# KEPLERIAN TRAJECTORIES

## INTRODUCTION

Space mechanics
- ASTRODYNAMICS (artificial bodies)
  - orbital mechanics
  - attitude dynamics
- CELESTIAL MECHANICS (natural bodies)

## FUNDAMENTAL PRINCIPLES

(A) Second Newton's law: $\underline{F} = \dfrac{d}{dt}(m\,\underline{v})$ (in an inertial frame)

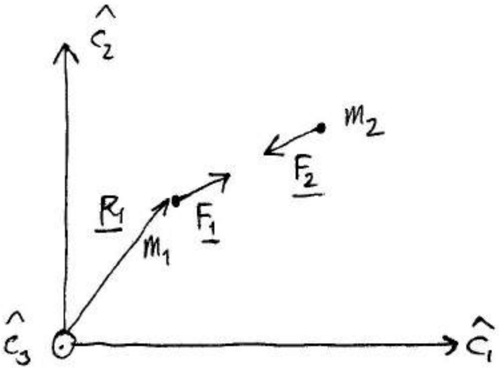
$m$ = mass $\quad \underline{v}$ = (inertial) velocity

(B) Gravitation law:

$$\underline{F_1} = \frac{G m_1 m_2}{R_{12}^3}\left(\underline{R_2} - \underline{R_1}\right)$$

$$\underline{F_1} = -\underline{F_2}$$

$$\underline{F_2} = \frac{G m_1 m_2}{R_{12}^3}\left(\underline{R_1} - \underline{R_2}\right)$$




$(\hat{c}_1, \hat{c}_2, \hat{c}_3)$ = inertial frame

where $\underline{R_1}$ = position vector of $m_1$

$\underline{R_2}$ = position vector of $m_2$

$R_{12} = \left|\underline{R_1} - \underline{R_2}\right|$

$G = 6.673 \cdot 10^{-11}\ m^3\, Kg^{-1}\, sec^{-2}$ is the universal constant of gravitation

The equivalence principle allows identifying the inertial and the gravitational mass.

## • TWO-BODY PROBLEM

Two masses subject to their mutual attraction obey the 2nd Newton's law and the gravitation law, thus

$$\underline{F_1} = m_1 \frac{d^2 \underline{R_1}}{dt^2} = \frac{G m_1 m_2}{R_{12}^3} \left( \underline{R_2} - \underline{R_1} \right)$$

$$\underline{F_2} = m_2 \frac{d^2 \underline{R_2}}{dt^2} = \frac{G m_1 m_2}{R_{12}^3} \left( \underline{R_1} - \underline{R_2} \right)$$

These relations lead to

$$\frac{d^2}{dt^2} \left( \underline{R_2} - \underline{R_1} \right) = - \frac{G (m_1 + m_2)}{R_{12}^3} \left( \underline{R_2} - \underline{R_1} \right)$$

The latter equation describes the motion of body 2 with respect to body 1

### • Restricted two-body problem

If $m_1 >> m_2$, then $m_1 + m_2 \simeq m_1$. Moreover, the effect of mass 2 on mass 1 can be neglected.

Letting $\underline{r} := \underline{R_2} - \underline{R_1}$ one obtains

$$\frac{d^2 \underline{r}}{dt^2} = - \frac{G m_1}{r^3} \underline{r} \longrightarrow \frac{d^2 \underline{r}}{dt^2} = - \frac{\mu}{r^3} \underline{r}$$

where $\mu$ is the gravitational parameter associated with the main attracting body.

For the Earth $\mu_\oplus = 398600.4 \frac{Km^3}{sec^2}$

## GRAVITATIONAL POTENTIAL

A conservative force can be obtained from the respective potential function $V$, i.e.

$$\underline{F} = \nabla V$$

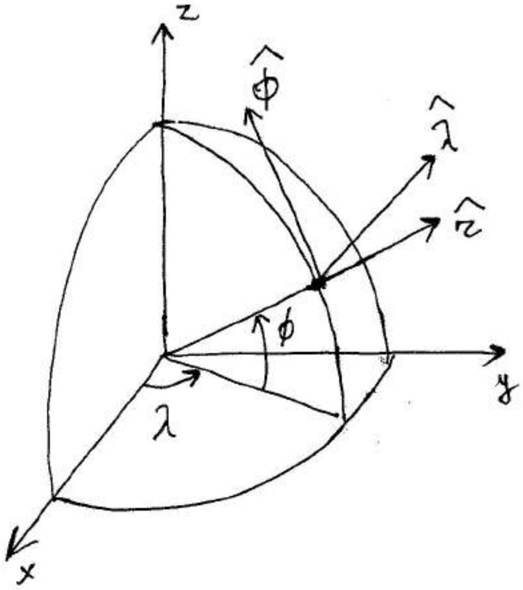
or from the potential energy $U$, i.e.

$$\underline{F} = -\nabla U$$

A possible choice for $V$ and $U$ (which are defined with an arbitrary additive constant, in general) is $V = -U$

If the potential function is written in spherical coordinates, then

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\lambda}}{r C\phi}\frac{\partial}{\partial \lambda} + \frac{\hat{\phi}}{r}\frac{\partial}{\partial \phi}$$




Because $\underline{F} = -\frac{G m_1 m_2}{r^3}\underline{r}$

it is straightforward to recognize that the potential associated with the gravitational field generated by a massive body (modeled as a point mass) is

$$V = \frac{G m_1 m_2}{r} \quad \text{whereas} \quad U = -\frac{G m_1 m_2}{r}$$

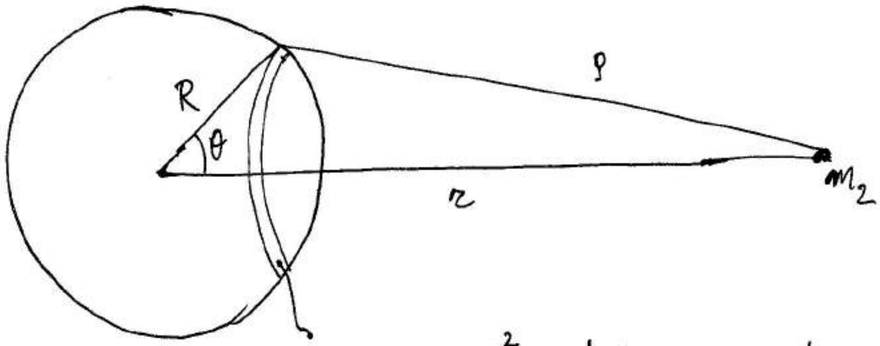
In orbital mechanics is more often used the potential per mass unit, i.e.

