COMPLEMENTS OF NEWTON MECHANICS

NEWTON'S LAWS

1st law: Unless acted upon by a force, a particle will maintain a straight line motion with constant inertial velocity

2nd law: Let $\underline{F}$ be the sum of all forces acting on a particle having mass m and inertial position vector $\underline{r}$. Assuming that I is an inertial reference frame,

$$\underline{F} = \frac{{}^{I}d}{dt}\left(m \dot{\underline{r}}\right)$$

or, letting $\underline{p} := m\dot{\underline{r}}$ (linear momentum)

$$\underline{F} = \frac{{}^{I}d}{dt}\left(\underline{p}\right)$$

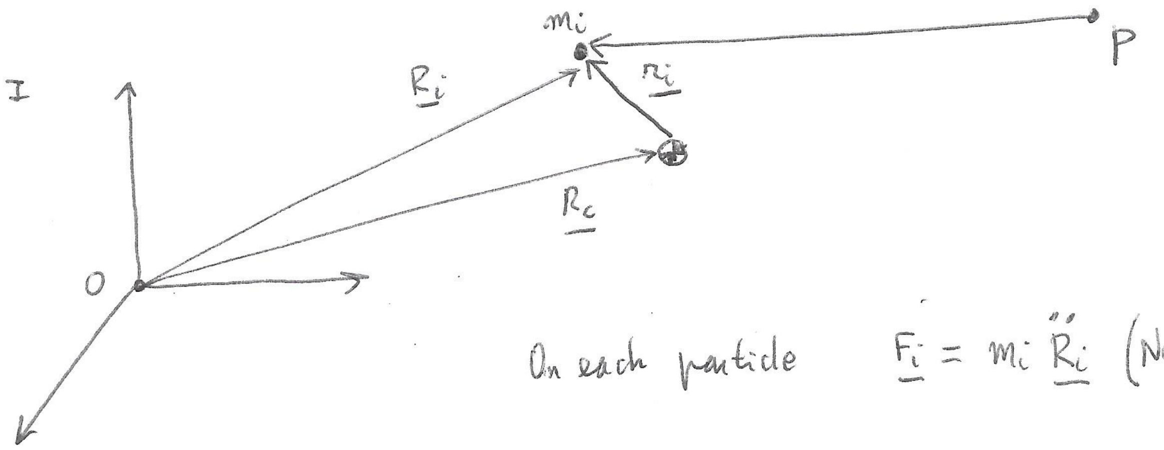
3rd law: If m_1 is exerting a force $\underline{F_{21}}$ on m_2, then m_1 experiences the force $\underline{F_{12}}$ due to interaction with m_2, and

$$\underline{F_{21}} + \underline{F_{12}} = \underline{0}$$

It is apparent that the 2nd law implies the 1st law

SYSTEMS OF PARTICLES

N particle with constant masses $\{m_i\}$; I = inertial frame



On each particle $\quad \underline{F_i} = m_i \underline{\ddot{R}_i}$ (Newton's law)

Letting
$$\begin{cases} \underline{F_{iE}} = \text{external force on particle } i \text{ (sum)} \\ \underline{F_{ij}} = \text{force on particle } i \text{ exerted by particle } j \end{cases}$$

$$\underline{F_i} = \underline{F_{iE}} + \sum_{j=1}^{N} \underline{F_{ij}}$$ is the force on particle i

The overall force $\underline{F}$ is

$$\underline{F} = \sum_{i=1}^{N} \underline{F_i} = \sum_{i=1}^{N} \underline{F_{iE}} + \sum_{i,j=1}^{N} \underline{F_{ij}} \underset{\substack{\uparrow \\ \text{Newton 3rd law}}}{=} \sum_{i=1}^{N} \underline{F_{iE}}$$

Letting $M = \sum_{i=1}^{N} m_i$ be the overall mass. By definition the center of mass C is identified by $\underline{R_c}$ such that

$$M\underline{R_c} = \sum_{i=1}^{N} m_i \underline{R_i} \longrightarrow \underline{R_c} = \frac{1}{M} \sum_{i=1}^{N} m_i \underline{R_i}$$ CENTER OF MASS

If $m_i = \text{const.}\ \forall i$, then $\left[M\underline{\ddot{R}_c} = \sum m_i \underline{\ddot{R}_i} = \underline{F} \right]$ SUPERPARTICLE THEOREM

