FUNDAMENTALS OF RIGID BODY DYNAMICS

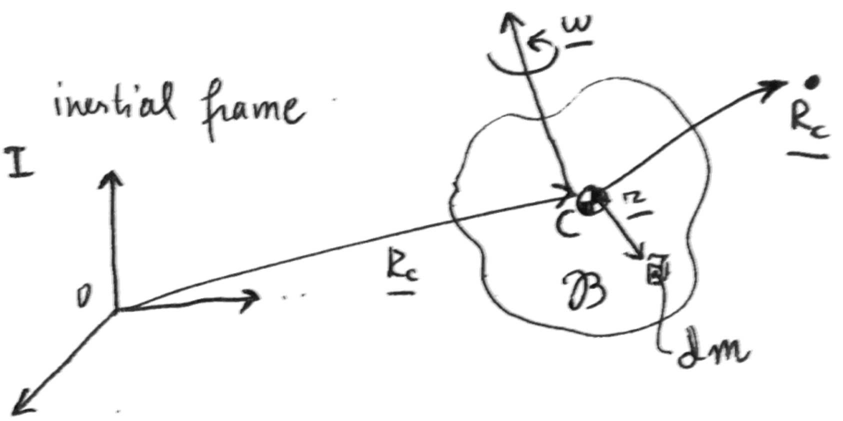
• INTRODUCTION TO RIGID BODY DYNAMICS

A rigid body is a continuous system with prescribed, time-invariant and space-invariant mass distribution (i.e. also constant shape).

If the point P has no inertial velocity or coincides with the center of mass, then one obtains the EULER EQUATION,

$$\dot{\underline{H}}_P = \underline{L}_P$$

Let $\underline{R}$ be the inertial position vector of the infinitesimal mass dm



• Angular momentum

· The angular momentum about O, (origin of the inertial frame I)

$$\underline{H}_O = \int_{\mathcal{B}} \underline{R} \times \dot{\underline{R}}\, dm$$

As $\underline{R} = \underline{R}_c + \underline{r}$ one obtains

$$\underline{H}_O = \int_{\mathcal{B}} (\underline{R}_c + \underline{r}) \times (\dot{\underline{R}}_c + \dot{\underline{r}})\, dm =$$

$$= \int_{\mathcal{B}} \underline{R}_c \times \dot{\underline{R}}_c\, dm + \underbrace{\int_{\mathcal{B}} \underline{r}\, dm}_{=0} \times \dot{\underline{R}}_c + \underline{R}_c \times \underbrace{\int_{\mathcal{B}} \dot{\underline{r}}\, dm}_{=0} + \int_{\mathcal{B}} \underline{r} \times \dot{\underline{r}}\, dm =$$

definition of mass center

$$= \underbrace{\underline{R_c} \times \underline{\dot{R}_c} M}_{①} + \underbrace{\int_{\mathcal{B}} \underline{r} \times \underline{\dot{r}}\, dm}_{②} = \underline{H_{MC}} + \underline{H_c}$$

① $\underline{H_{MC,O}}$ is the angular momentum related to motion of center of mass C with respect to O

② $\underline{H_c}$ is the angular momentum of the body w.r.t. C

- Inertia dyad

For a rigid body, $\underline{\dot{r}} = \frac{{}^N d\underline{r}}{dt}$ is given by

$$\underline{\dot{r}} = \frac{{}^B d\underline{r}}{dt} + {}^N\underline{\omega}^B \times \underline{r} = \underline{\omega} \times \underline{r}$$

Reference B moves together with the body ; $\underline{\omega} := {}^N\underline{\omega}^B$ hence forward

Therefore, the angular momentum $\underline{H_c}$ is

$$\underline{H_c} = \int_{\mathcal{B}} \underline{r} \times (\underline{\omega} \times \underline{r})\, dm = \int_{\mathcal{B}} \left[\underline{\omega}\, r^2 - \underline{r}(\underline{\omega} \cdot \underline{r})\right] dm =$$

$$= \underbrace{\int_{\mathcal{B}} \left[r^2 \underset{\rightarrow}{1} - \underline{r}\,\underline{r}\right] dm}_{\text{INERTIA DYAD } \underline{\underline{I_c}}} \cdot \underline{\omega}$$

, where $\underset{\rightarrow}{1}$ is the unit dyad such that $\underset{\rightarrow}{1} \cdot \underline{\omega} = \underline{\omega}$

$$\underset{\rightarrow}{1} = \hat{b}_1\hat{b}_1 + \hat{b}_2\hat{b}_2 + \hat{b}_3\hat{b}_3$$

Letting

$$\underline{r} = [x \quad y \quad z] \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} = [\hat{b}_1 \; \hat{b}_2 \; \hat{b}_3] \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad r^2 = x^2 + y^2 + z^2$$

$$\underline{\omega} = [\omega_1 \; \omega_2 \; \omega_3] \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} = [\hat{b}_1 \; \hat{b}_2 \; \hat{b}_3] \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} =: [\hat{b}_1 \; \hat{b}_2 \; \hat{b}_3]\, \underline{\omega}^{(B)}$$

one obtains the INERTIA DYAD, only related to mass distribution:

$$\underline{I}_c = [\hat{b}_1\ \hat{b}_2\ \hat{b}_3] \underbrace{\int_{\mathcal{B}} \left\{ (x^2+y^2+z^2)\, I_{3\times3} - \begin{bmatrix} x \\ y \\ z \end{bmatrix} [x\ \ y\ \ z] \right\} dm}_{\text{Inertia matrix } (3\times3)\ I_c^{(B)}} \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix}$$

Therefore, the inertia matrix is associated with the inertia dyad and can be regarded as the projection of $\underline{I}_c$ along $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$. Superscript (B) in $I_c^{(B)}$ specifies the frame to which the inertia matrix is referred. It is easy to check that

$$\begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} \cdot \underline{I}_c \cdot [\hat{b}_1\ \hat{b}_2\ \hat{b}_3] = I_c^{(B)} \quad \text{, because} \quad \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} \cdot [\hat{b}_1\ \hat{b}_2\ \hat{b}_3] = I_{3\times3}$$

and the general expression of $I_c^{(B)}$ is

$$I_c^{(B)} = \int_{\mathcal{B}} \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -yx & x^2+z^2 & -yz \\ -zx & -zy & x^2+y^2 \end{bmatrix} dm$$

Letting $\underline{H}_c = [\hat{b}_1\ \hat{b}_2\ \hat{b}_3] \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix} = [\hat{b}_1\ \hat{b}_2\ \hat{b}_3]\, \underline{H}^{(B)}$

$$\underline{H}_c = \underline{I}_c \cdot \underline{\omega} = [\hat{b}_1\ \hat{b}_2\ \hat{b}_3]\, I_c^{(B)} \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} \cdot [\hat{b}_1\ \hat{b}_2\ \hat{b}_3]\, \underline{\omega}^{(B)} =$$

$$= [\hat{b}_1\ \hat{b}_2\ \hat{b}_3]\, I_c^{(B)} \underline{\omega}^{(B)} \Rightarrow \underset{(3\times1)}{\underline{H}^{(B)}} = \underset{(3\times3)}{I_c^{(B)}}\ \underset{(3\times1)}{\underline{\omega}^{(B)}}$$

- <u>Properties of the inertia matrix</u>

1. SYMMETRY. From the definition it is apparent that off diagonal terms are equal in pairs, i.e. mirrored

2. TRIANGULAR INEQUALITIES. From the definition, it is relatively easy to prove the 3 triangular inequalities

(a) $I_{11} \leq I_{22} + I_{33}$ (b) $I_{22} \leq I_{11} + I_{33}$ (c) $I_{33} \leq I_{11} + I_{22}$

For (a), for instance

$$I_{22} + I_{33} = \int_{\mathcal{B}} (x^2 + z^2 + x^2 + y^2)\, dm \qquad I_{11} = \int_{\mathcal{B}} (z^2 + y^2)\, dm$$

$$\rightarrow \quad I_{22} + I_{33} - I_{11} = \int_{\mathcal{B}} 2x^2\, dm \geq 0$$

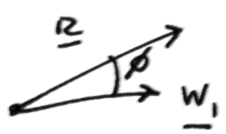
3. POSITIVE DEFINITENESS. $\underline{\underline{I}}_c^{(B)}$ satisfies the following inequality:

$$\underline{w}_1^{(B)T}\, \underline{\underline{I}}_c^{(B)}\, \underline{w}_1^{(B)} \geq 0 \qquad \forall\ \underline{w}_1^{(B)} \ (3\times1)\text{-column vector}$$

$$\underline{w}_1^{(B)T}\, \underline{\underline{I}}_c^{(B)}\, \underline{w}_1^{(B)} = 0 \quad \text{only if} \quad \underline{w}_1^{(B)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The preceding product can be rewritten as

$$\underline{w}_1^{(B)T}\, \underline{\underline{I}}_c^{(B)}\, \underline{w}_1^{(B)} = \underline{w}_1^{(B)T} \underbrace{\begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} \cdot \begin{bmatrix} \hat{b}_1 & \hat{b}_2 & \hat{b}_3 \end{bmatrix}}_{I_{3\times3}} \underline{\underline{I}}_c^{(B)} \underbrace{\begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} \cdot \begin{bmatrix} \hat{b}_1 & \hat{b}_2 & \hat{b}_3 \end{bmatrix}}_{I_{3\times3}} \underline{w}_1^{(B)} =$$



$$= \underline{w}_1 \cdot \underline{\underline{I}}_c \cdot \underline{w}_1 = \underline{w}_1 \cdot \int_{\mathcal{B}} \left[r^2 \underline{\underline{1}} - \underline{r}\,\underline{r} \right] dm \cdot \underline{w}_1 = \int_{\mathcal{B}} \left[w_1^2 r^2 - (\underline{w}_1 \cdot \underline{r})(\underline{r} \cdot \underline{w}_1) \right] dm$$

$$= \int_{\mathcal{B}} w_1^2 r^2 (1 - c_\phi^2)\, dm \geq 0$$

- Inertia matrix in different frames

If $\underline{\underline{I}}_c$ is expressed in $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$, i.e. if $I_c^{(B)}$ is Known one may want to get $I_c^{(A)}$ in another frame such that

$$\begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} = \underset{B \leftarrow A}{R} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{bmatrix} \iff [\hat{b}_1\ \hat{b}_2\ \hat{b}_3] = [\hat{a}_1\ \hat{a}_2\ \hat{a}_3] \underset{B \leftarrow A}{R}^T$$

Insertion of the preceding relations into $\underline{\underline{I}}_c$ leads to

$$\underline{\underline{I}}_c = [\hat{b}_1\ \hat{b}_2\ \hat{b}_3]\, I_c^{(B)} \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} = [\hat{a}_1\ \hat{a}_2\ \hat{a}_3] \underbrace{\underset{B \leftarrow A}{R}^T\ I_c^{(B)} \underset{B \leftarrow A}{R}}_{I_c^{(A)}} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{bmatrix}$$

and one recognizes $I_c^{(A)}$, i.e.

$$I_c^{(A)} = \underset{B \leftarrow A}{R}^T\ I_c^{(B)} \underset{B \leftarrow A}{R} = \underset{A \leftarrow B}{R}\ I_c^{(B)} \underset{B \leftarrow A}{R}$$

Therefore, the two matrices $I_c^{(A)}$ and $I_c^{(B)}$ are SIMILAR.

- principal axes of inertia

Because $I_c^{(B)}$ is symmetric :

→ there exist 3 real eigenvalues $\{\lambda_i\}_{i=1,2,3}$

→ there exist 3 orthonormal eigenvectors $\underline{w}_i^{(B)} \leftrightarrow \lambda_i \ (i=1,2,3)$

such that

$$I_c^{(B)}\, \underline{w}_i^{(B)} = \lambda_i\, \underline{w}_i^{(B)}$$

In order that the previous equation be solvable with $\underline{w}_i^{(B)} \neq \underline{0}$

$$\det\left(I_c^{(B)} - \lambda\, I_{3\times3}\right) = 0 \qquad \text{(eigenvalue equation)}$$

λ_i are the 3 (real) roots of the latter equation.

Moreover $\underline{w}_i^{(B)}$ are arranged in a matrix

$$T = \begin{bmatrix} | & | & | \\ \underline{w}_1 & \underline{w}_2 & \underline{w}_3 \\ | & | & | \end{bmatrix} \quad \text{such that} \quad \det T = 1$$

Because $\underline{w}_i^{(B)}$ are 3 orthonormal vectors and $\det T = 1$

T can also be regarded as a rotation matrix.

Then, the new inertia matrix is constructed, using the general property of similarity:

$$\underline{\underline{I}}_C^{(E)} = T^T \, \underline{\underline{I}}_C^{(B)} \, T = \begin{bmatrix} \text{---} \; \underline{w}_1^T \; \text{---} \\ \text{---} \; \underline{w}_2^T \; \text{---} \\ \text{---} \; \underline{w}_3^T \; \text{---} \end{bmatrix} I_C^{(B)} \begin{bmatrix} | & | & | \\ \underline{w}_1 & \underline{w}_2 & \underline{w}_3 \\ | & | & | \end{bmatrix} =$$

$$\underset{\substack{\uparrow \\ \text{eigenvalue} \\ \text{equation}}}{=} \begin{bmatrix} \text{---} \; \underline{w}_1^T \; \text{---} \\ \text{---} \; \underline{w}_2^T \; \text{---} \\ \text{---} \; \underline{w}_3^T \; \text{---} \end{bmatrix} \begin{bmatrix} | & | & | \\ \lambda_1 \underline{w}_1 & \lambda_2 \underline{w}_2 & \lambda_3 \underline{w}_3 \\ | & | & | \end{bmatrix} \underset{\substack{\uparrow \\ \text{orthonormality of } \{\underline{w}_i\}_{i=1,2,3}}}{=} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Therefore, for this particular choice of the rotation matrix T (with eigenvectors as columns) one obtains a diagonal inertia matrix $\underline{\underline{I}}_C^{(E)}$. Recalling the general similarity relation

$$\underline{\underline{I}}_C^{(E)} = \underset{E \leftarrow B}{R} \, I_C^{(B)} \underset{B \leftarrow E}{R}$$ one recognizes that in fact

$$\underset{B \leftarrow E}{R} \equiv T$$

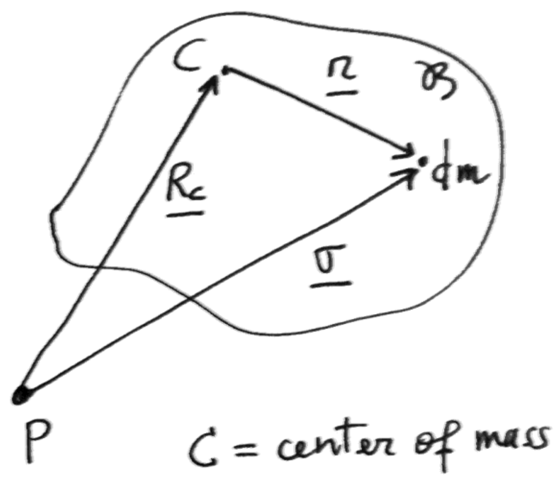
This allows identifying the 3 PRINCIPAL AXES of INERTIA

$$\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} = \underset{E \leftarrow B}{R} \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix}$$

along which the inertia matrix $I_C^{(E)}$ is diagonal.