$$V = \frac{\mu}{r} \quad \text{and} \quad U = -\frac{\mu}{r}$$

- <u>Spherical mass distribution</u>

A spherical mass distribution can be regarded as the combination of spherical hollows. Each hollow has uniform mass density

For a thin spherical shell (hollow) of radius R




$$dm = 2\pi R^2 S_\theta \, d\theta \, \sigma \quad , \text{where} \quad \sigma = \frac{m_1}{4\pi R^2}$$

(surface mass density)

$$dV = \frac{G m_2 \, dm}{\rho} = \frac{G m_2}{\rho} 2\pi R^2 S_\theta \sigma d\theta \longrightarrow$$

$$\rightarrow V = G m_2 2\pi R^2 \sigma \int_0^\pi \frac{S_\theta}{\rho} d\theta \qquad \text{where} \quad \rho^2 = r^2 + R^2 - 2rR\,C_\theta$$

Because $\quad 2\rho d\rho = 2rRS_\theta \, d\theta \quad$ one obtains

$$V = G m_2 2\pi R^2 \sigma \int_{r-R}^{r+R} \frac{S_\theta \, \rho \, d\rho}{rR\,S_\theta \, \rho} = \frac{4\pi R^2 \sigma \, G m_2}{r} \underset{\uparrow}{=} \frac{G m_1 m_2}{r}$$

$$m_1 = 4\pi R^2 \sigma$$

Therefore, for a shell the potential is the same as that generated by a mass particle of mass $m_1$ located at its center

The same conclusion holds for the whole spherical mass distribution, regarded as the sum of infinitesimal (thin) shells.

## FIRST INTEGRALS

In the restricted two-body problem, a spacecraft obeys the equation

$$\frac{d^2 \underline{r}}{dt^2} = -\frac{\mu}{r^2}\,\hat{r}$$

Integrals are quantities that preserve in time

### Angular momentum

$\underline{h} := \underline{r} \times \underline{v}$ where $\underline{v} = \dfrac{d\underline{r}}{dt}$ — $\underline{h}$ is the (specific) angular momentum

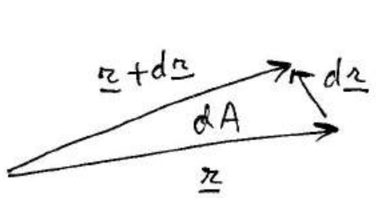
$$\frac{d\underline{h}}{dt} = \frac{d\underline{r}}{dt} \times \underline{v} + \underline{r} \times \frac{d^2\underline{r}}{dt^2} = 0 \quad \Rightarrow \quad \underline{h} = \text{constant}$$

Constancy of $\underline{h}$ has two implications

(a) Trajectory is planar, because $\hat{h} = $ constant. In fact, $\hat{h}$ is always orthogonal to the instantaneous plane of motion, therefore $\hat{h} = $ const $\Rightarrow$ planar motion

(b) Equal areas are swept in equal times ($2^{nd}$ KEPLER'S LAW).
In fact $\quad h\,dt = |\underline{r} \times \underline{v}|\,dt = |\underline{r} \times d\underline{r}|$

But $\quad |\underline{r} \times d\underline{r}| = 2\,dA$




and

$$h = 2\,\frac{dA}{dt} = \text{constant}$$

i.e. $\quad \dfrac{dA}{dt} = \text{constant} \quad$ (areolar velocity is constant)

• Eccentricity vector

The time derivative of the following vector is taken:

$$\frac{d}{dt}\left(\underline{h}\times\underline{v}\right) = \frac{d\underline{h}}{dt}\times\underline{v} + \underline{h}\times\frac{d\underline{v}}{dt} \underset{\substack{\uparrow\\ \underline{h}=\text{const}}}{=} \left(\underline{r}\times\underline{v}\right)\times\left(-\frac{\mu}{r^2}\hat{r}\right) =$$

$$= \left[\underline{r}\times\left(\dot{r}\hat{r}+\underline{\omega}\times\underline{r}\right)\right]\times\left(-\frac{\mu}{r^2}\hat{r}\right) =$$

$$= \left[\underline{\omega}r^2 - \underline{r}\left(\underline{\omega}\cdot\underline{r}\right)\right]\times\left(-\frac{\mu}{r^2}\hat{r}\right) = -\mu\underline{\omega}\times\hat{r} = -\mu\frac{d\hat{r}}{dt}$$

$$\Rightarrow \frac{d}{dt}\left[-\hat{r}+\frac{\underline{v}\times\underline{h}}{\mu}\right] = 0$$

Eccentricity vector is defined as $\underline{e} := -\hat{r}+\frac{\underline{v}\times\underline{h}}{\mu}$ and is another integral of motion, as well as $\underline{h}$

Eccentricity and angular momentum (per mass unit) are equivalent to 5 scalar integrals (i.e. 5 scalar quantities that do not change during the motion); not 6 because $\underline{h}$ and $\underline{e}$ are not independent (in fact $\underline{h}\bullet\underline{e}=0$)

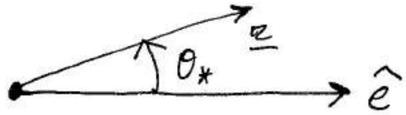
Because $\underline{h}\cdot\underline{e}=0$, $\underline{e}$ identifies a direction in the orbital plane. This direction is inertially fixed and it is associated with an important position of the spacecraft. This is being proven in the next pages.

## POSITION AND VELOCITY

Once the first integrals, $\underline{h}$ and $\underline{e}$, are identified, position and velocity along a (planar) Keplerian trajectory can be determined

### Polar equation

Eccentricity vector identifies a fixed direction in the orbit plane
The instantaneous position vector $\underline{r}$ forms the angle $\theta_*$ with $\underline{e}$
$\theta_*$ is termed TRUE ANOMALY




$$\underline{r}\cdot\underline{e} = \underline{r}\cdot\left[-\hat{r}+\frac{\underline{v}\times\underline{h}}{\mu}\right] = -r + \underline{h}\cdot\frac{\underline{r}\times\underline{v}}{\mu} = -r+\frac{h^2}{\mu}$$

i.e. $\quad r e c_{\theta_*} = -r+\dfrac{h^2}{\mu} \longrightarrow r = \dfrac{\frac{h^2}{\mu}}{1+e c_{\theta_*}}$ provided that $h \neq 0$

$\frac{h^2}{\mu}$ has the physical unit of [km] and is termed SEMILATUS RECTUM $p$ (or parameter), because $r = p = \frac{h^2}{\mu}$ when $\theta_* = \frac{\pi}{2}$

The polar equation $\quad r = \dfrac{p}{1+e c_{\theta_*}} \quad$ is associated with a conic arc, either an ellipse (1), a parabola (2), or a hyperbola (3).