- Linear momentum

Let $\underline{p}_i = m_i \dot{\underline{R}}_i$ denote the linear momentum of a single particle

$$\underline{p} = \sum_{i=1}^{N} \underline{p}_i = \sum m_i \dot{\underline{R}}_i$$

TOTAL LINEAR MOMENTUM of the system of masses

Due to the definition of center of mass,

$$\underline{R}_c = \frac{\sum_{i=1}^{N} m_i \underline{R}_i}{M}$$

one obtains $\quad \underline{p} = M \dot{\underline{R}}_c$

i.e. the total angular momentum equals that of a particle of mass M that moves like the center of mass. Due to this, the superparticle theorem may be rewritten as

$$\underline{F} = \frac{d\underline{p}}{dt}$$

where the derivative is taken with respect to an inertial frame

- Kinetic energy

The overall kinetic energy equals the sum of all the contributions:

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\underline{R}}_i \cdot \dot{\underline{R}}_i \underset{\underline{R}_i = \underline{R}_c + \underline{r}_i}{=} \frac{M}{2} \dot{\underline{R}}_c \cdot \dot{\underline{R}}_c + \dot{\underline{R}}_c \sum_{i=1}^{N} m_i \dot{\underline{r}}_i + \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\underline{r}}_i \cdot \dot{\underline{r}}_i =$$

$$= \underbrace{\frac{1}{2} M \dot{\underline{R}}_c \cdot \dot{\underline{R}}_c}_{\text{Traslational energy}} + \underbrace{\frac{1}{2} \sum_{i=1}^{N} m_i \dot{\underline{r}}_i \cdot \dot{\underline{r}}_i}_{\text{Rotational and deformation energy}}$$

In the previous steps the definition of mass center was used ;

$$\sum_{i=1}^{N} m_i \underline{r}_i = \sum_{i=1}^{N} m_i \underline{R}_i - \sum_{i=1}^{N} m_i \underline{R}_c = \underline{0} \Rightarrow \sum_{i=1}^{N} m_i \dot{\underline{r}}_i = \underline{0}$$

The kinetic energy rate is

$$\frac{dT}{dt} = M\underline{\ddot{R}}_c \cdot \underline{\dot{R}}_c + \sum_{i=1}^{N} m_i \, \underline{\ddot{r}}_i \cdot \underline{\dot{r}}_i = \underline{F} \cdot \underline{\dot{R}}_c + \sum_{i=1}^{N} m_i \left(\underline{\ddot{R}}_i - \underline{\ddot{R}}_c\right) \cdot \underline{\dot{r}}_i =$$

$$= \underline{F} \cdot \underline{\dot{R}}_c + \sum_{i=1}^{N} \underline{F}_i \cdot \underline{\dot{r}}_i \qquad \text{because} \quad \sum_{i=1}^{N} m_i \underline{\dot{r}}_i = 0$$

(due to definition of mass center)

The overall change in kinetic energy is

$$T(t_2) - T(t_1) = \underbrace{\int_{t_1}^{t_2} \underline{F} \cdot \underline{\dot{R}}_c \, dt}_{\text{Translational work}} + \underbrace{\sum_{i=1}^{N} \int_{t_1}^{t_2} \underline{F}_i \cdot \underline{\dot{r}}_i \, dt}_{\text{rotation and deformation work}}$$

- <u>Mechanical energy</u>

If a force can be written in the form of a gradient of a potential, then the force is CONSERVATIVE, i.e.

$$\underline{F}_{i,c} = -\nabla_{\underline{R}_i} U \qquad \text{where} \qquad U = U(\underline{R}_1, .., \underline{R}_N) \text{ is a function of the coordinates of the particles only}$$

The force components are written in the same frame as $\underline{R}_i$, i.e.

$$\underline{F}_{i,c} = F_{i1}\hat{b}_1 + F_{i2}\hat{b}_2 + F_{i3}\hat{b}_3 \qquad \underline{R}_i = R_{i1}\hat{b}_1 + R_{i2}\hat{b}_2 + R_{i3}\hat{b}_3$$

whereas in a Cartesian frame, $\nabla_{\underline{R}_i}$ assumes the form

$$\nabla_{\underline{R}_i} = \hat{b}_1 \frac{\partial}{\partial R_{i1}} + \hat{b}_2 \frac{\partial}{\partial R_{i2}} + \hat{b}_3 \frac{\partial}{\partial R_{i3}}$$