Due to symmetry of $I_C^{(B)}$, these principal inertia axes exist for all rigid bodies.

- <u>Parallel axis theorem</u>



If $\underline{I_C}$ is Known, one may want to get $\underline{I_P}$, defined as

$$\underline{I_P} = \int_{\mathcal{B}} \left[\sigma^2 \underset{\rightarrow}{1} - \underline{\sigma}\,\underline{\sigma}\right] dm$$

Because $\underline{\sigma} = \underline{R_c} + \underline{r}$

$$\underline{I_P} = \int_{\mathcal{B}} \left[(\underline{R_c}+\underline{r}) \cdot (\underline{R_c}+\underline{r})\, \underset{\rightarrow}{1} - (\underline{R_c}+\underline{r})(\underline{R_c}+\underline{r}) \right] dm =$$

$$= M^2 R_C^2 \underset{\rightarrow}{1} + 2\underline{R_c} \cdot \int_{\mathcal{B}} \underline{r}\, dm\, \underset{\rightarrow}{1} + \int_{\mathcal{B}} r^2 \underset{\rightarrow}{1}\, dm - M\underline{R_c}\,\underline{R_c} - \int_{\mathcal{B}} \underline{r}\,\underline{r}\, dm +$$

$$- \underline{R_c} \int_{\mathcal{B}} \underline{r}\, dm - \int_{\mathcal{B}} \underline{r}\, dm\, \underline{R_c}$$

All the terms $\int \underline{r}\, dm$ vanish because $\underline{r}$ is taken from C (center of mass)

Therefore

$$\underline{I_P} = M^2\left(R_C^2 \underset{\rightarrow}{1} - \underline{R_c}\,\underline{R_c}\right) + \int_{\mathcal{B}} \left(r^2 \underset{\rightarrow}{1} - \underline{r}\,\underline{r}\right) dm = \underline{I_c} + M^2\left(R_c^2 \underset{\rightarrow}{1} - \underline{R_c}\,\underline{R_c}\right)$$

parallel axis theorem (dyadic form)

Letting $\underline{R_c} = X_c\,\hat{b}_1 + Y_c\,\hat{b}_2 + Z_c\,\hat{b}_3$, projection of the dyadic relation along $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$ yields

$$\begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} \cdot \underline{I_P} \cdot \begin{bmatrix} \hat{b}_1 & \hat{b}_2 & \hat{b}_3 \end{bmatrix} = I_P^{(B)}$$

(left-hand side)

$$= \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} \cdot \left\{ \underline{I_c} + M^2\left(R_C^2\, \underset{\rightarrow}{1} - \underline{R_c}\,\underline{R_c}\right) \right\} \cdot \begin{bmatrix} \hat{b}_1 & \hat{b}_2 & \hat{b}_3 \end{bmatrix} =$$

$$= I_C^{(B)} + M^2\left(R_c^2\, I_{3\times3} - \begin{bmatrix} X_c \\ Y_c \\ Z_c \end{bmatrix} \begin{bmatrix} X_c & Y_c & Z_c \end{bmatrix} \right) =$$

$$= I_C^{(B)} + M^2 \begin{bmatrix} Y_c^2 + Z_c^2 & -X_c Y_C & -X_c Z_C \\ -Y_c X_C & X_c^2 + Z_c^2 & -Y_c Z_C \\ -Z_c X_c & -Z_c Y_C & X_c^2 + Y_c^2 \end{bmatrix}$$

Parallel axis theorem (matrix form)

- <u>Inertia matrix of symmetric spacecraft</u>

Letting again $\underline{r} = x\,\hat{b}_1 + y\,\hat{b}_2 + z\,\hat{b}_3$ taken from C (center of mass)

$$I_C^{(B)} = \int_{\mathcal{B}} \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -yx & x^2+z^2 & -yz \\ -zx & -zy & x^2+y^2 \end{bmatrix} dm$$

Diagonal terms I_{ii} are named MOMENTS of inertia

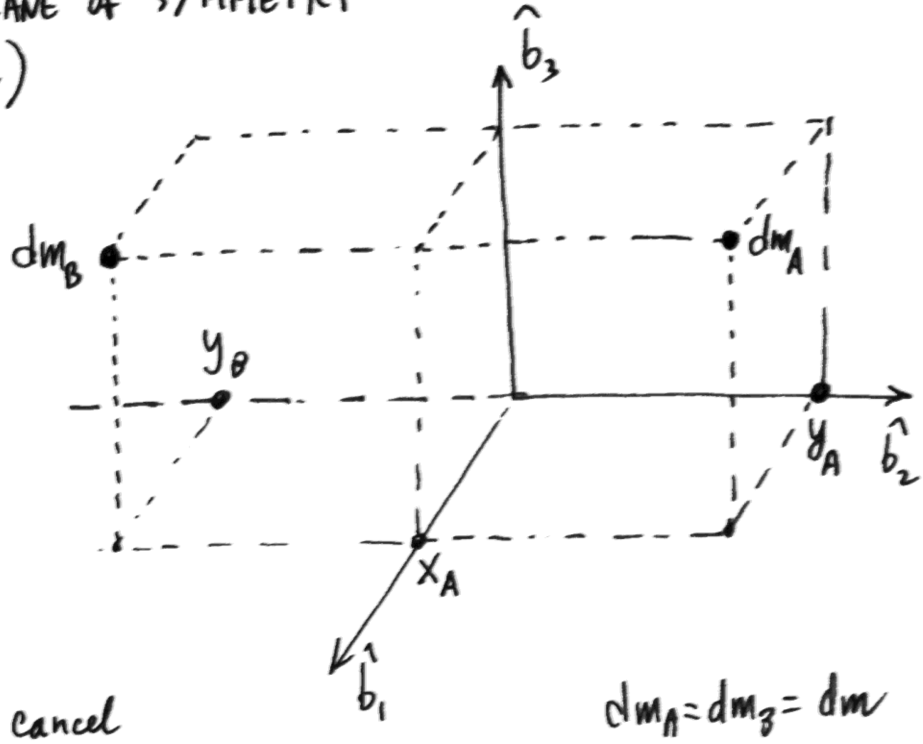
Off-diagonal terms I_{ij} $(j \neq i)$ are named PRODUCTS of inertia

(A) SPACECRAFT with a PLANE OF SYMMETRY
(e.g. winged spacecraft)

$dm_A \leftrightarrow (x_A, y_A, z_A)$

$dm_B \leftrightarrow (x_B, y_B, z_B)$

where $x_B = x_A$
$y_B = -y_A$
$z_B = z_A$



Therefore the infinitesimal contributions (1,2) and (2,3) cancel in pair. In fact

$$(1,2) \quad dm_A\,(-x_A y_A) + dm_B\,(-x_B\, y_B) = dm\,(-x_A\, y_A + x_A\, y_A) = 0$$

$$(2,3) \quad dm_A\,(-y_A z_A) + dm_B\,(-y_B z_B) = dm\,(-z_A\, y_A + z_A\, y_A) = 0$$

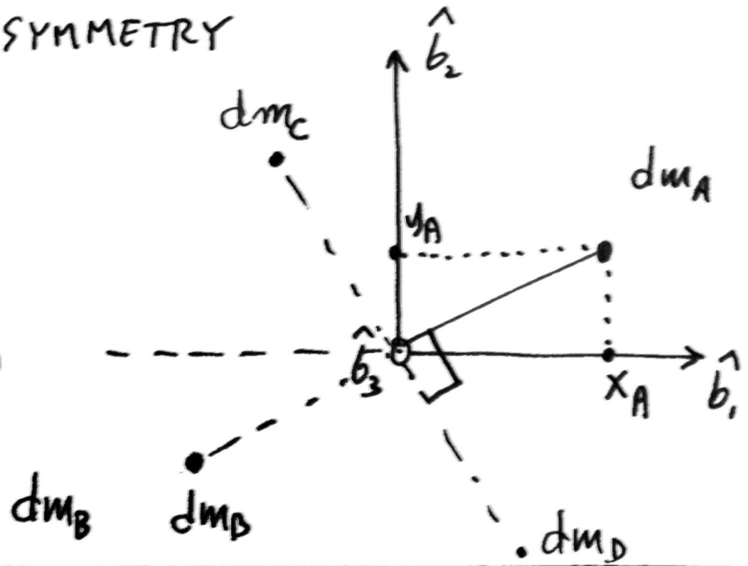
Therefore the typical inertia matrix has the form

$$\underline{\underline{I}}_c^{(B)} = \begin{bmatrix} I_{11} & 0 & I_{13} \\ 0 & I_{22} & 0 \\ I_{31} & 0 & I_{33} \end{bmatrix}$$

and one can recognize that
→ I_{22} is a principal inertia moment
→ $\hat{b}_2$ ($\perp$ symmetry plane) is a principal inertia axis.

(B) SPACECRAFT with an AXIS of SYMMETRY
(e.g. wingless rocket)

If $\hat{b}_3$ is the symmetry axis, the top view is on the right. All 4 elementary masses have same z, i.e.
$z_A = z_B = z_C = z_D$



4 equal small masses $dm_A = dm_B = dm_C = dm_D = dm$

$dm_A \leftrightarrow (x_A, y_A)$

$dm_B \leftrightarrow (x_B, y_B)$ such that $x_B = -x_A$, $y_B = -y_A$ $(z_B = z_A)$

$dm_C \leftrightarrow (x_C, y_C)$ such that $x_C = -y_A$, $y_C = +x_A$ $(z_C = z_A)$

$dm_D \leftrightarrow (x_D, y_D)$ such that $x_D = y_A$, $y_D = -x_A$ $(z_D = z_A)$

The infinitesimal contributions in $(1,2)$, $(2,3)$, $(1,3)$ cancel in pair

for instance $(1,3)$ $(-x_A z_A)dm_A + (-x_B z_B)dm_B + (-x_C z_C)dm_C + (-x_D z_D)dm_D$

$$= (-x_A z_A - x_A z_A + x_A z_A + x_A z_A)dm = 0$$

The two terms I_{11} and I_{22} equal. In fact

$(1,1)$ $(y_A^2 + z_A^2)dm_A + (y_B^2 + z_B^2)dm_B + (y_C^2 + z_C^2)dm_C + (y_D^2 + z_D^2)dm_D =$

$$= dm\left(y_A^2 + z_A^2 + y_A^2 + z_A^2 + x_A^2 + z_A^2 + x_A^2 + z_A^2\right) = dm\left(4z_A^2 + 2x_A^2 + 2y_A^2\right)$$

$(2,2)$ $(x_A^2 + z_A^2)dm_A + (x_B^2 + z_B^2)dm_B + (x_C^2 + z_C^2)dm_C + (x_D^2 + z_D^2)dm_D =$

$$= dm\left(x_A^2 + z_A^2 + x_A^2 + z_A^2 + y_A^2 + z_A^2 + y_A^2 + z_A^2\right) = dm\left(4z_A^2 + 2x_A^2 + 2y_A^2\right)$$

Therefore, the typical inertia matrix has the form

$$I_C^{(B)} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

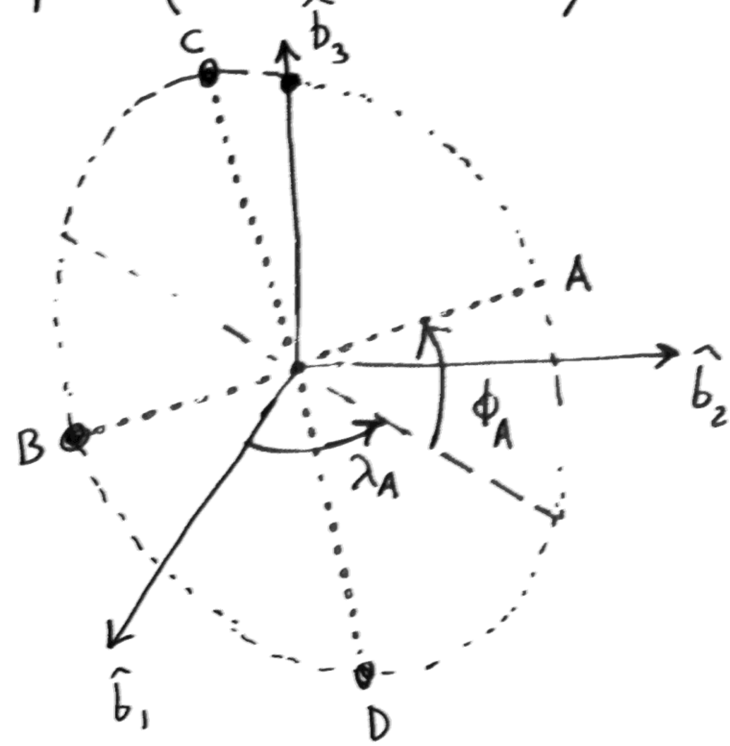
and one can recognize that

$\rightarrow$ $I_1, I_2 = I_1, I_3$ are principal inertia moments

$\rightarrow$ $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$ are principal inertia axes

In particular $\hat{b}_3$ is the symmetry axis and $(\hat{b}_1, \hat{b}_2)$ are two arbitrary axes orthogonal to $\hat{b}_3$ and such that $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$ is right-handed.