(1a) $e = 0 \Rightarrow r = p$ (circle)

(1) $0 < e < 1 \Rightarrow$ ellipse

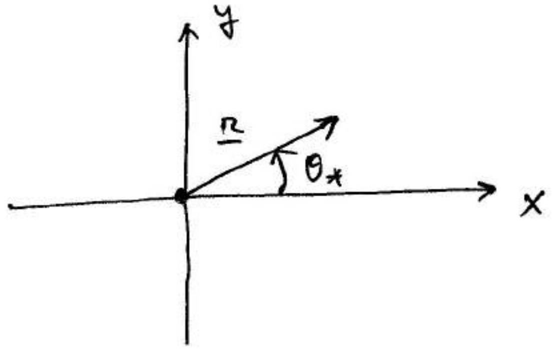
(2) $e = 1 \Rightarrow$ parabola $\qquad (h \neq 0)$

(3) $e > 1 \Rightarrow$ hyperbola

This is being proven in the next section.

- <u>Cartesian equation</u>

Let the reference system be centered in the center of the attracting body and with $x$ aligned with $\hat{e}$




$$\begin{cases} x = r\,c_{\theta_*} = \dfrac{p}{1+e\,c_{\theta_*}}\,c_{\theta_*} & (a) \\ y = r\,s_{\theta_*} = \dfrac{p}{1+e\,c_{\theta_*}}\,s_{\theta_*} & (b) \end{cases}$$

From (a): $\quad c_{\theta_*} = \dfrac{x}{p - xe}$

After inclusion of this relation into (b) one obtains

$$s_{\theta_*} = \frac{y}{p - xe}$$

Thus,

$$\left(\frac{x}{p-xe}\right)^2 + \left(\frac{y}{p-xe}\right)^2 = 1$$

$$x^2 + y^2 = (p - xe)^2$$

$$x^2(1-e^2) + y^2 + 2pex = p^2$$

The latter cartesian equation represents an

(1) Ellipse if $0 \le e < 1$

(2) Parabola if $e = 1$

(3) Hyperbola if $e > 1$

The special case $e=0$ is straightforward, i.e. a circle with center in the center of the attracting body

(A) $0 \leq e < 1$ (ELLIPSE)

Letting $\eta = x\sqrt{1-e^2}$ one obtains

$$\eta^2 + 2ep\frac{\eta}{\sqrt{1-e^2}} + y^2 = p^2 \quad \text{i.e.}$$

$$\eta^2 + 2ep\frac{\eta}{\sqrt{1-e^2}} + \frac{e^2p^2}{1-e^2} + y^2 = p^2 + \frac{e^2p^2}{1-e^2}$$

$$\left[\eta + \frac{ep}{\sqrt{1-e^2}}\right]^2 + y^2 = \frac{p^2}{1-e^2} \longrightarrow \left[x\sqrt{1-e^2} + \frac{ep}{\sqrt{1-e^2}}\right]^2 + y^2 = \frac{p^2}{1-e^2}$$

i.e. $$\left[x + \frac{ep}{1-e^2}\right]^2 + \frac{y^2}{1-e^2} = \frac{p^2}{(1-e^2)^2}$$

$$\longrightarrow \left[\frac{x + \frac{ep}{1-e^2}}{\frac{p}{1-e^2}}\right]^2 + \left[\frac{y}{\frac{p}{\sqrt{1-e^2}}}\right]^2 = 1$$

Cartesian equation of an ellipse, centred at $\left(\frac{-ep}{1-e^2}, 0\right)$

Semimajor axis is $a = \frac{p}{1-e^2}$

Semiminor axis is $b = \frac{p}{\sqrt{1-e^2}}$

Focal distance for an ellipse is $c = ea = \frac{ep}{1-e^2}$

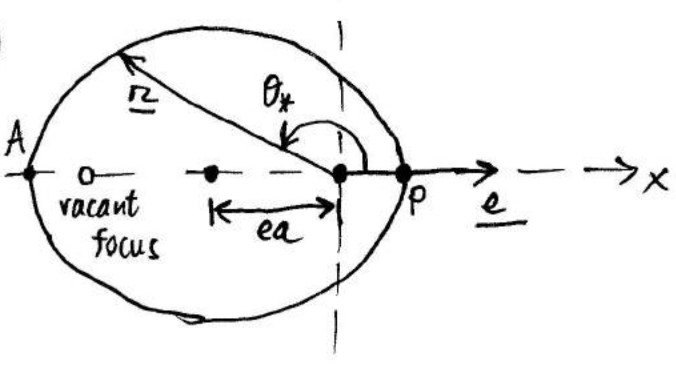
$\Rightarrow$ the center of the attracting body is at one of the two foci. The other one is termed the vacant focus (1st KEPLER'S LAW)

If $\theta_* = 0 \Rightarrow r = r_P = \frac{p}{1+e} = a(1-e)$

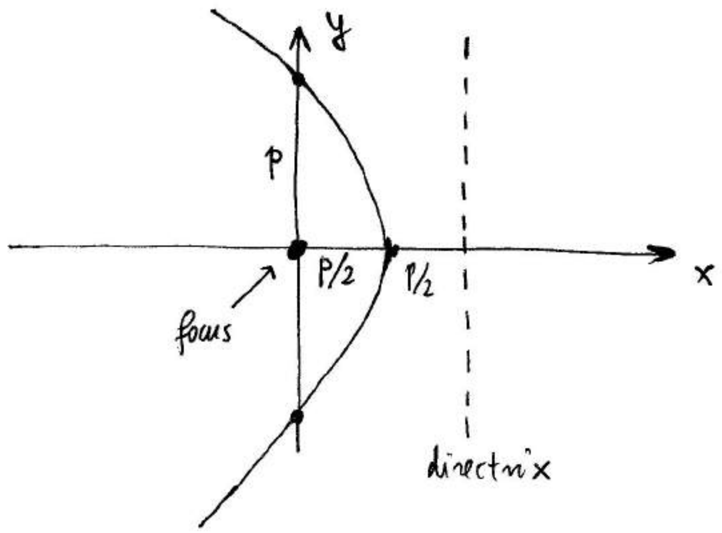
(PERIAPSE) $\Leftrightarrow$ P

If $\theta_* = \pi \Rightarrow r = r_A = \frac{p}{1-e} = a(1+e)$

(APOAPSE) $\Leftrightarrow$ A



(B) $e = 1$ (PARABOLA)




$$y^2 + 2px = p^2$$

At periapse

$$r_p = \frac{p}{2}$$

(C) $e > 1$ (HYPERBOLA)