In general, on each particle i the force includes two contributions

$$\underline{F_i} = \underline{F_{i,nc}} + \underline{F_{i,c}} \quad \text{where} \quad \begin{cases} \underline{F_{i,nc}} = \text{resultant of non-conservative forces} \\ \underline{F_{i,c}} = \text{resultant of conservative forces} \end{cases}$$

Due to the 3rd Newton law

$$\sum_{i=1}^{N} \underline{F_i} = \underline{F_{nc}} + \underline{F_c}$$

where $\underline{F_{nc}}$ and $\underline{F_c}$ are the sum of external forces, because the internal forces vanish while summing

The kinetic energy rate, $\frac{dT}{dt}$ can be rewritten as

$$\frac{dT}{dt} = \underline{F_{nc}} \cdot \underline{\dot{R}_c} + \underline{F_c} \cdot \underline{\dot{R}_c} + \sum_{i=1}^{N} \left\{ \underline{F_{i,nc}} \cdot \underline{\dot{r}_i} + \underline{F_{i,c}} \cdot \underline{\dot{r}_i} \right\}$$

Because $\underline{\dot{r}_i} = \underline{\dot{R}_i} - \underline{\dot{R}_c}$ one obtains

$$\frac{dT}{dt} = \underline{F_{nc}} \cdot \underline{\dot{R}_c} + \underline{F_c} \cdot \underline{\dot{R}_c} + \sum_{i=1}^{N} \left\{ \underline{F_{i,nc}} \cdot \underline{\dot{r}_i} + \underline{F_{i,c}} \cdot \underline{\dot{R}_i} \right\} - \sum_{i=1}^{N} \underline{F_{i,c}} \cdot \underline{\dot{R}_c}$$

$$= \underline{F_{nc}} \cdot \underline{\dot{R}_c} + \sum_{i=1}^{N} \underline{F_{i,nc}} \cdot \underline{\dot{r}_i} - \sum_{i=1}^{N} \nabla U_{\underline{R_i}} \cdot \underline{\dot{R}_i} =$$

$$\underline{F_c} \cdot \underline{\dot{R}_c} \equiv \sum_{i=1}^{N} \underline{F_{i,c}} \cdot \underline{\dot{R}_c} \qquad \sum_{i=1}^{N} \nabla U_{\underline{R_i}} \cdot \underline{\dot{R}_i} = \frac{dU}{dt}$$

$$= \underline{F_{nc}} \cdot \underline{\dot{R}_c} + \sum_{i=1}^{N} \underline{F_{i,nc}} \cdot \underline{\dot{r}_i} - \frac{dU}{dt} \longrightarrow$$

$$\rightarrow \frac{d\mathcal{E}}{dt} = \underline{F_{nc}} \cdot \underline{\dot{R}_c} + \sum_{i=1}^{N} \underline{F_{i,nc}} \cdot \underline{\dot{r}_i}$$

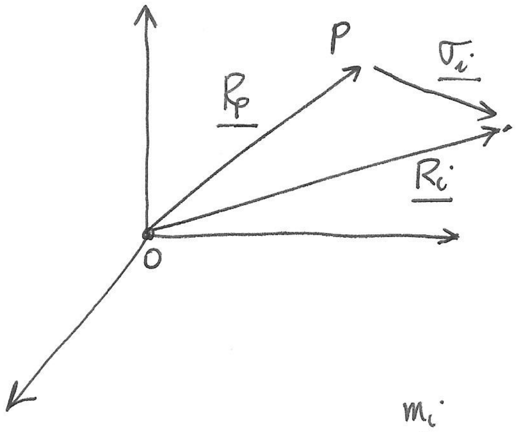
where $\mathcal{E} := T + U$ is the MECHANICAL ENERGY

The overall change in the mechanical energy is

$$\mathcal{E}(t_2)-\mathcal{E}(t_1)=\int_{t_1}^{t_2}\underline{F_{nc}}\cdot\underline{\dot{R}_c}\,dt+\sum_{i=1}^{N}\int_{t_1}^{t_2}\underline{F_{i,nc}}\cdot\underline{\dot{r}_i}\,dt$$

and is related uniquely to the action of non-conservative forces (e.g., friction). If the mechanical energy decreases due to dissipation, the TOTAL ENERGY = MECHANICAL EN. + INTERNAL ENERGY remains constant for an isolated system.
However, the previous relation is general for the mechanical energy of systems of particles in the presence of non-conservative forces (either external and internal).