(C) SPACECRAFT symmetric w.r.t. a point (SPHERICAL SYMMETRY)
(e.g. spherical satellite)



Spherical symmetry implies that 6 elementary masses can be chosen, in this way

$$dm_A: (\lambda_A, \phi_A) \rightarrow \begin{cases} x_A = R\, C\phi_A\, C\lambda_A \\ y_A = R\, C\phi_A\, S\lambda_A \\ z_A = R\, S\phi_A \end{cases}$$

$$dm_B: (\lambda_B, \phi_B) \rightarrow \begin{cases} x_B = -x_A \\ y_B = -y_A \\ z_B = -z_A \end{cases}$$

$$dm_C: (\lambda_C, \phi_C) = \left(\lambda_A + \pi, \frac{\pi}{2} - \phi_A\right) \rightarrow \begin{cases} x_C = -R\, S\phi_A\, C\lambda_A \\ y_C = -R\, S\phi_A\, S\lambda_A \\ z_C = R\, C\phi_A \end{cases}$$

$$dm_D: (\lambda_D, \phi_D) = (\lambda_A, -\phi_A) \rightarrow \begin{cases} x_D = -x_C \\ y_D = -y_C \\ z_D = -z_C \end{cases}$$

The remaining two masses have distance R from the origin and are orthogonal to the plane of A, B, C, D, i.e.

$$dm_E \begin{cases} x_E = R\, S\lambda_A \\ y_E = -R\, C\lambda_A \\ z_E = 0 \end{cases} \qquad dm_F \begin{cases} x_F = -x_E \\ y_F = -y_E \\ z_F = 0 \end{cases}$$

(masses not portrayed in figure)

Using these 6 masses one can prove that
→ all diagonal terms equal $4\,dm\,R^2$
→ all off-diagonal terms are 0

$$\rightarrow \underline{I}^{(B)} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_1 \end{bmatrix}$$

For a spherical mass distribution the 3 right-hand axes $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$ are arbitrary

- <u>Euler equation of attitude dynamics</u>

These are the fundamental equations of attitude dynamics. They relate the external action (torque) to the angular velocity. Let C be the center of mass. From previous developments:

$$\dot{\underline{H}}_c = \underline{L}_c \rightarrow \frac{{}^{B}d\underline{H}_c}{dt} + \underline{\omega} \times \underline{H}_c = \underline{L}_c$$

Using $\underline{H}_c = \underline{\underline{I}}_c \cdot \underline{\omega} = [\hat{b}_1\ \hat{b}_2\ \hat{b}_3]\, \underline{\underline{I}}_c^{(B)}\, \underline{\omega}^{(B)}$ one obtains

$$\frac{{}^{B}d\underline{H}_c}{dt} = [\hat{b}_1\ \hat{b}_2\ \hat{b}_3]\, \underline{\underline{I}}_c^{(B)}\, \dot{\underline{\omega}}^{(B)}$$

$$\underline{\omega} \times \underline{H}_c = \begin{vmatrix} \hat{b}_1 & \hat{b}_2 & \hat{b}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ H_1 & H_2 & H_3 \end{vmatrix} = [\hat{b}_1\ \hat{b}_2\ \hat{b}_3] \begin{bmatrix} \omega_2 H_3 - \omega_3 H_2 \\ \omega_3 H_1 - \omega_1 H_3 \\ \omega_1 H_2 - \omega_2 H_1 \end{bmatrix} =$$

$$= [\hat{b}_1\ \hat{b}_2\ \hat{b}_3] \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix} = [\hat{b}_1\ \hat{b}_2\ \hat{b}_3]\, \tilde{\underline{\omega}}\, \underline{H}_c^{(B)}$$

But $\underline{H}_c^{(B)} = \underline{\underline{I}}_c^{(B)} \underline{\omega}^{(B)}$, thus the previous relations lead to

$$\boxed{\begin{array}{l} \underline{\underline{I}}_c^{(B)} \dot{\underline{\omega}}^{(B)} + \tilde{\underline{\omega}}\, \underline{\underline{I}}_c^{(B)} \underline{\omega} = \underline{L}_c^{(B)} \quad , \text{where} \quad \underline{L}_c = [\hat{b}_1\ \hat{b}_2\ \hat{b}_3]\, \underline{L}_c^{(B)} \\ \text{EULER EQUATION of attitude dynamics} \end{array}}$$

If the principal axes are chosen, then $\underline{\underline{I}}_c^{(E)}$ is diagonal and the EULER equation reduces to its well-known simplified form

$$\begin{cases} I_1 \dot{\omega}_1 = (I_2 - I_3)\,\omega_2 \omega_3 + L_1 \\ I_2 \dot{\omega}_2 = (I_3 - I_1)\,\omega_1 \omega_3 + L_2 \\ I_3 \dot{\omega}_3 = (I_1 - I_2)\,\omega_1 \omega_2 + L_3 \end{cases}$$

where
$$\underline{\underline{I}}_c^{(E)} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \text{ and } \underline{L}_c^{(B)} = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}$$

- Kinetic energy

In general, for a continuous system

$$T = \frac{1}{2} M \dot{\underline{R}}_c \cdot \dot{\underline{R}}_c + \frac{1}{2}\int_{\mathcal{B}} \dot{\underline{r}} \cdot \dot{\underline{r}}\, dm = T_{transl} + T_{rot}$$

In the presence of a rigid body $\dot{\underline{r}} = \underline{\omega} \times \underline{r}$, therefore

$$T_{rot} = \frac{1}{2}\int_{\mathcal{B}} (\underline{\omega} \times \underline{r}) \cdot (\underline{\omega} \times \underline{r})\, dm = \frac{1}{2}\int_{\mathcal{B}} \underline{r} \cdot [(\underline{\omega} \times \underline{r}) \times \underline{\omega}]\, dm =$$

$$= \frac{1}{2}\underline{\omega} \cdot \int_{\mathcal{B}} \underline{r} \times (\underline{\omega} \times \underline{r})\, dm = \frac{1}{2}\underline{\omega} \cdot \underline{H}_c = \frac{1}{2}\left[\underline{\omega}^{(B)}\right]^T \underline{I}_c^{(B)} \underline{\omega}^{(B)}$$

where $\underline{\omega}^{(B)}$ is a (3×1) vector including the components of $\underline{\omega}$ in B

The work done on B is given by two terms, integrated from

$$\dot{T}_{transl} = \frac{1}{2} M \dot{\underline{R}}_c \cdot \ddot{\underline{R}}_c + \frac{1}{2} M \ddot{\underline{R}}_c \cdot \dot{\underline{R}}_c = M \ddot{\underline{R}}_c \cdot \dot{\underline{R}}_c = \underline{F} \cdot \dot{\underline{R}}_c$$

$$\dot{T}_{rot} = \frac{1}{2}\frac{d}{dt}\left\{\left[\underline{\omega}^{(B)}\right]^T \underline{I}_c^{(B)} \underline{\omega}^{(B)}\right\} \underset{\uparrow}{=} \left[\underline{\omega}^{(B)}\right]^T \underline{I}_c^{(B)} \dot{\underline{\omega}}^{(B)} = \underline{\omega} \cdot \dot{\underline{H}}_c \underset{\uparrow}{=} \underline{\omega} \cdot \underline{L}_c$$

$\underline{I}_c^{(B)}$ is constant and symmetric   Euler eq.

If both $\underline{F}$ and $\underline{L}_c$ are conservative, one can define a potential function.

Regardless of this property, in general the work is

$$W = \int_{t_1}^{t_2} \left\{ \underline{F} \cdot \dot{\underline{R}}_c + \underline{L}_c \cdot \underline{\omega} \right\} dm$$

• TORQUE-FREE MOTION

• Axisymmetric body with no external torque

An axisymmetric body (with $\hat{b}_3$ as symmetry axis) has two coincident principal inertia moments: $I_1 = I_2 = I_T \neq I_3$

The Euler equations become (with $L_1 = L_2 = L_3 = 0$, no torque)

$$\begin{cases} I_T \dot{\omega}_1 = (I_T - I_3)\omega_2 \omega_3 \longrightarrow \dot{\omega}_1 = -\omega_p \omega_2 \\ I_T \dot{\omega}_2 = (I_3 - I_T)\omega_1 \omega_3 \longrightarrow \dot{\omega}_2 = \omega_p \omega_1 \qquad \text{letting } \omega_p := \omega_{30}\left(\frac{I_3}{I_T} - 1\right) \\ I_3 \dot{\omega}_3 = 0 \Rightarrow \omega_3 = \text{const} = \omega_{30} \end{cases}$$

Taking the time derivative of the preceding equations, one obtains

$$\begin{cases} \ddot{\omega}_1 + \omega_p^2 \omega_1 = 0 \\ \ddot{\omega}_2 + \omega_p^2 \omega_2 = 0 \end{cases}$$

The general solution of these two harmonic equations is

$$\omega_1(t) = A_1 \cos(\omega_p t) + B_1 \sin(\omega_p t)$$

$$\omega_2(t) = A_2 \cos(\omega_p t) + B_2 \sin(\omega_p t)$$

These constants A_1, B_1, A_2, B_2 can be found in terms of ω_{10}, ω_{20}, the initial values of ω_1 and ω_2. The respective derivatives $\dot{\omega}_1(t_0)$ and $\dot{\omega}_2(t_0)$ (where t_0 is set to 0) are evaluated by using

$$\dot{\omega}_1(t_0) = -\omega_p \omega_2(t_0) \quad \text{and} \quad \dot{\omega}_2(t_0) = \omega_p \omega_1(t_0)$$

First, ω_1 and ω_2 are evaluated at $t = 0$, yielding

$$\omega_{1,0} = A_1 \qquad \text{and} \qquad \omega_{2,0} = A_2$$

while the time derivatives of $\omega_1(t)$ and $\omega_2(t)$ are

$$\dot{\omega}_1 = -A_1 \omega_p \sin(\omega_p t) + B_1 \omega_p \cos(\omega_p t)$$

$$\dot{\omega}_2 = -A_2 \omega_p \sin(\omega_p t) + B_2 \omega_p \cos(\omega_p t)$$

Evaluated at $t=0$,

$$\dot{\omega}_{1,0} = B_1 \omega_p = -\omega_p \omega_{2,0} \Rightarrow B_1 = -\omega_{2,0}$$

$$\dot{\omega}_{2,0} = B_2 \omega_p \underset{\uparrow}{=} \omega_p \omega_{1,0} \Rightarrow B_2 = \omega_{1,0}$$

Euler equations

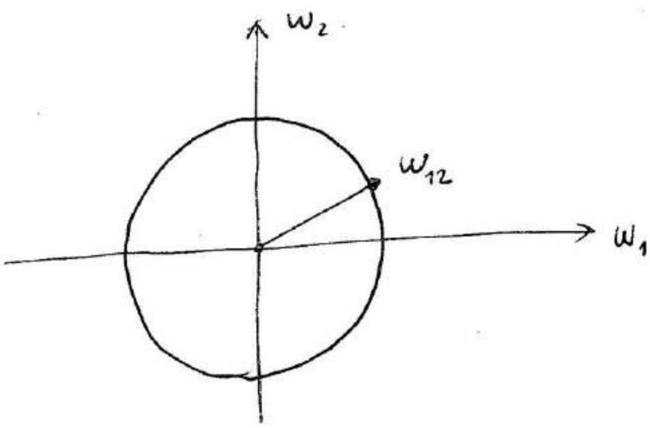
Finally, the complete solution is

$$\begin{cases} \omega_1(t) = \omega_{1,0} \cos(\omega_p t) - \omega_{2,0} \sin(\omega_p t) \\ \omega_2(t) = \omega_{2,0} \cos(\omega_p t) + \omega_{1,0} \sin(\omega_p t) \\ \omega_3(t) = \omega_{3,0} \end{cases}$$

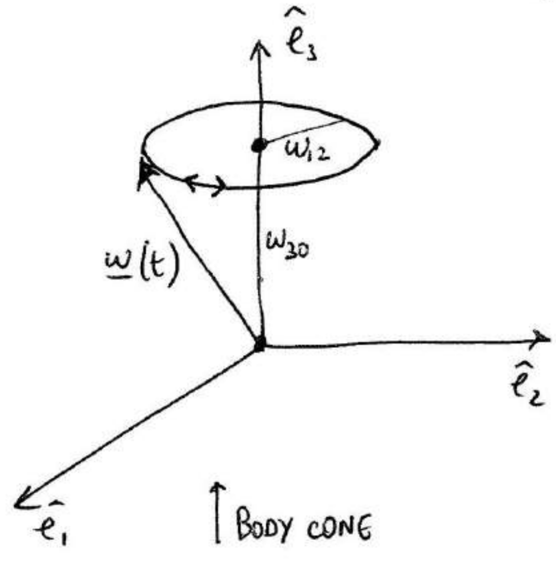
Letting $\omega_{12} = \sqrt{\omega_{1,0}^2 + \omega_{2,0}^2}$ and $\begin{cases} \cos\varphi = \dfrac{\omega_{10}}{\omega_{12}} \\ \sin\varphi = \dfrac{\omega_{20}}{\omega_{12}} \end{cases}$

the previous components become

$$\begin{cases} \omega_1(t) = \omega_{12} \cos(\omega_p t + \varphi) \\ \omega_2(t) = \omega_{12} \sin(\omega_p t + \varphi) \\ \omega_3(t) = \omega_{30} \end{cases}$$



This means that the angular velocity $\underline{\omega}$ describes a cone in the $\{\hat{e}_i\}$ reference frame. This is spanned in



$$\begin{cases} \text{counterclockwise sense if } \omega_p = \left(\frac{I_3}{J_T} - 1\right)\omega_{30} > 0 \\ \text{clockwise sense if } \omega_p = \left(\frac{I_3}{J_T} - 1\right)\omega_{30} < 0 \end{cases}$$

For the instantaneous orientation, one can choose the Euler angles (3·1·3), associated with the following kinematics equations

$$\dot{\psi} = \frac{1}{S_\theta}\left[\omega_1 S_\phi + \omega_2 C_\phi\right] \longrightarrow \dot{\psi} = \frac{\omega_{12}}{S_\theta} \sin\left(\omega_p t + \varphi + \phi\right)$$

$$\dot{\theta} = \omega_1 C_\phi - \omega_2 S_\phi \longrightarrow \dot{\theta} = \omega_{12} \cos\left(\omega_p t + \varphi + \phi\right)$$

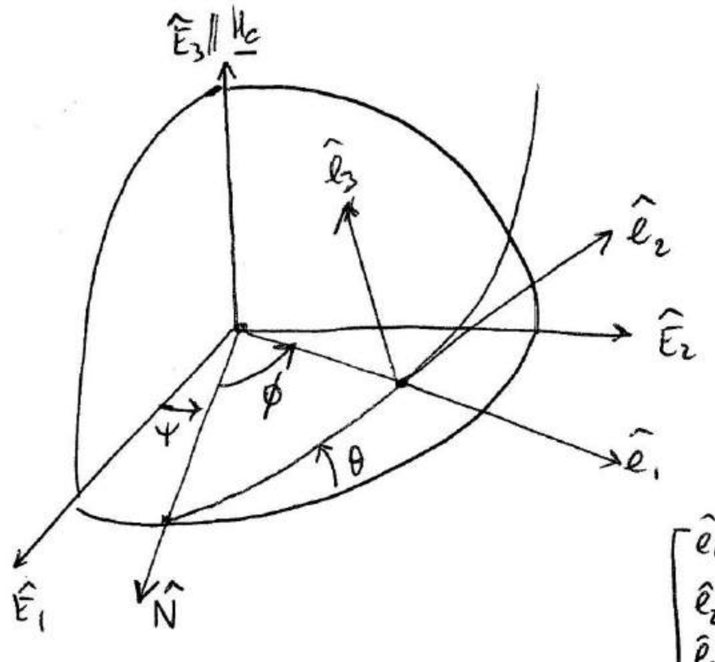
$$\dot{\phi} = \omega_3 - \frac{\omega_1 S_\phi + \omega_2 C_\phi}{\tan\theta} \longrightarrow \dot{\phi} = \omega_{30} - \frac{\omega_{12}}{\tan\theta} \sin\left(\omega_p t + \varphi + \phi\right)$$

where in the last three relations the explicit form of ω_1, ω_2 are used.