Letting $\eta = x\sqrt{e^2-1}$ one obtains

$$\eta^2 - 2ep\frac{\eta}{\sqrt{e^2-1}} - y^2 = -p^2 \quad \text{i.e.}$$

$$\eta^2 - 2ep\frac{\eta}{\sqrt{e^2-1}} + \frac{e^2p^2}{e^2-1} - y^2 = -p^2 + \frac{e^2p^2}{e^2-1}$$

$$\left[\eta - \frac{ep}{\sqrt{e^2-1}}\right]^2 - y^2 = \frac{p^2}{e^2-1} \longrightarrow \left[x\sqrt{e^2-1} - \frac{ep}{\sqrt{e^2-1}}\right]^2 - y^2 = \frac{p^2}{e^2-1}$$

$$\text{i.e.} \quad \left[x - \frac{ep}{e^2-1}\right]^2 - \frac{y^2}{e^2-1} = \frac{p^2}{(e^2-1)^2}$$

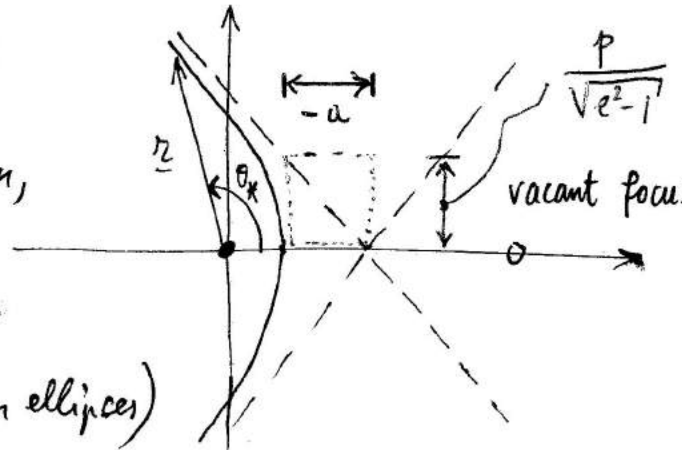
$$\longrightarrow \left[\frac{x - \frac{ep}{e^2-1}}{\frac{p}{e^2-1}}\right]^2 - \left[\frac{y}{\frac{p}{\sqrt{e^2-1}}}\right]^2 = 1$$

Cartesian equation of a hyperbola, centered at $\left(\frac{ep}{e^2-1}, 0\right)$

The two axes are $\left(\frac{p}{e^2-1}, \frac{p}{\sqrt{e^2-1}}\right)$

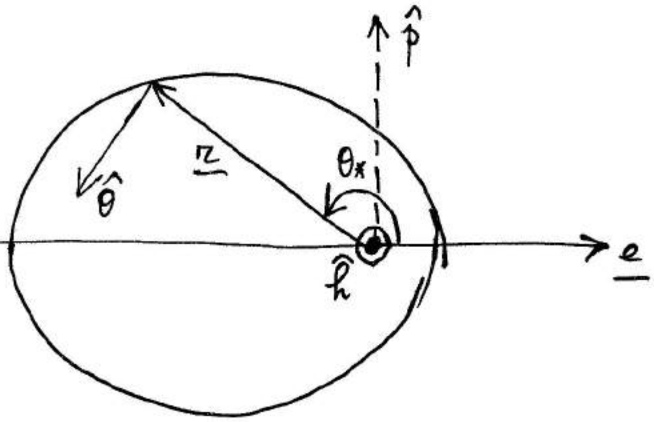
In orbital mechanics, by convention, the semimajor axis of a hyperbola is $a = p\frac{1}{1-e^2} < 0$

(formally, same definition as that for ellipses)



- Velocity

Along a Keplerian path of any kind, the velocity is found as follows




$$\underline{h} \times \underline{e} = \underline{h} \times \left[ -\hat{r} + \frac{\underline{v} \times \underline{h}}{\mu} \right]$$

$$h e \hat{p} = -h \hat{\theta} + \frac{1}{\mu} \left[ \underline{v} h^2 - \underline{h} (\underline{v} \cdot \underline{h}) \right]$$

$$h e \hat{p} = -h \hat{\theta} + \frac{h^2}{\mu} \underline{v}$$

$$\rightarrow \quad \underline{v} = \frac{\mu}{h} \left[ e \hat{p} + \hat{\theta} \right] = \sqrt{\frac{\mu}{p}} \left[ e \hat{p} + \hat{\theta} \right]$$

In terms of radial and horizontal components, $v_r$ and $v_\theta$

$$\begin{bmatrix} \hat{r} \\ \hat{\theta} \end{bmatrix} = \begin{bmatrix} c_{\theta_*} & s_{\theta_*} \\ -s_{\theta_*} & c_{\theta_*} \end{bmatrix} \begin{bmatrix} \hat{e} \\ \hat{p} \end{bmatrix} \rightarrow \begin{bmatrix} \hat{e} \\ \hat{p} \end{bmatrix} = \begin{bmatrix} c_{\theta_*} & -s_{\theta_*} \\ s_{\theta_*} & c_{\theta_*} \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\theta} \end{bmatrix}$$

After replacing $\hat{p}$ with

$\hat{p} = s_{\theta_*} \hat{r} + c_{\theta_*} \hat{\theta}$ one obtains

$$\underline{v} = \frac{\mu}{h} \left[ e \left( s_{\theta_*} \hat{r} + c_{\theta_*} \hat{\theta} \right) + \hat{\theta} \right] =$$

$$= \sqrt{\frac{\mu}{p}} \left[ e s_{\theta_*} \hat{r} + \left( 1 + e c_{\theta_*} \right) \hat{\theta} \right]$$

Hence, the radial and horizontal components are

$v_r = \sqrt{\frac{\mu}{p}} \, e s_{\theta_*}$ and $v_\theta = \sqrt{\frac{\mu}{p}} \left( 1 + e c_{\theta_*} \right)$

whereas the velocity magnitude is $v = \sqrt{v_r^2 + v_\theta^2} = \sqrt{\frac{\mu}{p}} \sqrt{1 + e^2 + 2 e c_{\theta_*}}$

(A) CIRCULAR ORBITS $(e=0)$

$$r = p = R \quad \text{and} \quad v = \sqrt{\frac{\mu}{p}} = \sqrt{\frac{\mu}{R}} = \text{constant}$$

where $R$ is the radius of the circular orbit

Moreover
$$\begin{cases} v_r = 0 & : \text{ in fact the radius does not change} \\ v_\theta = \sqrt{\frac{\mu}{R}} \end{cases}$$