- Angular momentum

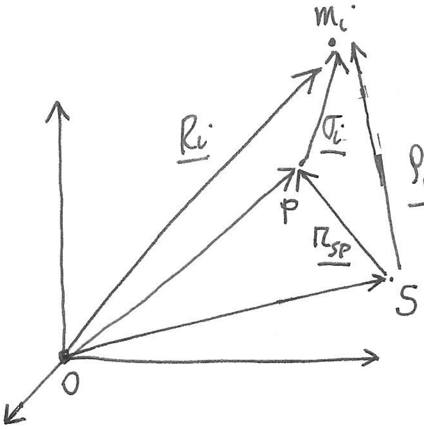


For each particle i, the angular momentum with respect to a generic point P is

$$\underline{H_{iP}}=\underline{\sigma_i}\times m_i\underline{\dot{R}_i}$$

The total angular momentum is

$$\underline{H_P}=\sum_{i=1}^{N}\underline{\sigma_i}\times m_i\underline{\dot{R}_i}$$



If another point S is chosen

$$\underline{H_S}=\sum_{i=1}^{N}\underline{\rho_i}\times m_i\underline{\dot{R}_i}=\sum_{i=1}^{N}\left(\underline{r_{SP}}+\underline{\sigma_i}\right)\times m_i\underline{\dot{R}_i}=$$

$$\left(\underline{\rho_i}=\underline{r_{SP}}+\underline{\sigma_i}\right)$$

$$=\underline{r_{SP}}\times M\underline{\dot{R}_c}+\underline{H_P}$$

$$\sum_{i=1}^{N}m_i\underline{\dot{R}_i}=M\underline{\dot{R}_c}$$ due to definition of mass center

The derivative of $\underline{H_P}$ is

$$\dot{\underline{H_P}} = \sum_{i=1}^{N} \dot{\underline{\sigma_i}} \times m_i (\dot{\underline{R_P}} + \dot{\underline{\sigma_i}}) + \sum_{i=1}^{N} \underline{\sigma_i} \times m_i \ddot{\underline{R_i}} =$$

$$\underline{F_i} = m_i \ddot{\underline{R_i}}$$

$$= \left\{ \sum_{i=1}^{N} m_i \dot{\underline{\sigma_i}} \right\} \times \dot{\underline{R_P}} + \underline{L_P}$$

The term $\underline{L_P}$ is the total TORQUE $\underline{L_P} = \sum_{i=1}^{N} \underline{\sigma_i} \times \underline{F_i}$ with respect to P.

The first term vanishes in two cases:

(1) $\dot{\underline{R_P}} = 0$, i.e. point P is at rest in the inertial frame

(2) $P \equiv C$, i.e. point P is the center of mass; in fact, in this case $\underline{\sigma_i} \equiv \underline{r_i}$ and

$$\sum_{i=1}^{N} \dot{\underline{\sigma_i}} m_i \equiv \sum_{i=1}^{N} \dot{\underline{r_i}} m_i = 0$$