As the choice of the inertial sequence $(\hat{E}_1, \hat{E}_2, \hat{E}_3)$ is arbitrary one can select $\hat{E}_3$ such that $\hat{E}_3 \uparrow\uparrow \underline{H_c}$, because $\underline{H_c}$ = constant for a torque-free body.

Moreover, one can also choose $\hat{e}_3$ such that $\omega_{30} \geq 0$.

These two assumptions will hold in the next 4 pages



$\hat{E}_3$ is aligned with $\underline{H}_c$

$$\underline{H}_c^{(B)} = \begin{bmatrix} I_1 w_1 \\ I_2 w_2 \\ I_3 w_3 \end{bmatrix} \uparrow\uparrow \hat{E}_3$$

Moreover, in general

$$\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} = \underbrace{R_3(\phi) R_1(\theta) R_3(\psi)}_{\underset{B\leftarrow I}{R}} \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \end{bmatrix}$$

and the last column of $\underset{B\leftarrow I}{R}$ reports the components of $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ along $\hat{E}_3 = \begin{bmatrix} I_1 w_1 & I_2 w_2 & I_3 w_3 \end{bmatrix} \frac{1}{H_c} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}$

Therefore, letting r_{ij} be the elements of $\underset{B\leftarrow I}{R}$ one obtains

$$\begin{cases} r_{13} = S_\phi S_\theta = \dfrac{I_1 w_1}{H_c} \\ r_{23} = C_\phi S_\theta = \dfrac{I_2 w_2}{H_c} \\ r_{33} = C_\theta = \dfrac{I_3 w_3}{H_c} \end{cases} \qquad H_c = \sqrt{(I_1 w_1)^2 + (I_2 w_2)^2 + (I_3 w_3)^2}$$

$\Longrightarrow \quad \theta = \text{constant}$ because $w_3 = w_{30} (\geq 0)$

Due to the equations on $\dot{\psi}, \dot{\theta}, \dot{\phi}$, one obtains

$$\begin{cases} \dot{\psi} = \dfrac{I_1 w_1^2}{H_c S_\theta^2} + \dfrac{I_2 w_2^2}{H_c S_\theta^2} = \dfrac{I_T w_{12}^2}{H_c S_\theta^2} > 0 \\ \dot{\theta} = 0 \\ \dot{\phi} = w_{30} - C_\theta \dfrac{I_T w_{12}^2}{H_c S_\theta^2} \end{cases}$$

In order that $\dot{\theta} = 0$, from $\dot{\theta} = \omega_{12} \cos(\omega_p t + \varphi + \phi) = 0$

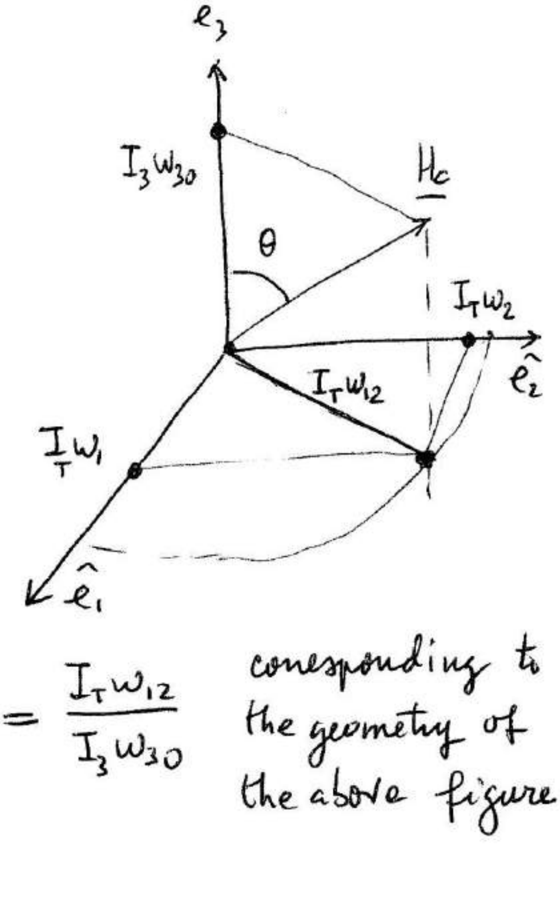
and this equation yields $\omega_p t + \varphi + \phi = \frac{\pi}{2} + k\pi \quad (k \in \mathbb{Z})$

But only $\omega_p t + \varphi + \phi = \frac{\pi}{2} + 2l\pi \quad (l \in \mathbb{Z})$ is acceptable because

$\dot{\psi} = \frac{\omega_{12}}{S_\theta} \sin(\omega_p t + \varphi + \phi) > 0$ (because $\dot{\psi} = \frac{I_1 \omega_{12}^2}{H_c S_\theta^2} > 0$)

Therefore $\phi = \frac{\pi}{2} + 2l\pi - \omega_p t - \varphi \Rightarrow \dot{\phi} = -\omega_p = -\omega_{30}\left[\frac{I_3}{I_T} - 1\right]$

The latter equation is consistent with that found on page 13 and regarding $\dot{\phi}$. In fact



$$\omega_{30} - C_\theta \frac{I_T \omega_{12}^2}{H_c S_\theta^2} = -\omega_{30}\left[\frac{I_3}{I_T} - 1\right]$$

$$S_\theta I_3 \omega_{30} = \frac{I_T^2 \omega_{12}^2}{\tan\theta \, H_c}$$

$$S_\theta I_3 \omega_{30} = \underbrace{\frac{I_T \omega_{12}}{H_c}}_{S_\theta} \frac{I_T \omega_{12}}{\tan\theta}$$

$\Rightarrow \quad I_3 \omega_{30} S_\theta = I_T \omega_{12} C_\theta \rightarrow \tan\theta = \frac{I_T \omega_{12}}{I_3 \omega_{30}}$ corresponding to the geometry of the above figure

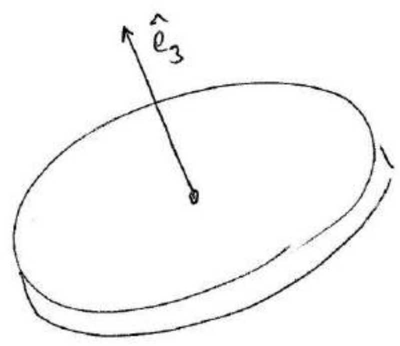
It is apparent that

$$\begin{cases} \dot{\psi} > 0 \\ \dot{\theta} = 0 \\ \dot{\phi} \begin{cases} > 0 & \text{if } I_3 < I_T \quad \text{prolate body} \rightarrow \text{prograde precession} \\ < 0 & \text{if } I_3 > I_T \quad \text{oblate body} \rightarrow \text{retrograde precession} \end{cases} \end{cases}$$

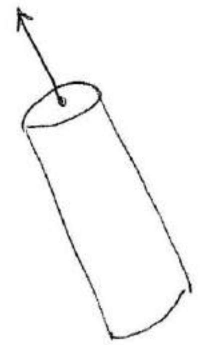
In the retrograde precession

$$\dot{\phi} = \omega_{30} - C_\theta \dot{\psi} < 0 \quad \left(\text{where } C_\theta > 0 \text{ because } \omega_{30} \geq 0\right)$$

this means that ψ increases faster than the rate ω_{30} (>0)



oblate body

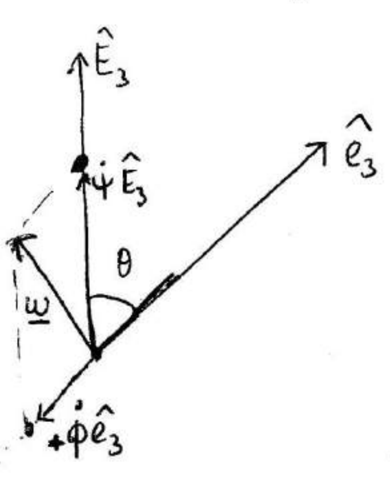


prolate body

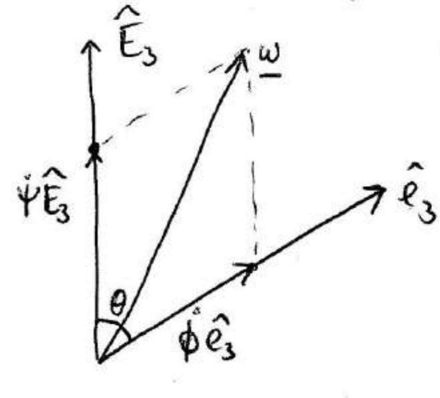
Moreover, the angular velocity may be written also as

$$\underline{\omega} = \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3 = \dot{\psi} \hat{E}_3 + \dot{\theta} \hat{N} + \dot{\phi} \hat{e}_3 \underset{\substack{\uparrow \\ \dot{\theta}=0}}{=} \dot{\psi} \hat{E}_3 + \dot{\phi} \hat{e}_3$$

(A) OBLATE BODY $(I_3 > I_T)$



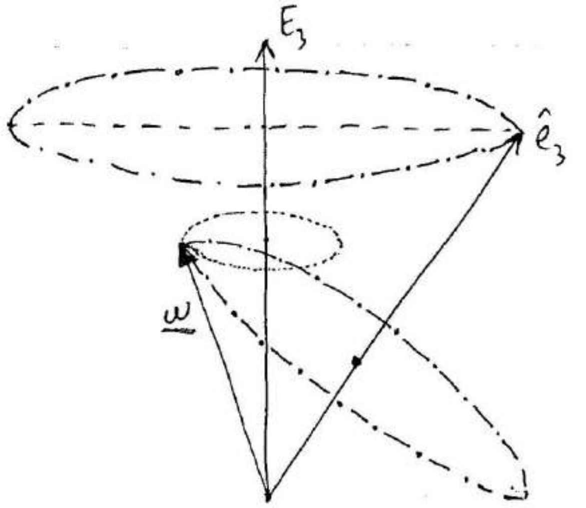
(B) PROLATE BODY $(I_T > I_3)$



In both cases $\hat{e}_3$ rotates in counterclockwise sense around $\hat{E}_3$ together with $\underline{\omega}$ that rotates as well.

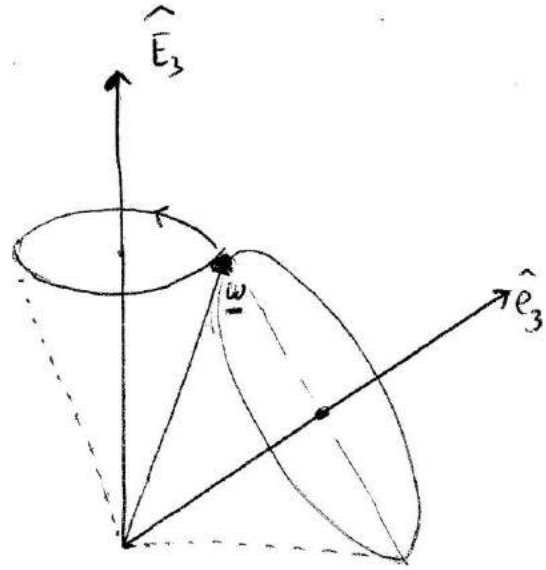
<u>Remark</u> Axial symmetry is a sufficient condition in order to have $I_1 = I_2 = I_T$. For instance, also a homogeneous parallelepiped with a square face has two identical principal inertia moments

(A) OBLATE BODY $(I_3 > I_T)$



While $\underline{\omega}$ and $\hat{e}_3$ rotate together $\hat{e}_3$ is the axis of a cone that is internally tangential to the cone described by $\underline{\omega}$ around $\hat{E}_3$
The latter is the SPACE CONE

(B) PROLATE BODY $(I_T > I_3)$



While $\underline{\omega}$ and $\hat{e}_3$ rotate together $\hat{e}_3$ is the axis of a cone that is externally tangential to the cone described by $\underline{\omega}$ around $\hat{E}_3$
The latter is the SPACE CONE

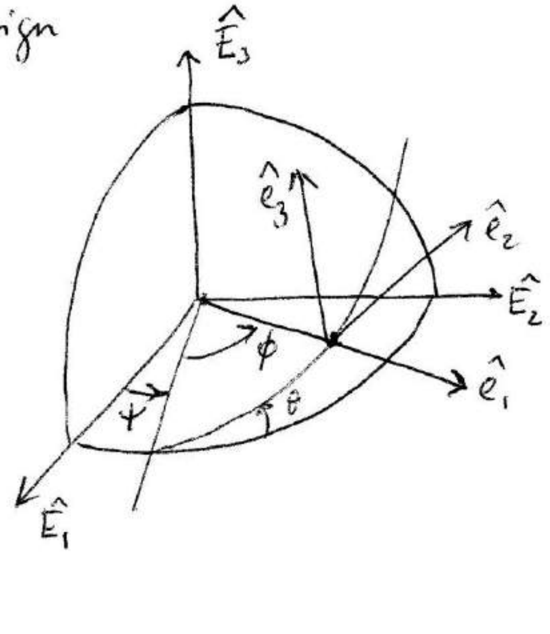
- General body with no external torque

The general solution for ψ, θ, ϕ under the assumption that $\hat{E}_3$ is aligned with $\underline{H_c}$ can be written in terms of elliptic integrals. This approach is omitted here, however.
It is remarkable the fact that $\dot{\psi} > 0$ in any case whereas $\dot{\phi}$ and $\dot{\theta}$ can have either sign

$$\dot{\psi} = \frac{1}{S_\theta}\left[\omega_1 S_\phi + \omega_2 C_\phi\right] = \frac{I_1 \omega_1^2 + I_2 \omega_2^2}{S_\theta^2 H_c} > 0$$

$$\dot{\theta} = \omega_1 \frac{I_2 \omega_2}{H_c S_\theta} - \omega_2 \frac{I_1 \omega_1}{H_c S_\theta} = \frac{\omega_1 \omega_2}{H_c S_\theta}\left(I_2 - I_1\right)$$

$$\dot{\phi} = \omega_3 - C_\theta \frac{I_1 \omega_1^2 + I_2 \omega_2^2}{H_c S_\theta^2}$$



- Equilibrium solutions

If the angular velocity remains aligned with a direction $\hat{a}$ attached to the body frame, one has

$$\underline{\omega} = \omega(t) \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}$$

where a_i are the components of $\hat{a}$ along the principal axes of inertia $\hat{e}_i$.

