace{\left[-\int_t^T \sigma(t,s)\,ds\right]}_{=: S(t,T)} dW_t$$

$$\Rightarrow d y(t,T) = r_t\,dt + A(t,T)\,dt + S(t,T)\,dW_t$$

Applichiamo Itô a $f(y) = e^{y}$

$$\Rightarrow d g = \frac{\partial f}{\partial t}\,dt + \frac{\partial f}{\partial y}\,dy + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}\langle dy, dy\rangle$$

poiché $\dfrac{\partial f}{\partial t} = 0$, $\dfrac{\partial f}{\partial y} = \dfrac{\partial^2 f}{\partial y^2} = g$, allora

$$\underset{\displaystyle d\left(\exp\{y(t,T)\}\right)}{\overset{\displaystyle d p(t,T)}{\shortparallel}} = p(t,T)\,dy(t,T) + \frac{1}{2}p(t,T)\langle dy(t,T), dy(t,T)\rangle$$

$$= p(t,T)\left[r_t\,dt + A(t,T)\,dt + S(t,T)\,dW_t\right] +$$

$$+ \frac{1}{2}p(t,T)\langle r_t\,dt + A(t,T)\,dt + S(t,T)\,dW_t, r_t\,dt + A(t,T)\,dt + S(t,T)\,d$$

12b

$$= p(t,T)\left[r_t\, dt + A(t,T)\, dt + S(t,T)\, dW_t + \frac{1}{2}\left(S(t,T)\right)^2 dt \right]$$

$$\Rightarrow \frac{dp(t,T)}{p(t,T)} = \left[r_t + A(t,T) + \frac{1}{2}\left(S(t,T)\right)^2 \right] dt + S(t,T)\, dW_t \quad (\mathbb{P})$$

RECALL: $p(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t \right]$

DINAMICA PER SPOT RATE:

HP: r_t proc. di Markov $\Rightarrow p(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_s ds} \middle| r_t \right] =: F(t, r_t)$ in $\mathbb{Q}$

$\hookrightarrow$ fissato r_t, incrementi indipendenti (dipendono solo da W_t)

$$\Rightarrow \boxed{dr_t = \mu(t, r_t)\, dt + \sigma(t, r_t)\, dW_t}$$

1° TENTATIVO: costruzione di ptf autofinanziante

$\Rightarrow$ NO, perché r_t non ha mercato.

$\Rightarrow$ 2° TENTATIVO: consideriamo due bonds, con scadenze differenti e costruiamo un portafoglio di titoli. (in $\mathbb{P}$)

$$p(t,T) =: F^T(t, r_t) \quad , \quad p(t,S) = F^S(t, r_t)$$

Siano u^T, u^S le quote di ricchezza investite nei due titoli:

$$\frac{dV_t}{V_t} = u^T \frac{dF_t^T}{F_t^T} + u^S \frac{dF_t^S}{F_t^S}$$

$\Rightarrow$ dobbiamo conoscere le dinamiche di F^T, F^S: $\underline{YLCC}$

$$\overset{(ITO)}{\Rightarrow} dF^x = \frac{\partial F^x}{\partial t} dt + \frac{\partial F^x}{\partial r} dr_t + \frac{1}{2} \frac{\partial^2 F^x}{\partial r^2} \langle dr_t, dr_t \rangle$$

$$= \frac{\partial F^x}{\partial t} dt + \frac{\partial F^x}{\partial r} \left[\mu^x(t, r_t) dt + \sigma^x(t, r_t) dW_t \right] +$$

$$+ \frac{1}{2} \frac{\partial^2 F^x}{\partial r^2} \left(\sigma^x(t, r_t) \right)^2 dt \quad , \quad \forall x \in \{T, S\}$$

$$\Rightarrow \frac{dF_t^x}{F_t^x} = \overbrace{\frac{1}{F_t^x}\left[\frac{\partial F^x}{\partial t} + \mu^x(t, r_t) \frac{\partial F^x}{\partial r} + \frac{1}{2} \left(\sigma^x(t, r_t) \right)^2 \right]}^{=: \alpha^x(t, T_t)} dt +$$

$$+ \underbrace{\frac{1}{F_t^x} \sigma^x(t, r_t) \frac{\partial F^x}{\partial r}}_{=: \sigma^x(t, T_t)} dW_t \quad , \quad \forall x \in \{T, S\} \; (**)$$

$$\Rightarrow \begin{cases} \dfrac{dF_t^T}{F_t^T} = \alpha^T(t, T) dt + \sigma^T(t, T) dW_t \\[2ex] \dfrac{dF_t^S}{F_t^S} = \alpha^S(t, T) dt + \sigma^S(t, T) dW_t \end{cases}$$

$$\Rightarrow \frac{dV_t}{V_t} = u^T \left[\alpha^T(t, T) dt + \sigma^T(t, T) dW_t \right] +$$

$$+ u^S \left[\alpha^S(t, T) dt + \sigma^S(t, T) dW_t \right]$$

$$\Rightarrow \frac{dV_t}{V_t} = \left[u^T \alpha^T(t, T) + u^S \alpha^S(t, T) \right] dt + \left[u^T \sigma^T(t, T) + u^S \sigma^S(t, T) \right] dW$$