(B) ELLIPTIC ORBITS $(0<e<1)$

$$r = \frac{p}{1+e\cos\theta_*} \qquad \begin{cases} r_A = \dfrac{p}{1-e} = a(1+e) & \text{apoapse radius} \\ r_P = \dfrac{p}{1+e} = a(1-e) & \text{periapse radius} \end{cases}$$

$$v = \sqrt{\frac{\mu}{p}}\sqrt{1+e^2+2e\cos\theta_*} \qquad \begin{cases} v_A = \sqrt{\dfrac{\mu}{p}}(1-e) = \sqrt{\dfrac{\mu}{a}}\sqrt{\dfrac{1-e}{1+e}} \\ v_P = \sqrt{\dfrac{\mu}{p}}(1+e) = \sqrt{\dfrac{\mu}{a}}\sqrt{\dfrac{1+e}{1-e}} \end{cases}$$

Of course $v_P > v_A$, which is consistent with the 2nd KEPLER'S LAW

Moreover:

At periapse
$$\begin{cases} v_r = 0 \\ v_\theta = \sqrt{\dfrac{\mu}{a}}\sqrt{\dfrac{1+e}{1-e}} \end{cases}$$

At apoapse
$$\begin{cases} v_r = 0 \\ v_\theta = \sqrt{\dfrac{\mu}{a}}\sqrt{\dfrac{1-e}{1+e}} \end{cases}$$

(B) PARABOLIC TRAJECTORIES $(e=1)$

$$r = \frac{p}{1+e\,c_{\theta_*}} = \frac{p}{1+c_{\theta_*}} \quad \text{and} \quad v = \sqrt{\frac{\mu}{p}}\sqrt{2+2c_{\theta_*}}$$

At periapse : $\quad r_p = \frac{p}{2} \quad$ and $\quad v_p = 2\sqrt{\frac{\mu}{p}}$

The $2^{nd}$ relation can also be rewritten as

$$v_p = \sqrt{\frac{2\mu}{r_p}} \quad \text{using} \quad p = 2r_p$$

Moreover $\quad r \rightarrow \infty \quad$ as $\quad \theta_* \longrightarrow \pm\pi$

If $\theta_* \rightarrow \pm\pi \Rightarrow v \rightarrow 0$

i.e. the asymptotic velocity along a parabola equals 0

Finally, it is straightforward to obtain the radial and horizontal components at periapse,

$$\text{At periapse} \quad \begin{cases} v_r = 0 \\ v_\theta = 2\sqrt{\frac{\mu}{p}} = \sqrt{\frac{2\mu}{r_p}} \end{cases}$$

(C) HYPERBOLIC TRAJECTORIES $(e>1)$

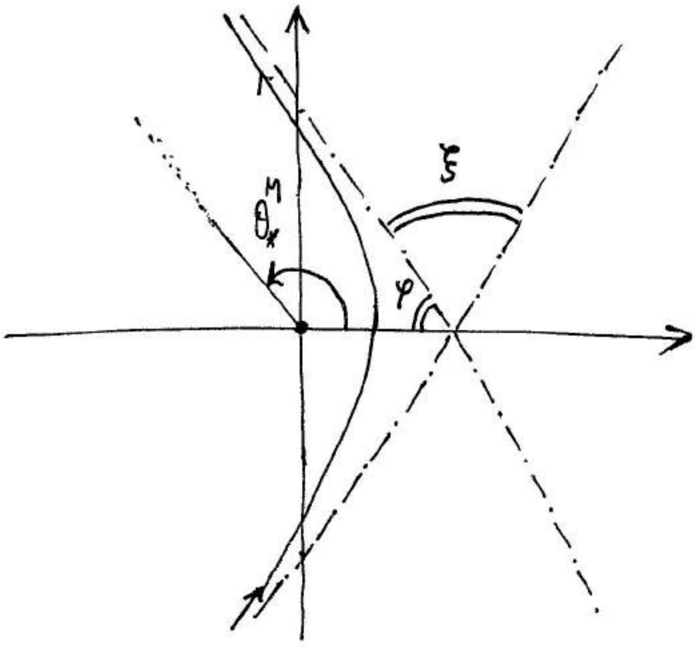
$$r = \frac{p}{1+e\,c_{\theta_*}} \quad \text{where} \quad e>1$$

This implies that the true anomaly is constrained to

$$-\text{acos}\left(-\frac{1}{e}\right) \leq \theta_* \leq \text{acos}\left(-\frac{1}{e}\right)$$

In fact, at infinite distance before approaching

$$\theta_* = -\operatorname{acos}\left(-\frac{1}{e}\right)$$




At infinite distance while going far away

$$\theta_* = \operatorname{acos}\left(-\frac{1}{e}\right)$$

In both cases, at infinite distance the velocity is aligned with the two asymptotes. In the figure $\theta_*^M$ is the limiting value for $\theta_*$, i.e. $\theta_*^M = \operatorname{acos}\left(-\frac{1}{e}\right)$

The angle $\varphi$ is given by $\qquad \varphi = \pi - \theta_*^M$

The deflection angle $\xi$ is $\qquad \xi = \pi - 2\varphi = 2\theta_*^M - \pi$

It is termed deflection angle because a spacecraft that travels far from the attracting body, then reaches the body and finally goes far away is "deflected", i.e. its velocity vector at infinite distance while leaving is rotated by the angle $\xi$ with respect to the direction of the velocity vector while approaching

At periapse

$$\begin{cases} r_p = \dfrac{p}{1+e} = a(1-e) \quad (\text{where } e>1 \text{ and } a<0) \\[2ex] v_p = \sqrt{\dfrac{\mu}{p}}\,(1+e) = \sqrt{\dfrac{\mu}{a}\dfrac{1+e}{1-e}} \\[2ex] v_r = 0 \quad \text{and} \quad v_\theta = v_p \end{cases}$$

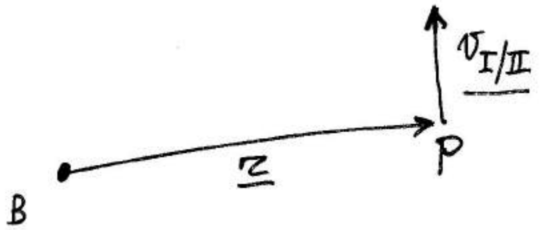
A infinite distance ( $r \to \infty$ and $\theta_* = \pm \operatorname{acos}\left(-\frac{1}{e}\right)$ )

$$\begin{cases} v_\infty = \sqrt{\dfrac{\mu}{p}}\sqrt{e^2-1} = \sqrt{-\dfrac{\mu}{a}} \\[2ex] v_\theta = 0 \\[1ex] v_r = v_\infty = \sqrt{-\dfrac{\mu}{a}} \end{cases}$$