Insertion of this relation into the Euler equations yields

$$\begin{aligned} I_1 \dot{\omega} a_1 &= (I_2 - I_3) a_2 a_3 \omega^2 \\ I_2 \dot{\omega} a_2 &= (I_3 - I_1) a_1 a_3 \omega^2 \qquad \Rightarrow \\ I_3 \dot{\omega} a_3 &= (I_1 - I_2) a_2 a_1 \omega^2 \end{aligned}$$

$$\dot{\omega} = \frac{I_2 - I_3}{I_1} \frac{a_2 a_3}{a_1} \omega^2 = \frac{I_3 - I_1}{I_2} \frac{a_1 a_3}{a_2} \omega^2 = \frac{I_1 - I_2}{I_3} \frac{a_2 a_1}{a_3} \omega^2$$

or, equivalently

$$\omega^2 a_2^2 a_3^2 (I_2 - I_3) I_2 I_3 = \omega^2 a_1^2 a_3^2 (I_3 - I_1) I_1 I_3 = a_1^2 a_2^2 (I_1 - I_2) I_1 I_2 \omega^2$$

These 2 equalities are satisfied if

(a) $\omega = 0 \Rightarrow$ no rotation

(b) If $I_1 = I_2 = I_3$ (mass distribution with spherical symmetry) then the ~~th~~ two equalities are satisfied regardless of a_1, a_2, a_3

(c) If $\omega \neq 0$ and $I_1 = I_2 \neq I_3$ (mass distribution with axial symmetry, i.e. axisymmetric body with axis $\hat{e}_3$)
then $a_3 = 0 \Rightarrow$ fulfillment regardless of a_1, a_2
$\Rightarrow$ rotation axis $\perp$ to $\hat{e}_3$, i.e. $\perp$ to the symmetry axis

(d) If $\omega \neq 0$ and $I_1 < I_2 < I_3$, no solution if only a component $a_i \neq 0$. In fact

$$\left.\begin{array}{ll} (d1) & a_1 = 0,\ a_2 \neq 0,\ a_3 \neq 0 \\ (d2) & a_1 \neq 0,\ a_2 = 0,\ a_3 \neq 0 \\ (d3) & a_1 \neq 0,\ a_2 \neq 0,\ a_3 = 0 \end{array}\right\} \Rightarrow \text{One term equals } 0 \text{, the remaining two are not } 0$$

If 2 components (a_1, a_2) or (a_1, a_3) or (a_2, a_3) are 0 then one obtains an equilibrium solution. In this case the angular velocity is aligned with one of the principal axes of inertia

Hence, in general, pure spin about a principal axis of inertia is an equilibrium solution.

Moreover, spherical-mass-distribution spacecraft preserve $\underline{\omega}$ along the body axes regardless of $\{a_i\}$

- Stability of pure spin

Without any assumption on I_1, I_2, I_3

$$\begin{cases} \omega_1 = 0 \\ \omega_2 = 0 \\ \omega_3 = \omega_{30} \end{cases} \qquad \text{Pure spin about axis } \hat{e}_3$$

In the presence of a small perturbation about this equilibrium solution,

$$\begin{cases} \omega_1 = \delta\omega_1 \\ \omega_2 = \delta\omega_2 \\ \omega_3 = \omega_{30} + \delta\omega_3 \end{cases}$$

Insertion of these components into the Euler equations leads to

$$\delta\dot{\omega}_1 = \frac{I_2 - I_3}{I_1}\left(\omega_{30} + \delta\omega_3\right)\delta\omega_2 = \frac{I_2 - I_3}{I_1}\,\omega_{30}\,\delta\omega_2$$

$$\delta\dot{\omega}_2 = \frac{I_3 - I_1}{I_2}\left(\omega_{30} + \delta\omega_3\right)\delta\omega_1 = \frac{I_3 - I_1}{I_2}\,\omega_{30}\,\delta\omega_1$$

$$\delta\dot{\omega}_3 = \frac{I_1 - I_2}{I_3}\,\delta\omega_1\,\delta\omega_2 \simeq 0$$

The first two equations are linear in $\delta\omega_1, \delta\omega_2$. In compact form

$$\begin{bmatrix} \delta\omega_1 \\ \delta\omega_2 \end{bmatrix}^{\circ} = \begin{bmatrix} 0 & \frac{I_2 - I_3}{I_1} \\ \frac{I_3 - I_1}{I_2} & 0 \end{bmatrix} \omega_{30} \begin{bmatrix} \delta\omega_1 \\ \delta\omega_2 \end{bmatrix} = A \begin{bmatrix} \delta\omega_1 \\ \delta\omega_2 \end{bmatrix}$$

For linear time-invariant differential systems of the 1st order stability depends on the real part of the eigenvalues.

If $\quad Re(\lambda_i) \le 0 \ \forall i \ \Rightarrow$ simple stability

If $\quad Re(\lambda_i) < 0 \ \forall i \ \Rightarrow$ asymptotic "

In this case the eigenvalue equation is

$$\lambda^2 - \frac{(I_2 - I_3)(I_3 - I_1)\omega_{30}^2}{I_2 I_1} = 0 \qquad \lambda_{1,2} = \pm \omega_{30} \sqrt{\frac{(I_2 - I_3)(I_3 - I_1)}{I_1 I_2}}$$

Both eigenvalues are imaginary if $(I_2 - I_3)(I_3 - I_1) < 0$

$\Rightarrow$ I_3 is maximum or minimum

If I_3 is the intermediate moment $(I_1 < I_3 < I_2$ or $I_2 < I_3 < I_1)$ then an eigenvalue exists that has positive real part and the solution for $\delta\omega_1$, $\delta\omega_2$ diverges.

In general, the solution of the linear system is

$$\delta x_i = k_{i1} e^{\lambda_1 t} + k_{i2} e^{\lambda_2 t}$$

, k_{i1} and k_{i2} depend on the initial conditions.

In the end, pure spin about

(a) the maximal or minimal inertia axis is simply stable

(b) the intermediate inertia axis is unstable

From a rigorous point of view, however, stability of the spin about the minimal and maximal inertia axes is not proven by linearizing the system, because the linear approximation yields imaginary eigenvalues. In this case, when the linearized model yields imaginary eigenvalues, the linear analysis of the related nonlinear system is inconclusive.
More insightful conclusions can be derived by considering the conservation of angular momentum and energy.

- Energy and momentum integrals

In the absence of external torques $\underline{H_c} =$ constant and also

$$\dot{T}_{rot} = \underline{\omega} \cdot \underline{\dot{H}_c} = 0 \quad \Rightarrow \quad T_{rot} = \text{constant}$$

If $\underline{H_c}$ is expressed in the principal axes of inertia, then

$$\underline{H_c}^{(B)} = \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix} \quad \text{and} \quad \underline{H_c} = \underbrace{\begin{bmatrix} H_1 & H_2 & H_3 \end{bmatrix}}_{\left[\underline{H_c}^{(B)}\right]^T} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}, \quad H_i = I_i \omega_i$$

and $\underline{H_c} =$ constant $\Rightarrow$ $H_1^2 + H_2^2 + H_3^2 = H^2$ $(=\text{constant})$

This equation identifies the locus of all the angular velocities that fulfill the angular momentum magnitude relation: it is a SPHERE of radius H

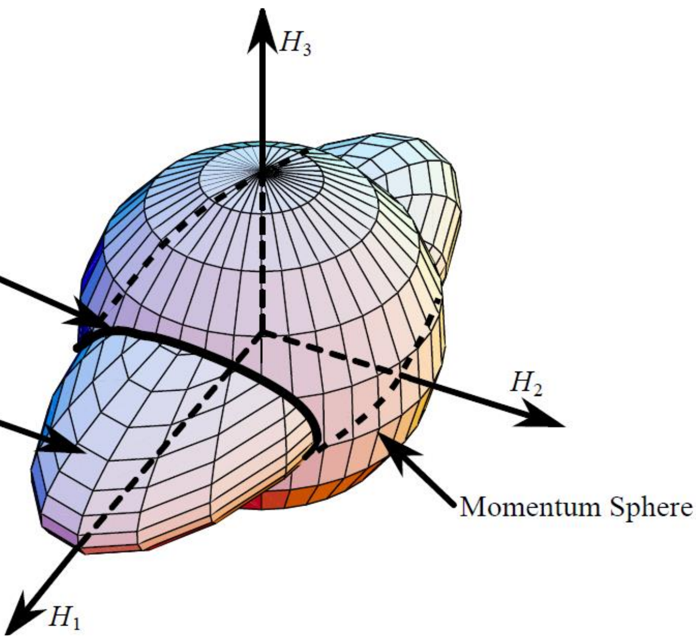
Moreover $\dot{T}_{rot} = 0 \Rightarrow \frac{1}{2}\omega_1^2 I_1 + \frac{1}{2}\omega_2^2 I_2 + \frac{1}{2}\omega_3^2 I_3 = T_{rot}$ $(=\text{constant})$

The preceding relation is rewritten in terms of H_1, H_2, H_3, to yield

$$2T_{rot} = \frac{H_1^2}{I_1} + \frac{H_2^2}{I_2} + \frac{H_3^2}{I_3} \xrightarrow[a_i = \sqrt{2 I_i T_{rot}}]{} \frac{H_1^2}{a_1^2} + \frac{H_2^2}{a_2^2} + \frac{H_3^2}{a_3^2} = 1$$

This equation identifies the POINSOT ELLIPSOID, related to T_{rot}

The trajectory of all possible $\underline{\omega}$ corresponds to the intersection of the energy ellipsoid with the momentum sphere



Given $\underline{H} = \text{const}$, $|\underline{H}|^2 = \text{const}$, only a given range of energies corresponds to feasibility.

Assuming $I_1 > I_2 > I_3$, the energy ellipsoid has

(i) greatest axis along 1

(ii) intermediate axis along 2

(iii) smallest axis along 3

and the 3 special cases shown in next pages can occur

(A) MAXIMUM ENERGY, (B) INTERMEDIATE ENERGY, (C) MINIMUM ENERGY

(A) MAXIMUM ENERGY : 2 intersections, at points $(0, 0, \pm H)$

$T_{max} = \frac{H^2}{2 I_3}$ i.e. pure spin about axis of min inertia 3

(B) INTERMEDIATE ENERGY: 2 closed circles, which intersect at

$T_{int} = \frac{H^2}{2 I_2}$ 2 points $(0, \pm H, 0)$

the 2 circles are the separatrices

the 2 points correspond to pure spin about axis of intermediate inertia 2

(C) MINIMUM ENERGY : 2 intersections, at points $(\pm H, 0, 0)$

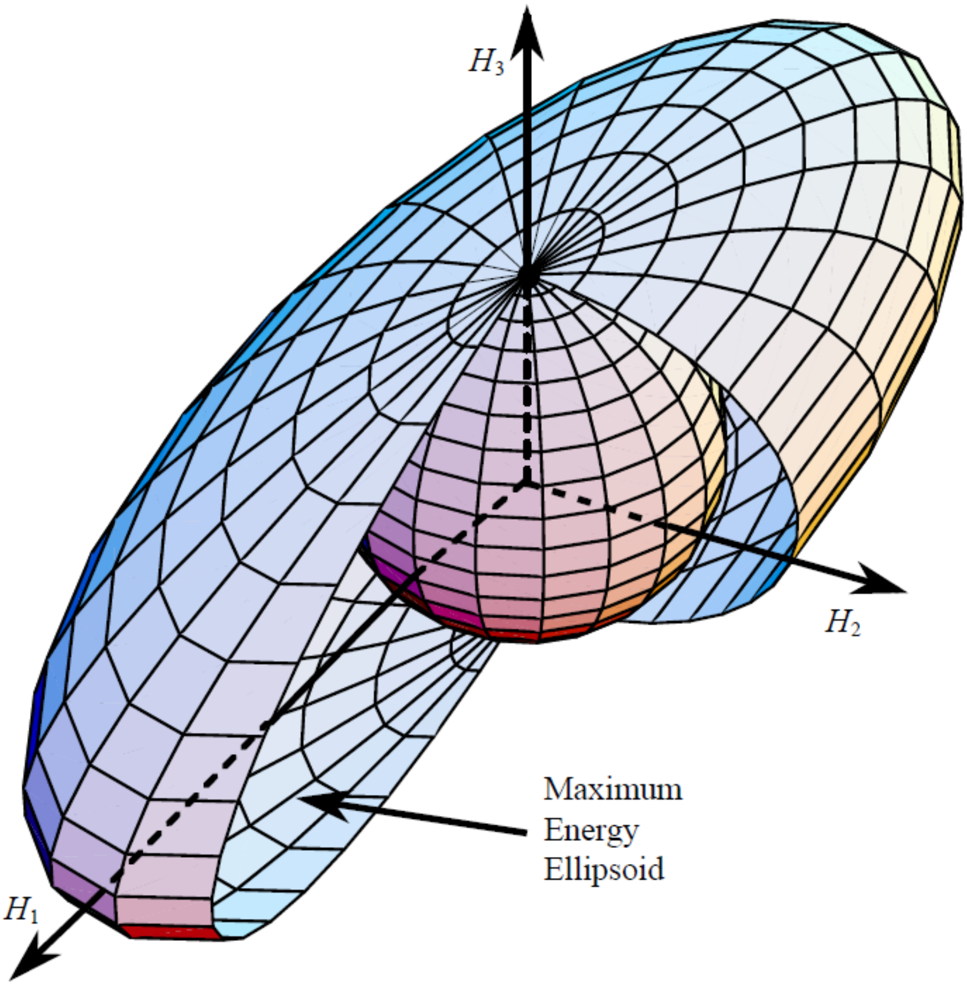
$T_{min} = \frac{H^2}{2 I_1}$ i.e. pure spin about axis of max inertia 1