Imponiamo che la ricchezza totale da investire sia unitaria e che il ptf sia risk-free: 11a

$$\begin{cases} u^T + u^S = 1 \\ u^T\sigma^T + u^S\sigma^S = 0 \end{cases}$$

$$\Rightarrow \begin{cases} u^T = 1 - u^S \\ (1-u^S)\sigma^T + u^S\sigma^S = 0 \end{cases} \Rightarrow \begin{cases} u^T = 1 - u^S \\ u^S\left(\sigma^S - \sigma^T\right) + \sigma^T = 0 \end{cases}$$

$$\Rightarrow \begin{cases} u^T = 1 - \dfrac{\sigma^T}{\sigma^T - \sigma^S} = -\dfrac{\sigma^S}{\sigma^T - \sigma^S} \\ u^S = \dfrac{\sigma^T}{\sigma^T - \sigma^S} \end{cases}$$

$$\Rightarrow \text{vale } \frac{dV_t}{V_t} = \frac{dB_t}{B_t} \quad (\text{ptf risk-free})$$

$$\Rightarrow u^T\alpha^T + u^S\alpha^S = r_t$$

$$\Rightarrow -\frac{\sigma^S}{\sigma^T - \sigma^S}\alpha^T + \frac{\sigma^T}{\sigma^T - \sigma^S}\alpha^S = r_t$$

$$\Rightarrow \frac{-\sigma^S\alpha^T + \sigma^T\alpha^S}{\sigma^T - \sigma^S} = r_t$$

$$\Rightarrow \sigma^T\alpha^S - \sigma^S\alpha^T = r_t\left(\sigma^T - \sigma^S\right)$$

$$\Rightarrow \sigma^T\alpha^S - r_t\sigma^T = \sigma^S\alpha^T - r_t\sigma^S$$

$$\Rightarrow \sigma^T\left(\alpha^S - r_t\right) = \sigma^S\left(\alpha^T - r_t\right)$$

$$\Rightarrow \frac{\alpha^S - r_t}{\sigma^S} = \frac{\alpha^T - r_t}{\sigma^T} =: \lambda(t, r_t) \rightarrow \boxed{\text{PREMIO AL RISCHIO}}$$

Si ricava, $\forall x \in \{S,T\}$,

$$\alpha^x(t,r_t) = \sigma^x(t,r_t)\,\lambda(t,r_t) + r_t$$

(**)
$\Rightarrow$ otteniamo la seguente PDE: imponiamo uguaglianza delle relazioni di $\alpha^x(t,T)$

$$\frac{1}{F^x}\left[\frac{\partial F^x}{\partial t} + \mu^x(t,r_t)\frac{\partial F^x}{\partial r} + \frac{1}{2}\left(\sigma^x(t,r_t)\right)^2\frac{\partial^2 F^x}{\partial r^2}\right] = \left(\frac{1}{F^x}\sigma^x(t,r_t)\frac{\partial F^x}{\partial r}\right)\cdot\lambda + r$$

$$\Rightarrow \frac{\partial F^x}{\partial t} + \mu^x(t,r_t)\frac{\partial F^x}{\partial r} + \frac{1}{2}\left(\sigma^x(t,r_t)\right)^2\frac{\partial^2 F^x}{\partial r^2} - \lambda\sigma^x(t,r_t)\frac{\partial F^x}{\partial r} - r_t F^x_t = 0$$

$$\Rightarrow \boxed{\begin{array}{l} \dfrac{\partial F^x}{\partial t} + \left[\mu^x(t,r_t) - \lambda(t,r_t)\,\sigma^x(t,r_t)\right]\dfrac{\partial F^x}{\partial r} + \dfrac{1}{2}\left(\sigma^x(t,r_t)\right)^2\dfrac{\partial^2 F^x}{\partial r^2} - r_t F_t = 0 \\ \text{con condizione terminale } F^x(T,r_T) = 1 \end{array}}$$

↳ EQ. FONDAMENTALE DELLA STRUTTURA A TERMINE

Inoltre:

$\nexists$ arbitraggi $\Rightarrow \exists\, \mathbb{Q} \sim \mathbb{P}$:

$$F(t,r_t) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left\{-\int_t^T r_s\,ds\right\}\,\middle|\,\mathcal{F}_t\right],$$

dove la dinamica di r_t sotto $\mathbb{Q}$ è

$$dr_t = \left[\mu(t,r_t) - \lambda(t,r_t)\,\sigma(t,r_t)\right]dt + \sigma(t,r_t)\,d\tilde{W}_t^{\mathbb{Q}}$$

$$\left(\text{i.e., } \tilde{W}_t = W_t + \lambda(t,r_t)\,dt\right).$$