Unlike parabolic paths, along the hyperbola the velocity does not vanish as $r \to \infty$

- Cosmic velocities

Given an attracting body B, at a generic point P (associated with $\underline{r}$) two cosmic velocities are defined:




(I) FIRST COSMIC VELOCITY $\quad v_I = \sqrt{\dfrac{\mu}{r}}$

If a horizontal velocity with this magnitude is provided at P, then the resulting motion occurs along a circular orbit of radius $r$

(II) SECOND COSMIC VELOCITY $\quad v_{II} = \sqrt{\dfrac{2\mu}{r}}$

If a horizontal velocity with this magnitude is provided at P, the the resulting motion occurs along a parabolic path with periapse radius equal to $r$

This velocity is also termed ESCAPE VELOCITY, as it allows escaping from the gravitational field of B.

## • ENERGY (PER MASS UNIT)

For a spacecraft the specific energy (i.e., energy per mass unit) is the sum of two contributions:

$$\mathcal{E} = \underbrace{-\frac{\mu}{r}}_{\text{Potential energy}} + \underbrace{\frac{v^2}{2}}_{\text{Kinetic energy}}$$

Using the two expressions for $r$ and $v$ along Keplerian paths,

$$r = \frac{p}{1+e c_{\theta_*}} \qquad \text{and} \qquad v = \sqrt{\frac{\mu}{p}}\sqrt{1+e^2+2e c_{\theta_*}}$$

one obtains

$$\mathcal{E} = \frac{\mu}{2p}\left[1+e^2+2e c_{\theta_*}\right] - \frac{\mu}{p}\left(1+e c_{\theta_*}\right) = -\frac{\mu}{2p}\left(1-e^2\right)$$

Hence:

(1) $0 \le e < 1$ (ellipse) $\quad \mathcal{E} < 0 \quad$ |potential energy| > Kinetic energy

(2) $e = 1$ (parabola) $\quad \mathcal{E} = 0 \quad$ |potential energy| = Kinetic energy

(3) $e > 1$ (hyperbola) $\quad \mathcal{E} > 0 \quad$ |potential energy| < Kinetic energy

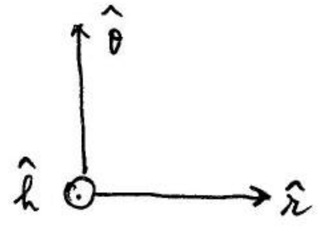
Because $\quad a = \frac{p}{1-e^2} \quad$ for hyperbola and ellipses,

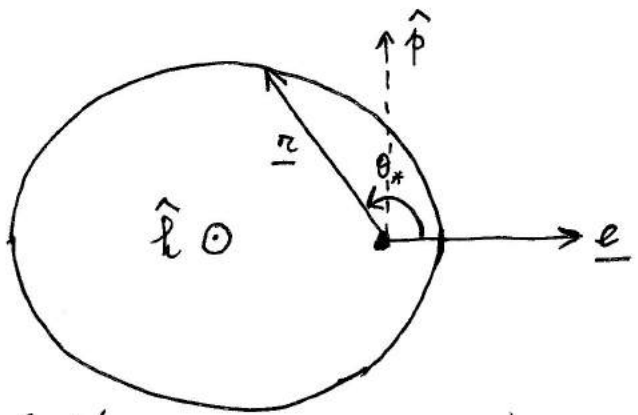
$$\mathcal{E} = -\frac{\mu}{2a}$$

This expression holds for parabolas, too, if, by convention $a \to \infty$ (parabolas)

- POSITION IN TIME

So far, no relation was found for position and time. However, the position is expressed as a function of $\theta_*$. Hence, it is desirable to relate $\theta_*$ with $t$.




The rotating frame has angular velocity $\underline{\omega}$, which can be written as $\underline{\omega} = \dot{\theta}_* \hat{h}$ (angular velocity w.r.t. inertial frame $(\hat{e}, \hat{p}, \hat{h})$, termed PERIFOCAL)

Moreover, from the definition

$$\underline{h} = \underline{r} \times \underline{v} = \underline{r} \times \left[ \dot{r}\hat{r} + \underline{\omega} \times \underline{r} \right] = \underline{\omega}\, r^2 - \underline{r} \underbrace{\left(\underline{\omega} \cdot \underline{r}\right)}_{=0 \text{ for Keplerian trajectories}} =$$

$$= \underline{\omega}\, r^2$$

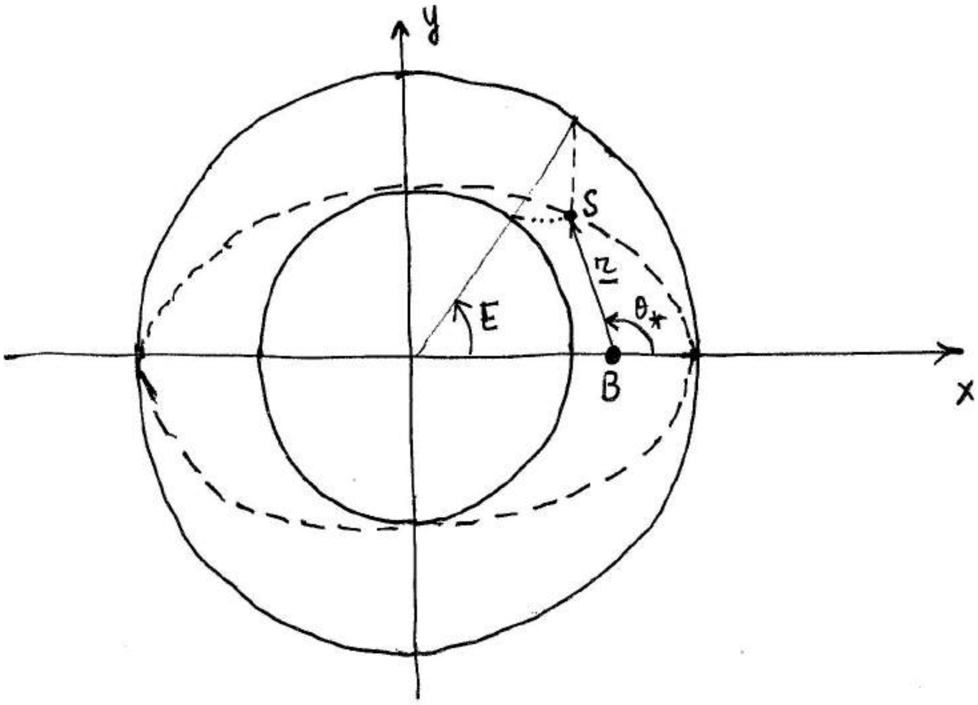
i.e.
$$h\hat{h} = \dot{\theta}_* r^2 \hat{h} \longrightarrow \dot{\theta}_* = \frac{h}{r^2} \Rightarrow$$

$$\Rightarrow \dot{\theta}_* = \frac{\sqrt{\mu p}}{p^2}\left(1 + e\, c_{\theta_*}\right)^2 = \sqrt{\frac{\mu}{p^3}}\left(1 + e\, c_{\theta_*}\right)^2$$

Numerical integration of this differential equation, yields $\theta_*(t)$, provided that $\theta_*$ is known at the initial time.