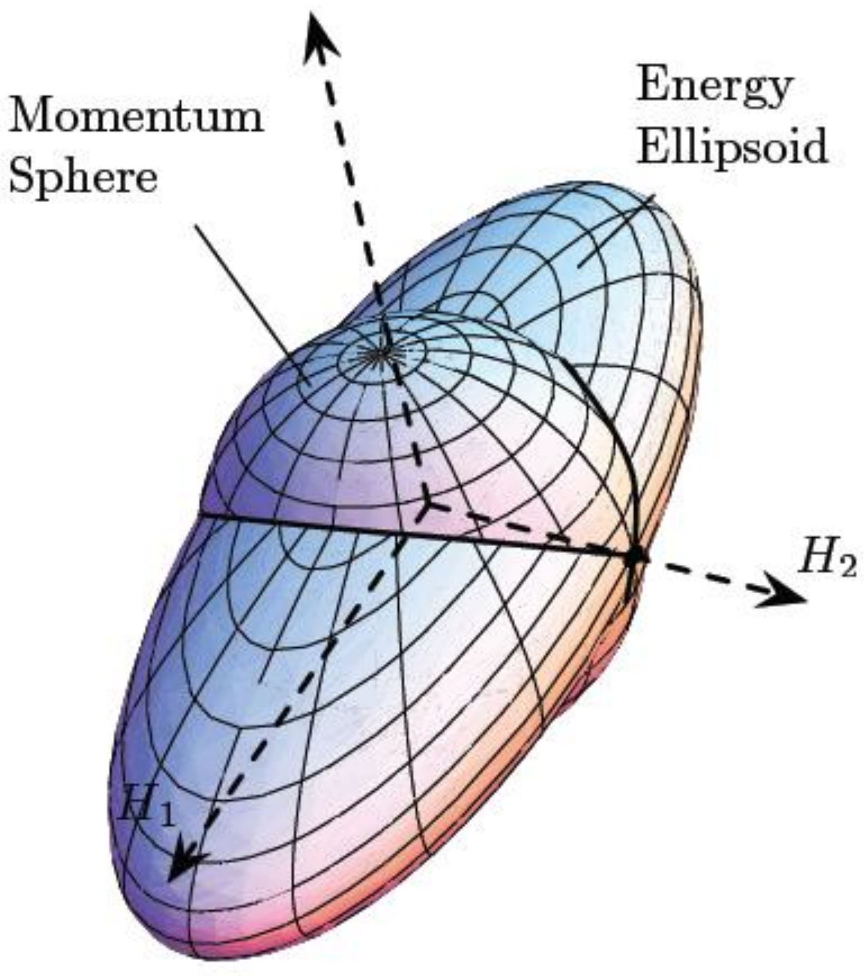
The last figure shows polhodes (i.e. the curves on the sphere) corresponding to different energies

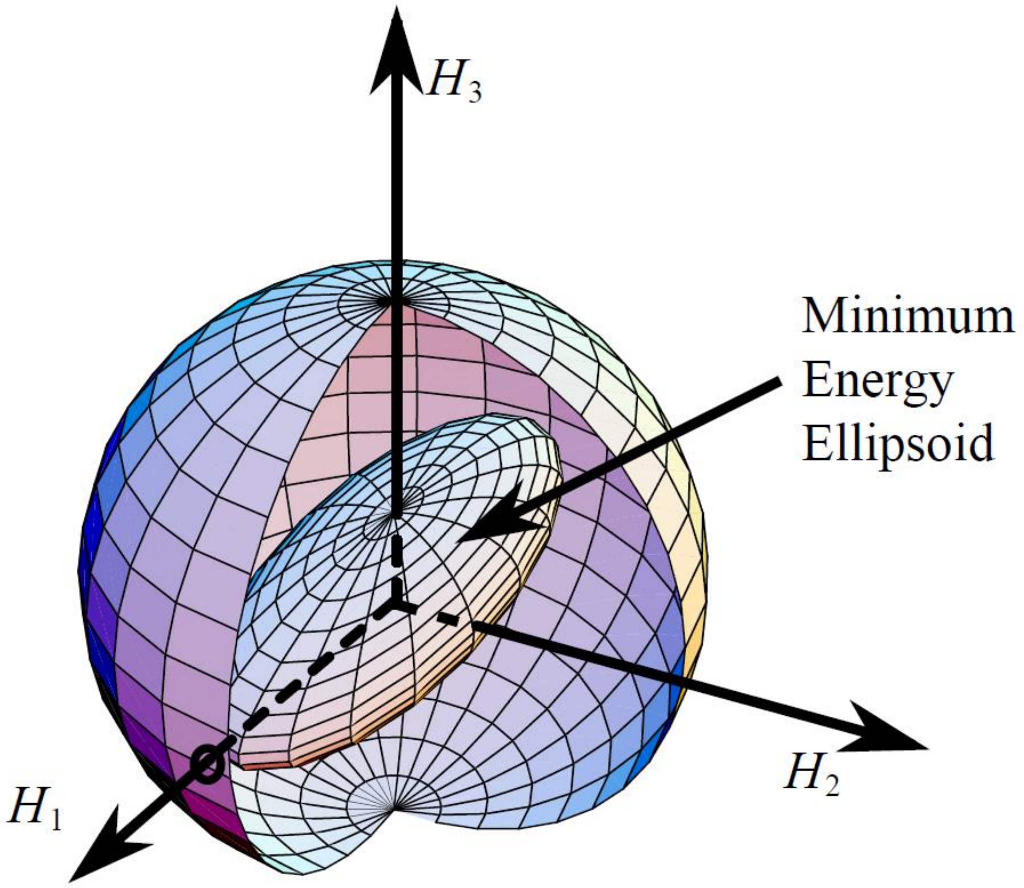
All curves are closed, except the 2 separatrices, which divide the closed curves about axis of min inertia from the closed curves about axis of max inertia.

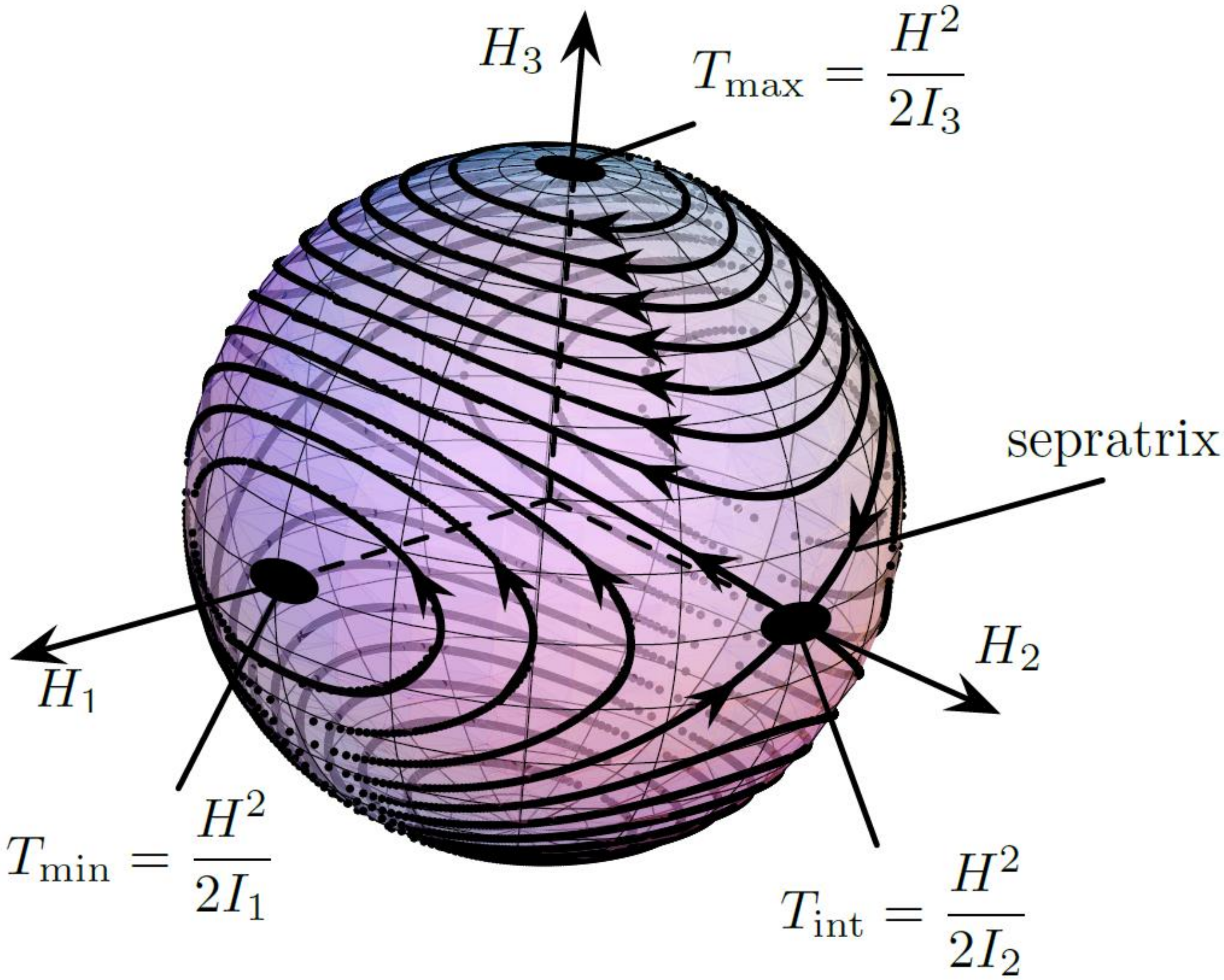
The fact that closed curves surround the axes of max and min inertia is an indication of neutral stability of pure spin about these 2 axes, unlike what occurs around the axis of intermediate inertia.

However, in real cases, $\dot{T}_{rot} < 0$ (energy dissipation), while $|\underline{H}| = const$, due to imperfect rigidity or internal damping (related to devices or dissipation of different nature). This means that $T_{rot} \downarrow$ up to reaching the minimum value. This is shown in the last figure, where the separatrices are crossed as T_{rot} reduces. Thus, the pure spin about 3 is unstable with dissipation.


H_3
H_2
Maximum
Energy
Ellipsoid
H_1


Momentum
Sphere
Energy
Ellipsoid
H_2
H_1

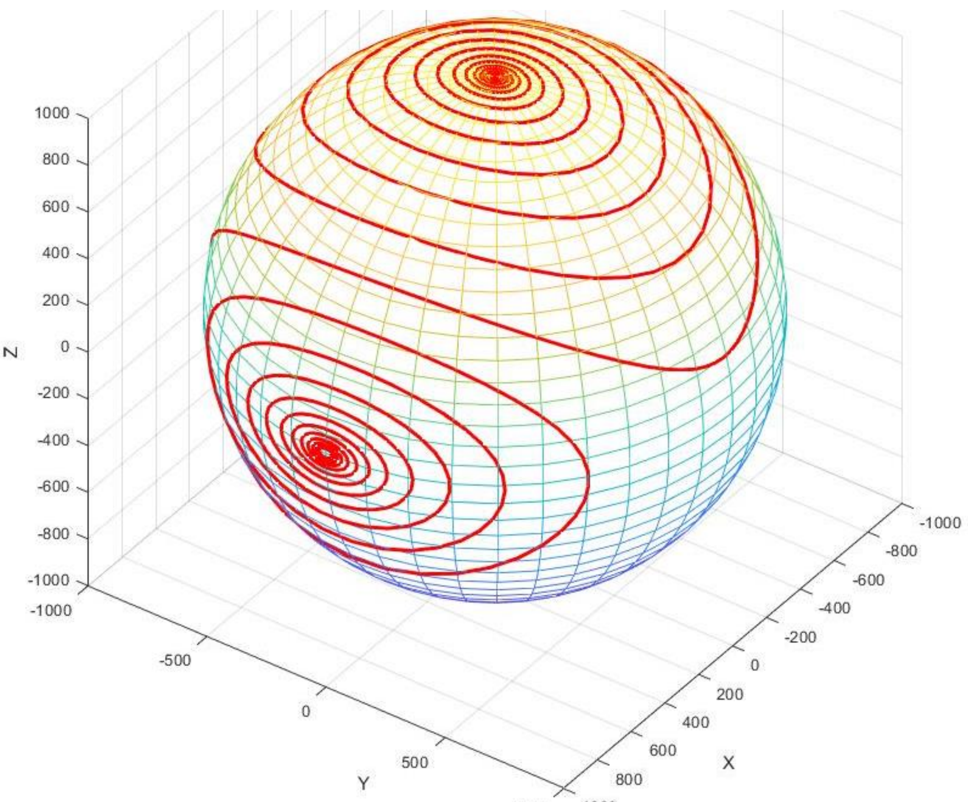

H_3
Minimum
Energy
Ellipsoid
H_2
H_1


H_3
$T_{\max} = \frac{H^2}{2I_3}$
sepratrix
H_1
H_2
$T_{\min} = \frac{H^2}{2I_1}$
$T_{\mathrm{int}} = \frac{H^2}{2I_2}$

Finally, as T_{rot} approaches T_{min}, the rotation tends to become pure spin about the maximal inertia axis, which is the only stable configuration when dissipation is considered.

SPIN ABOUT	NO DISSIPATION	DISSIPATION
MIN INERTIA AXIS	(Neutrally) stable	Unstable
INTERM INERTIA AXIS	Unstable	Unstable
MAX INERTIA AXIS	(Neutrally) stable	(Asymptotically) stable

After crossing the separatrix, ~~spin~~ rotation tends to occur about axis $\hat{e}_1$ (i.e. the maximal inertia axis)



This explains why celestial bodies (like the Earth) rotate about the maximal inertia axis.

This explains also why Explorer 1 started to tumble after being launched into orbit spinning about its axis of least inertia.