Moreover, if $P \equiv C$

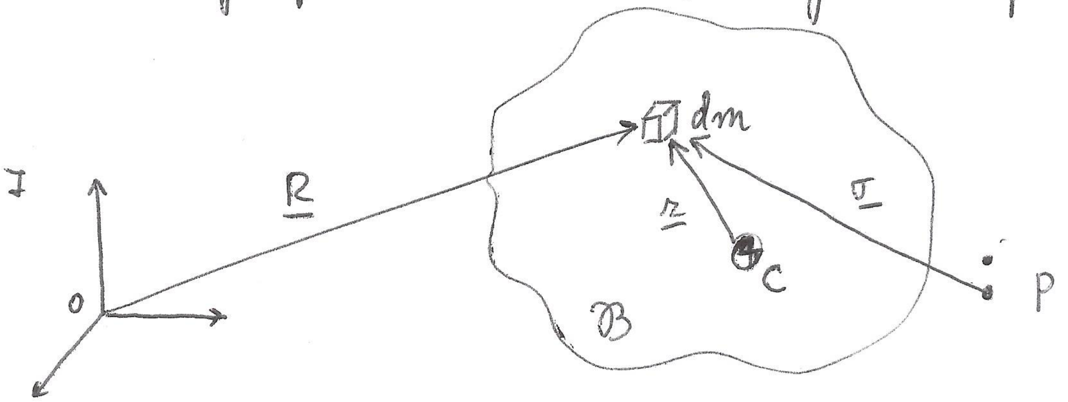
$$\dot{\underline{H_C}} = \underline{L_C} = \sum_{i=1}^{N} \underline{r_i} \times m_i \ddot{\underline{R_i}} = \sum_{i=1}^{N} \underline{r_i} \times m_i (\ddot{\underline{R_C}} + \ddot{\underline{r_i}}) =$$

$$= \sum_{i=1}^{N} \underline{r_i} \times m_i \ddot{\underline{r_i}}$$

because $\left\{ \sum_{i=1}^{N} m_i \underline{r_i} \right\} \times \ddot{\underline{R_C}} = \underline{0}$ due to definition of center of mass

CONTINUOUS SYSTEMS

This can be regarded as the integral sum of infinitesimal elementary particles. Hence, integrals replace sums



- Superparticle theorem

$$M\ddot{\underline{R}}_c = \underline{F} \qquad \text{where} \quad \underline{R}_c = \frac{\int_{\mathcal{B}} \underline{R}\, dm}{M} \quad \text{is the center of mass}$$

- Kinetic energy

$$T = \frac{1}{2} M \dot{\underline{R}}_c \cdot \dot{\underline{R}}_c + \frac{1}{2} \int_{\mathcal{B}} \dot{\underline{r}} \cdot \dot{\underline{r}}\, dm$$

- Kinetic energy rate

$$\frac{dT}{dt} = \underline{F} \cdot \dot{\underline{R}}_c + \int_{\mathcal{B}} d\underline{F} \cdot \dot{\underline{r}}$$

- Mechanical energy rate

$$\frac{d\mathcal{E}}{dt} = \underline{F}_{nc} \cdot \dot{\underline{R}}_c + \int_{\mathcal{B}} d\underline{F}_{nc} \cdot \dot{\underline{r}}$$

- Linear momentum

$$\underline{dp} = \dot{\underline{R}}\, dm \qquad \text{for each infinitesimal mass}$$

$$\underline{p} = \int_{\mathcal{B}} \underline{dp} = \int_{\mathcal{B}} \dot{\underline{R}}\, dm = M\dot{\underline{R}}_c$$

In this context the body is not necessarily rigid

The inertial derivative of $\underline{p}$ is related to the force through the 2nd and 3rd Newton law,

$$\dot{\underline{p}} = \int_{\mathcal{B}} \ddot{\underline{R}}\, dm = \int_{\mathcal{B}} d\underline{F} = \underline{F}$$

- Angular momentum

With respect to P, mass dm has position $\underline{\sigma}$

Steps similar to those valid for systems of particles lead to

$$\underline{H}_P = \int_{\mathcal{B}} \underline{\sigma} \times \dot{\underline{R}}\, dm \qquad \underline{H}_S = \underline{r}_{SP} \times M\dot{\underline{R}}_c + \underline{H}_P$$

$$\dot{\underline{H}}_P = \underline{L}_P + \left\{ \int_{Body} \underline{\sigma}\, dm \right\} \times \ddot{\underline{R}}_P \qquad \text{where } \underline{L}_P = \int_{\mathcal{B}} \underline{\sigma} \times d\underline{F}$$

is the total torque

$$\dot{\underline{H}}_P = \underline{L}_P \quad \text{if} \quad (1)\ \ddot{\underline{R}}_P = 0 \quad \text{or} \quad (2)\ P \equiv C \ (\text{P is the center of mass})$$

Moreover, if $P \equiv C$

$$\dot{\underline{H}}_c = \underline{L}_c = \int_{\mathcal{B}} \underline{r} \times \ddot{\underline{R}}\, dm = \int_{\mathcal{B}} \underline{r} \times \ddot{\underline{r}}\, dm$$