However, an alternative (less computationally expensive method) exists for elliptic orbits, as well as for parabolas and hyperbolas

- Eccentric anomaly (ellipses)




The spacecraft is located at $S$, at a given time

$$S\begin{cases} x = ea + \dfrac{a(1-e^2)c_{\theta_*}}{1+ec_{\theta_*}} \\ y = \dfrac{a(1-e^2)}{1+ec_{\theta_*}} s_{\theta_*} \end{cases}$$

in terms of TRUE ANOMALY

$$S\begin{cases} x = a c_E \\ y = b s_E = a\sqrt{1-e^2}\, s_E \end{cases}$$

in terms of ECCENTRIC ANOMALY

B = attracting body

In terms of true anomaly,

$x = ea + r c_{\theta_*}$ and $y = r s_{\theta_*}$, yielding

$$r^2 = (x-ea)^2 + y^2 \underset{x=ac_E,\ y=bs_E}{=} a^2(c_E - e)^2 + a^2(1-e^2)s_E^2 =$$

$$= a^2 + a^2e^2 - 2a^2 c_E e - a^2e^2s_E^2 = a^2\left[1+e^2-2ec_E - e^2(1-c_E^2)\right] =$$

$$= a^2\left[1 - ec_E\right]^2 \Rightarrow r = a(1-ec_E) \quad (1)$$

Moreover, the eccentric anomaly can be found as a function of $\theta_*$,

$$\tan\frac{E}{2} = \frac{s_E}{1+c_E} = \frac{\frac{r}{b}s_{\theta_*}}{1+e+\frac{(1-e^2)c_{\theta_*}}{1+ec_{\theta_*}}} = \frac{\frac{a\sqrt{1-e^2}}{a}\,\frac{s_{\theta_*}}{1+ec_{\theta_*}}}{\frac{1+ec_{\theta_*}+e+e^2c_{\theta_*}+c_{\theta_*}-e^2c_{\theta_*}}{1+ec_{\theta_*}}} =$$

$$= \frac{\sqrt{1-e^2}\; s_{\theta_*}}{1+e c_{\theta_*} + e + c_{\theta_*}} = \frac{\sqrt{1-e^2}\; s_{\theta_*}}{(1+e)(1+c_{\theta_*})} = \sqrt{\frac{1-e}{1+e}} \tan\frac{\theta_*}{2}$$

i.e. $$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}} \tan\frac{\theta_*}{2} \qquad (2)$$

(1) and (2) are two fundamental relations for $E$ and $\theta_*$

- Kepler's equation (ellipses)

Two expressions were found for $r$:

$$r = \frac{a(1-e^2)}{1+e c_{\theta_*}} \qquad \text{and} \qquad r = a(1-e c_E)$$

The derivatives are

$$\dot r = \frac{a(1-e^2)\,\dot\theta_*\, s_{\theta_*} e}{(1+e c_{\theta_*})^2} = \frac{ea(1-e^2) s_{\theta_*}}{\cancel{(1+e c_{\theta_*})^2}} \sqrt{\frac{\mu}{a^3(1-e^2)^3}}\,\cancel{(1+e c_{\theta_*})^2} =$$

$$= \sqrt{\frac{\mu}{a^3}}\; \frac{ae s_{\theta_*}}{\sqrt{1-e^2}}$$

$$\dot r = + a e \dot E s_E$$

Equating these two expressions and using $r s_{\theta_*} = b s_E$,

$$\cancel{ae}\,\dot E\, \cancel{s_E} = \sqrt{\frac{\mu}{a^3}}\; \frac{\cancel{ae}}{\cancel{\sqrt{1-e^2}}}\; \frac{\cancel{a}\cancel{\sqrt{1-e^2}}\,\cancel{s_E}}{\cancel{a}(1-e c_E)}$$

$$\dot E(1-e c_E) = \sqrt{\frac{\mu}{a^3}} \longrightarrow \Big[E - e s_E\Big]_{E_0}^{E} = \sqrt{\frac{\mu}{a^3}}\,(t-t_0)$$

i.e. $$(E - e s_E) - (E_0 - e s_{E_0}) = \sqrt{\frac{\mu}{a^3}}\,(t-t_0) \qquad \text{KEPLER'S EQUATION}$$

This equation is a trascendental equation in $E$.
If $t$ is given and $E_0$ is specified, then $E$ can be found by solving numerically this equation.

By definition, the mean anomaly $M$ is $M := E - e\,S_E$

Thus, the Kepler's equation may be rewritten as

$$M - M_0 = \sqrt{\frac{\mu}{a^3}}\,(t - t_0)$$

For a circular orbit $(e=0)$: $\theta_* \equiv E \equiv M$ and orbit motion is straightforward. In fact $\theta_* = \theta_{*0} + \sqrt{\frac{\mu}{a^3}}\,(t - t_0)$
(circular orbits only)