• Nutation of axisymmetric bodies

When $I_1 = I_2 = I_T$ one can easily find that

$$T_{rot} = \underbrace{\frac{1}{2} I_T \left(\omega_1^2 + \omega_2^2\right)}_{\text{transverse}} + \underbrace{\frac{1}{2} I_3 \omega_3^2}_{\text{spin}}$$

$$\underline{H} = I_T \omega_1 \hat{e}_1 + I_T \omega_2 \hat{e}_2 + I_3 \omega_3 \hat{e}_3 \rightarrow H^2 = I_T^2 \left(\omega_1^2 + \omega_2^2\right) + I_3^2 \omega_3^2$$

Now, the quantity $2 I_3 T_{rot} - H^2$ equals

$$2 I_3 \left\{ \frac{1}{2} I_T \left(\omega_1^2 + \omega_2^2\right) + \frac{1}{2} I_3 \omega_3^2 \right\} - I_T^2 \left(\omega_1^2 + \omega_2^2\right) - I_3^2 \omega_3^2 =$$

$$= I_T \left(I_3 - I_T\right)\left(\omega_1^2 + \omega_2^2\right)$$

As $H =$ constant (in the absence of external torques) one obtains

$$\frac{d}{dt}\left(2 I_3 T_{rot} - H^2\right) = 2 I_3 \dot{T}_{rot} = I_T \left(I_3 - I_T\right) \frac{d}{dt} \left(\omega_1^2 + \omega_2^2\right)$$

If $\dot{T}_{rot} = 0 \Rightarrow \omega_1^2 + \omega_2^2 =$ constant (as already proven)

However, if $\dot{T}_{rot} < 0 \Rightarrow$

(a) $I_3 - I_T > 0$ (oblate body) : $\omega_1^2 + \omega_2^2 \longrightarrow 0$

i.e. the body tends to assume a pure spin rotation

(b) $I_3 - I_T < 0$ (prolate body): $(\omega_1^2 + \omega_2^2)$ increases toward the maximal value compatible with $H =$ const

i.e. the body ~~the~~ tends to assume a flat spin rotation

flat spin for a prolate body

As H preserves in both cases, one can find easily the final angular velocity components:

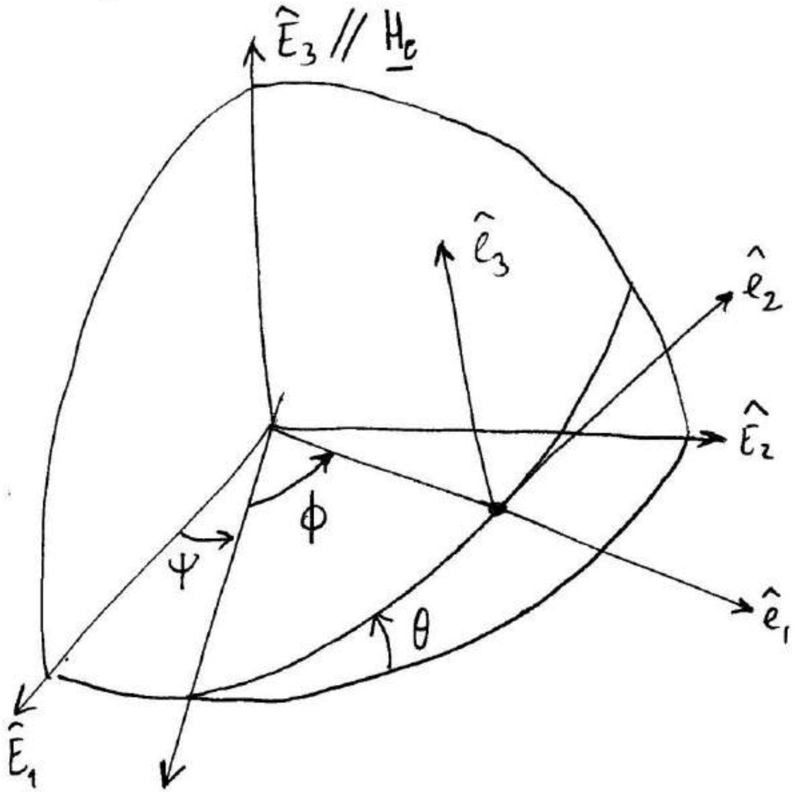
$$H^2 = I_T^2 (\omega_{10}^2 + \omega_{20}^2) + I_3^2 \omega_{30}^2 = I_3 \omega_{3f}^2 \rightarrow \omega_{3f}^2 \quad \text{(oblate body)}$$

$$H^2 = I_T^2 (\omega_{10}^2 + \omega_{20}^2) + I_3^2 \omega_{30}^2 = I_T^2 (\omega_{1f}^2 + \omega_{2f}^2) \rightarrow \omega_{1f}^2 + \omega_{2f}^2 \quad \text{(prolate body)}$$

It is worth stressing that these two relations hold in concrete situations where dissipation ($\dot{T}_{rot} < 0$) is considered

In ideal situations (where $\dot{T}_{rot} = 0$) the previous formulas for the angular velocity hold (see pages 10-16)

The intrinsic inertial frame, was already defined as follows $\hat{E}_3$ aligned with the initial angular momentum $\underline{H}_c(t_0)$



$(\hat{E}_1, \hat{E}_2, \hat{E}_3)$ inertial frame

$(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ body axes (aligned with the principal axes of inertia)

As remarked, internal energy dissipation does not change $\underline{H}_c$, only reduces the Kinetic energy

(a) OBLATE BODY $(I_3 > I_T)$

with dissipation $\underline{\omega} \longrightarrow \omega_{3f}\, \hat{e}_3$ as $t \longrightarrow \infty$.

Hence $\underline{H}_c$ tends to be aligned with $\hat{e}_3$ as $t \to \infty$ and this means that $\theta \longrightarrow 0$ (pure spin about $\hat{e}_3$)

(b) PROLATE BODY $(I_3 < I_T)$

with dissipation $\underline{\omega} \longrightarrow \omega_{1f}\, \hat{e}_1 + \omega_{2f}\, \hat{e}_2$

Hence $\underline{H}_c$ tends to be coplanar with $(\hat{e}_1, \hat{e}_2)$, or, in other words, $\underline{H}_c$ tends to have component $=0$ along $\hat{e}_3$

This means that $\theta \to \frac{\pi}{2}$ as $t \to \infty$ (flat spin)

• ATTITUDE MANEUVERS OF SPINNING SATELLITES

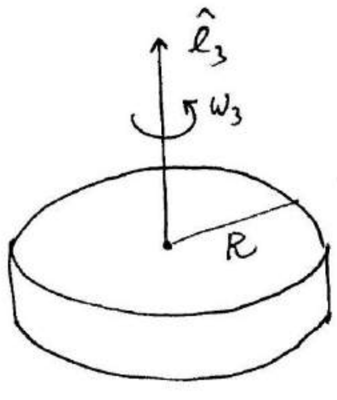
Spinning satellites about the axis of maximal inertia have an intrinsic stabilization. In some cases, maneuvers are needed, mainly for two reasons:

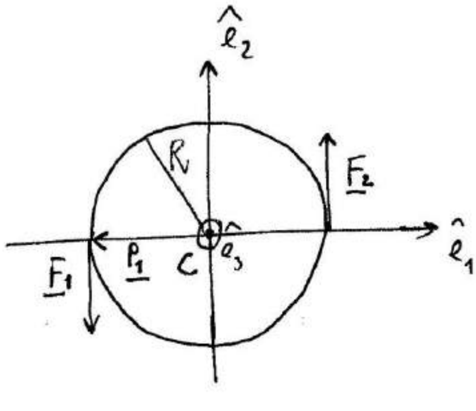
(a) SPIN-UP or SPIN-DOWN (DE-SPIN) the satellite

(b) CHANGE ORIENTATION of the spacecraft

• Spin-up and spin-down maneuvers

(A) Both spinup and spindown maneuvers can be performed using thrusters, which yield a torque aligned with axis $\hat{e}_3$





Thrusters for attitude are ignited in pairs, like in the above figure. In this way, no effect on the motion of the center of mass is produced. Conversely, the two forces $\underline{F}_1, \underline{F}_2$ yield a torque about the mass center C

$$\underline{L}_c = \underline{P}_1 \times \underline{F}_1 + \underline{P}_2 \times \underline{F}_2 = 2RF\,\hat{e}_3 \qquad (F = |\underline{F}|)$$

where $\underline{F} = \underline{F}_1 = -\underline{F}_2$ was used.

In the previous figures $\underline{L}_c$ has positive component along $\hat{e}_3$, thus the torque has the effect of increasing ω_3. In fact, considering that $\omega_1 = 0$, $\omega_2 = 0$, one obtains

$$\begin{cases} I_1 \dot{\omega}_1 = 0 \\ I_2 \dot{\omega}_2 = 0 \\ I_3 \dot{\omega}_3 = L_C = 2RF \end{cases}$$

in consideration that
$I_1 = I_2 = I_T$ and $\omega_1 = \omega_2 = 0$

If the direction of $\underline{F_1}$ and $\underline{F_2}$ is reversed, one obtains a despin (spindown) maneuver

$$I_3 \dot{\omega}_3 = -2RF$$

In both cases, the time variation for ω_3 is easy to find if F is assumed as constant

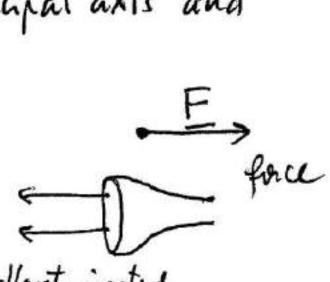
$$\begin{cases} \omega_3 = \omega_{30} + 2\dfrac{RF}{I_3}t \\ \text{(spinup maneuver)} \\ \omega_3 = \omega_{30} - 2\dfrac{RF}{I_3}t \\ \text{(despin maneuver)} \end{cases}$$

where $\omega_{30} = \omega_3(0)$ $(t_0 = 0)$

(and ω_{30} is assumed as positive)

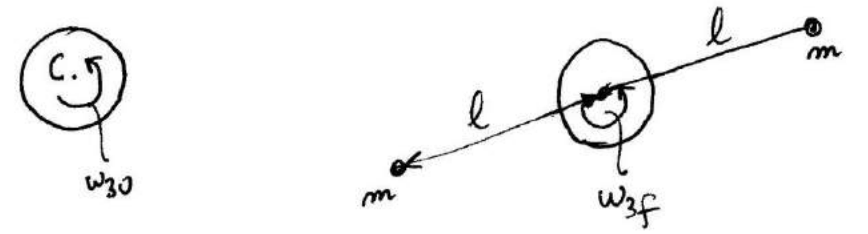
The corresponding attitude is found by integrating the kinematics equation, using any of the available representations (either sequences of angles, Euler parameters, or principal axis and angle).

Thrusters yield force by ejecting the propellant in the opposite direction



(B) For spindown maneuvers, yoyo systems can be used. They consist in (at least) two appendages that are deployed and have masses at their extremal point

Spinning sat at $t_0 = 0$ Spinning sat at t_f



l = distance of m from C

Due to conservation of angular momentum, after deployment the angular momentum is preserved, thus

$$\begin{cases} H_0 = I_3 \omega_{30} \\ H_f = (I_3' + 2ml^2)\,\omega_{3f} \end{cases} \quad \text{and} \quad H_0 = H_f$$

where $I_3' \leq I_3$ (because I_3' is the inertia matrix of the axisymmetric body without the two masses m, but these masses are relatively small)

In the end,

$$\omega_{3f} = \frac{I_3 \omega_{30}}{(I_3' + 2ml^2)} < \omega_{30}$$

The effect increases as l increases.

It is worth remarking that two equal masses located at equal distances from C are used, in order that the center of mass at t_f remains located at its original point (i.e. the center of the cylinder satellite).

- <u>Impulsive attitude maneuvers</u>

Impulsive maneuvers represent an approximation of real, finite-time maneuvers that change attitude.

With reference to the spin up maneuver performed with thrusters

$$w_3 = w_{30} + \frac{L_3}{I_3} t \qquad L_3 = 2RF \; (= \text{const})$$

and the corresponding Euler angle ϕ (i.e. the spin angle) can be found by integrating the respective kinematics equation,

$$\dot{\tilde{\phi}} = \dot{\psi} + \dot{\phi} \underset{\uparrow_{\theta=0}}{=} w_3 \qquad \Rightarrow \qquad \tilde{\Phi} = \tilde{\Phi}_0 + w_{30} t + \frac{1}{2}\frac{L_3}{I_3} t^2$$

The maneuver is assumed to take the time Δt. At the final time $t_f = t_0 + \Delta t = \Delta t$ $(t_0 = 0)$, one has

$$\begin{cases} w_{3f} = w_{30} + \dfrac{L_3}{I_3} \Delta t \\ \tilde{\Phi}_f = \tilde{\Phi}_0 + w_{30} \Delta t + \dfrac{1}{2}\dfrac{L_3}{I_3} (\Delta t)^2 \end{cases}$$

An impulsive maneuver can be regarded as the limiting situation where $\Delta t \to 0$ while $\frac{L_3}{I_3} \Delta t$ remains constant. This quantity represents the angular velocity increase $\Delta w_3 = w_{3f} - w_{30} = \frac{L_3}{I_3} \Delta t$

It is apparent that as $\Delta t \to 0$ $\frac{L_3}{I_3} \to \infty$ and $\frac{L_3}{I_3}$ can be regarded as a Dirac delta, from a rigorous point of view.

In the above equations, in the limit as $\Delta t \to 0$,

$$\begin{cases} w_{3f} = w_{30} + \Delta w_3 \\ \tilde{\Phi}_f = \tilde{\Phi}_0 + w_{30} \Delta t + \frac{1}{2} \Delta w_3 \, \Delta t \xrightarrow[\Delta t \to 0]{} \tilde{\Phi}_0 \end{cases}$$

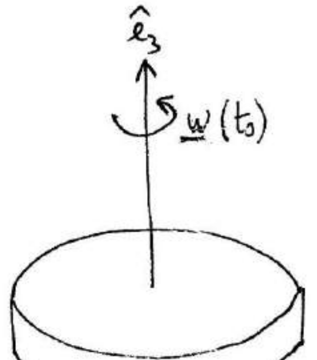
In the end, the overall effect of an impulsive torque is

a) instantaneous change of the angular velocity Δw_3

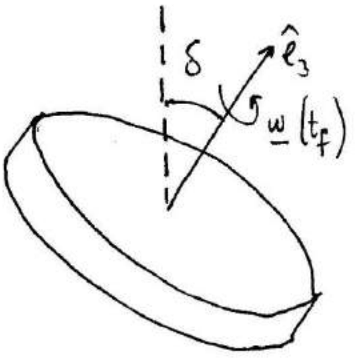
b) no change of the attitude orientation

- <u>Reorientation maneuvers</u>

Spinning satellites around the maximal inertia axis have inertial stabilization. In some cases, one wishes to change the inertial orientation of $\hat{e}_3$ by angle δ.



This maneuver can be performed using the so called "impulsive" torques, that are high-magnitude torques concentrated in very short time intervals



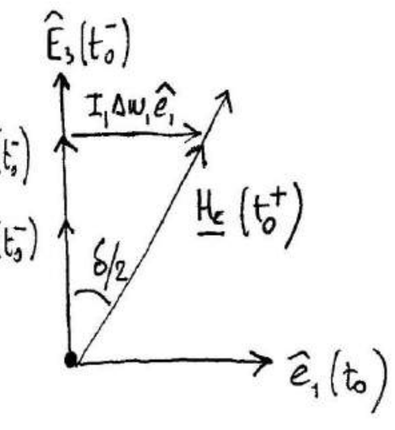
(1) First impulsive torque at t_0 to change $\underline{H}_c$ from $\underline{H}_c(t_0^-)$ to $\underline{H}_c(t_0^+)$

$$\underline{H}_c^{(E)}(t_0^-) = \begin{bmatrix} 0 \\ 0 \\ I_3 w_{30} \end{bmatrix}$$

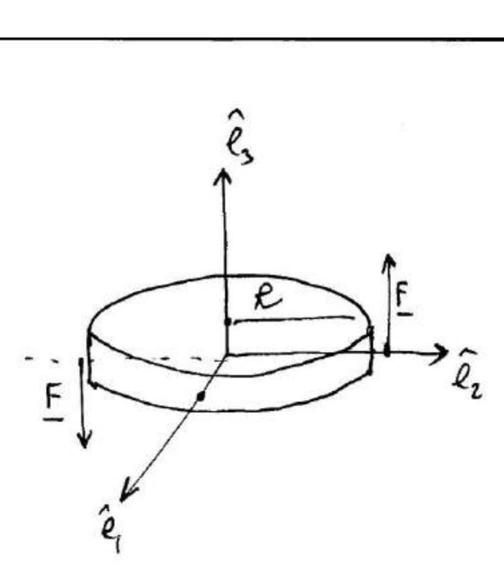
$$\underline{H}_c^{(E)}(t_0^+) = \begin{bmatrix} I_1 \Delta w_1 \\ 0 \\ I_3 w_{30} \end{bmatrix}$$

such that the angle $\frac{\delta}{2}$ is between $\underline{H}_c(t_0^-)$ and $\underline{H}_c(t_0^+)$

(1)



$$\underline{H}_c(t_0^+) = \underline{H}_c(t_0^-) + I_1 \Delta w_1 \hat{e}_1$$



In order to provide the impulsive change $I_1 \Delta w_1$ the impulsive torque is $2FR$

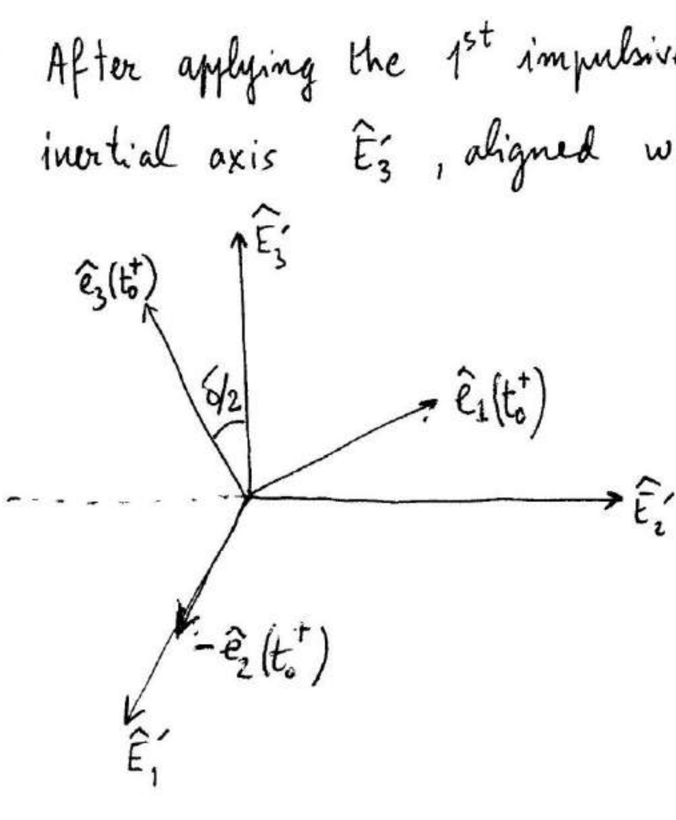
$$2FR\Delta t = I_1 \Delta w_1$$

applied during Δt (short)

The value of the impulsive change of angular velocity is found using the figure on page 7:

$$\frac{I_1 \Delta w_1}{I_3 w_{30}} = \tan\frac{\delta}{2} \quad \rightarrow \quad \Delta w_1 = \frac{I_3 w_{30}}{I_1} \tan\frac{\delta}{2}$$

(2) After applying the 1st impulsive torque, one can define the inertial axis $\hat{E}_3'$, aligned with $\underline{H}_c(t_0^+)$, while $\hat{e}_3(t_0^+) \equiv \hat{e}_3(t_0^-)$



whereas $\hat{E}_1'$ and $\hat{E}_2'$ are defined such that

$$\hat{E}_1' = -\hat{e}_2(t_0)$$

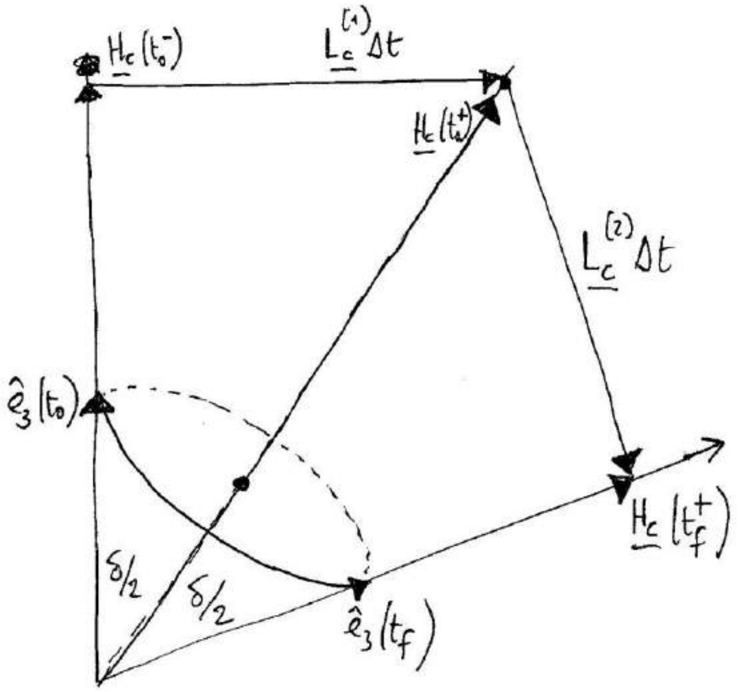
After t_0, the spinning spacecraft starts to have a retrograde precession (because it is oblate)

During this precession the nutation angle θ is equal to $\frac{\delta}{2}$ and remains constant. Conversely, ψ and ϕ vary

(a) $\dot{\psi} = \frac{\omega_{12}}{S_\theta} > 0$ where $\theta = \frac{\delta}{2}$ and $\omega_{12} = \Delta\omega_1$ (transverse component)

(b) $\dot{\phi} = \omega_{30}\left(1 - \frac{I_3}{I_T}\right) < 0$ for an oblate axisymmetric spacecraft

These equations hold in $[t_0^+, t_f^-]$, interval in which the spacecraft is subject to torque-free attitude motion, and $\hat{e}_3$ precedes about $\hat{E}_3' \parallel \underline{H_c}(t_0^+) \equiv \underline{H_c}(t_f^-)$, as shown in next figure



In $[t_0^+, t_f^-]$ also the three components are subject to

$$\begin{cases} \omega_1(t) = \omega_{10}\cos(\omega_p t) - \omega_{20}\sin(\omega_p t) \\ \omega_2(t) = \omega_{20}\cos(\omega_p t) + \omega_{10}\sin(\omega_p t) \\ \omega_3(t) = \omega_{30} \end{cases}$$

where $\omega_p = \omega_{30}\left(\frac{I_3}{I_T} - 1\right)$

and one can define $\underline{\omega_{12}} = \omega_1\hat{e}_1 + \omega_2\hat{e}_2$, recalling that $|\underline{\omega_{12}}| = \omega_{12} = \sqrt{\omega_{10}^2 + \omega_{20}^2} = \text{const}$ (for an axisymmetric spacecraft)

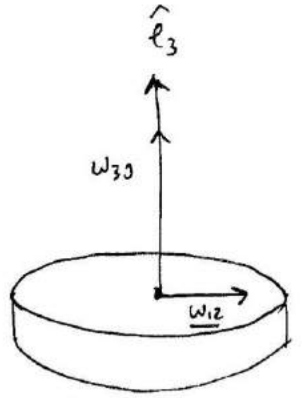
After half of the precession period, denoted with T_ψ, $\hat{e}_3$ has the correct orientation, displaced by δ from $\hat{e}_3(t_0)$.

The precession period T_ψ equals $T_\psi = \frac{2\pi}{\dot{\psi}} = \frac{2\pi S_\theta}{\omega_{12}}$

therefore $t_f^- - t_0^+ = \frac{\pi \sin\frac{\delta}{2}}{\omega_{12}}$

(3) The final impulsive torque, $\underline{L}_c^{(2)}$, has the final effect of aligning $\underline{H}_c$ with $\hat{e}_3$, i.e. also $\underline{\omega}$ with $\hat{e}_3(t_f)$

At t_f^-, the angular velocity components are given by



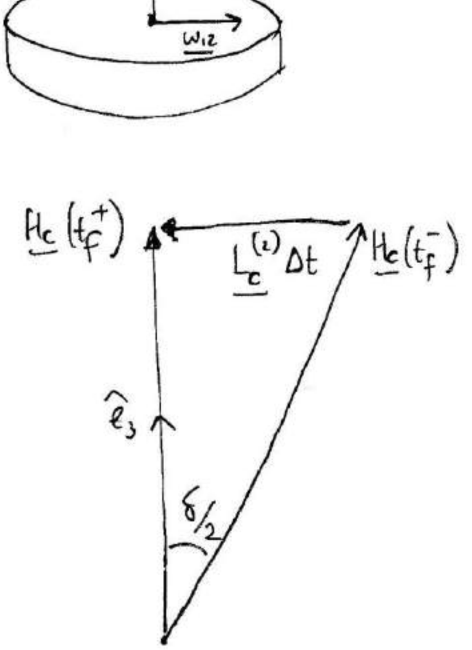
$$\underline{\omega}(t_f^-) = \omega_{30}\hat{e}_3 + \underline{\omega_{12}}(t_f^-) \quad , \text{where}$$

$$\underline{\omega_{12}} = \omega_1(t_f^-)\hat{e}_1 + \omega_2(t_f^-)\hat{e}_2 \quad \text{is}$$

the transversal angular velocity

The impulsive torque $\underline{L}_c^{(2)}$ is applied at t_f and is such that

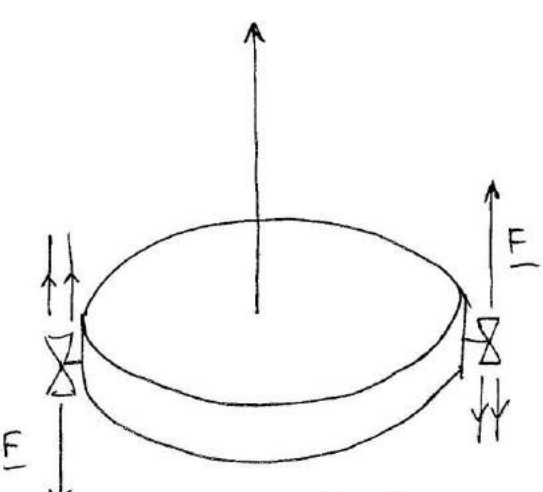
$$L_c^{(2)}\Delta t = I_T \omega_{12} = I_3 \omega_{30} \tan\frac{\delta}{2}$$



From the figure of previous page it is apparent that the first torque has the same magnitude, i.e.

$$L_c^{(1)}\Delta t = I_T \omega_{12} = I_3 \omega_{30} \tan\frac{\delta}{2}$$

In both cases, side jets (thrusters) are used to perform this maneuver. So far, one has assumed that sidejets are available to perform this second maneuver that are not necessarily the same ones employed for the first maneuver. If one wants to use the same side jets the correct phasing between ϕ and ψ must be achieved



⊙ direction of $\underline{L}_c$

For ψ and ϕ the following periods can be identified

$$\begin{cases} \omega_\psi = \dfrac{\omega_{12} I_T}{H_c} \dfrac{\omega_{12}}{S_\theta^2} = \dfrac{\omega_{12}}{S_\theta} = \dfrac{I_3}{I_T} \dfrac{\omega_{30}}{C_\theta} \\ \omega_\phi = \omega_{30}\left(1 - \dfrac{I_3}{I_T}\right) \equiv -\omega_p \qquad (\omega_p \text{ is the angular rate of } \underline{\omega_{12}}) \end{cases}$$

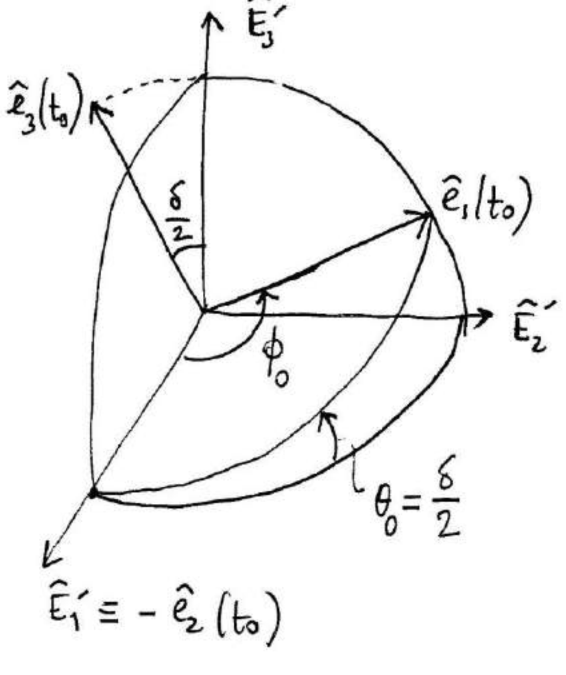
Hence, the periods that ψ and ϕ take to vary by 2π are

$$T_\psi = \frac{2\pi}{\omega_\psi} \qquad \text{and} \qquad T_\phi = \frac{2\pi}{\omega_\phi}$$

Correct phasing corresponds to

$$\bar{t} = \frac{T_\psi}{2} + k_1 T_\psi = k_2 T_\phi \qquad (k_1, k_2 \text{ integer})$$

<u>Remark</u> After the first impulsive torque, which is assumed to be aligned with $\hat{e}_1(t_0)$, axis $\hat{E}_3' \equiv \hat{H}_c(t_0^+)$ forms the angle $\frac{\delta}{2}$ with $\hat{e}_3(t_0)$ and $\left(\frac{\pi}{2} - \frac{\delta}{2}\right)$ with $\hat{e}_1(t_0)$, as shown in page 7.



The choice of $\hat{E}_2'$ and $\hat{E}_3'$ is arbitrary. However, if $\hat{E}_1'$ is chosen as aligned with $-\hat{e}_2(t_0)$, one can easily recognize that the values of the Euler angles at t_0 are

$$\begin{cases} \psi_0 = 0 \\ \theta_0 = \dfrac{\delta}{2} \\ \phi_0 = \dfrac{\pi}{2} \end{cases}$$