If $e \neq 0$, numerical solution is mandatory. The Newton method can be applied. Letting

$$f(E) = (E - e S_E) - (E_0 - e S_{E_0}) - \sqrt{\frac{\mu}{a^3}}\,(t - t_0)$$

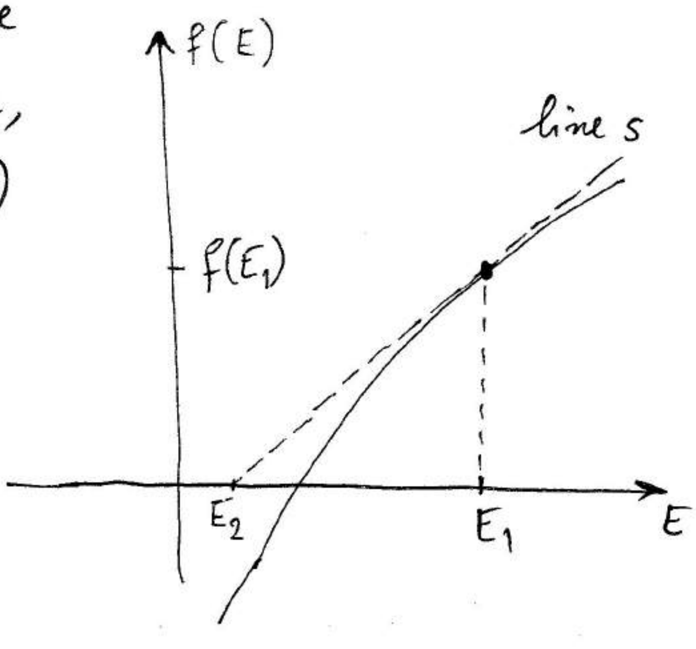
(with $E_0$, $t_0$, $t$ specified)

the zero of this function is to be sought. If $E_K$ is a guess solution, the straight line tangent to $f(E)$ has equation

$$\frac{f(E) - f(E_K)}{E - E_K} = f'(E_K) \quad \text{(line s)}$$

and the new (tentative) solution is

$$E_{K+1} = E_K - \frac{f(E_K)}{f'(E_K)}$$



This process is repeated iteratively, up to obtaining a refined solution. The initial guess $E_1$ is usually chosen: $E_1 \equiv M_1$ (i.e. $M_1 = \sqrt{\frac{\mu}{a^3}}\,(t-t_0)$)

– As a consequence of the Kepler's equation, after one orbit period,

$$\left(E - e\,s_E\right) - \left(E_0 - e\,s_{E_0}\right) = \sqrt{\frac{\mu}{a^3}}\,(t - t_0)$$

$$2\pi = \sqrt{\frac{\mu}{a^3}}\,T \rightarrow \quad T = 2\pi\sqrt{\frac{a^3}{\mu}}$$

This implies also $\frac{T^2}{a^3} = \frac{4\pi^2}{\mu}$ ($3^{rd}$ KEPLER'S LAW), i.e. for a given attracting body (associated with $\mu$) the ratio $\frac{T^2}{a^3}$ is constant

- <u>Barker's equation (parabolas)</u>

For parabolas, the following equation holds for $\theta_*$:

$$\dot{\theta}_* = \sqrt{\frac{\mu}{p^3}}\left(1 + c_{\theta_*}\right)^2$$

$$\rightarrow \quad \int_{\theta_{*0}}^{\theta_*} \frac{d\theta_*}{\left(1 + c_{\theta_*}\right)^2} = \sqrt{\frac{\mu}{p^3}}\,(t - t_0) \rightarrow \int_{\theta_{*0}}^{\theta_*} \frac{d\theta_*}{4\cos^4\frac{\theta_*}{2}} = \sqrt{\frac{\mu}{p^3}}\,(t - t_0)$$

$$\rightarrow \quad \int \frac{\sin^2\frac{\theta_*}{2} + \cos^2\frac{\theta_*}{2}}{2\cos^2\frac{\theta_*}{2}}\, d\left(\tan\frac{\theta_*}{2}\right) = \sqrt{\frac{\mu}{p^3}}\,(t - t_0)$$

$$\rightarrow \frac{1}{2}\int_{\theta_{*0}}^{\theta_*} \left[\tan^2\frac{\theta_*}{2} + 1\right] d\left(\tan\frac{\theta_*}{2}\right) = \sqrt{\frac{\mu}{p^3}}\,(t - t_0)$$

$$\rightarrow \frac{1}{2}\left[\frac{1}{3}\tan^3\frac{\theta_*}{2} + \tan\frac{\theta_*}{2}\right]_{\theta_{*0}}^{\theta_*} = \sqrt{\frac{\mu}{p^3}}\,(t - t_0)$$

and finally, the following trascendental equation holds:

$$\left(\frac{1}{6}\tan^3\frac{\theta_*}{2} + \frac{1}{2}\tan\frac{\theta_*}{2}\right) - \left(\frac{1}{6}\tan^3\frac{\theta_{*0}}{2} + \frac{1}{2}\tan\frac{\theta_{*0}}{2}\right) = \sqrt{\frac{\mu}{p^3}}\,(t - t_0)$$

- <u>Solution for hyperbolas</u>

Without providing proofs, the following relations hold:

$$r = a\,(1 - e\cosh H)$$

$$\tanh\frac{H}{2} = \sqrt{\frac{e-1}{e+1}}\,\tan\frac{\theta_*}{2}$$

$$(e\sinh H - H) - (e\sinh H_0 - H_0) = \sqrt{\frac{\mu}{-a^3}}\,(t - t_0)$$

The last one is again a trascendental equation in $H$, once $H_0$ is specified and $t$ is given.

## SPECIAL KEPLERIAN TRAJECTORIES

There exist some special Keplerian trajectories such as

(a) CIRCULAR ORBITS

(b) GEOSYNCHRONOUS ORBITS (elliptic-type orbits)

(c) BALLISTIC TRAJECTORIES (elliptic-type orbits)

(d) RECTILINEAR TRAJECTORIES

### Circular orbits

These orbits have $r = p = R$ and $v = \sqrt{\frac{\mu}{R}}$, i.e. both the radius and the orbital velocity are constant.

In general the (inertial) acceleration $\frac{d^2\underline{r}}{dt^2}$ is given by

$$\frac{d^2\underline{r}}{dt^2} = \frac{d}{dt}\frac{d\underline{r}}{dt} = \frac{d}{dt}\left[\dot{r}\hat{r} + \underline{\omega}\times\underline{r}\right] = \ddot{r}\hat{r} + 2\underline{\omega}\times\dot{\underline{r}} + \dot{\underline{\omega}}\times\underline{r} +$$

$$+\underline{\omega}\times(\underline{\omega}\times\underline{r})$$

where the $2^{nd}$ term is the CORIOLIS ACCELERATION
$3^{rd}$ term is the EULER ACCELERATION
$4^{th}$ term is the CENTRIPETAL ACCELERATION

In the rotating frame $(\hat{r}, \hat{\theta}, \hat{h})$ the radial acceleration felt by the space vehicle is $\ddot{r}\hat{r}$ and equals 0 along a circular orbit, because $r = \text{const}$. Moreover, also $\underline{\omega} = \text{const} \Rightarrow \dot{\underline{\omega}} = 0$.

This can be proven easily. In fact

$$\underline{h} = \underline{r}\times\underline{v} = \underline{r}\times\left[\dot{r}\hat{r} + \underline{\omega}\times\underline{r}\right] = \underline{\omega}r^2 \text{ and } \underline{\omega} = \text{const because}$$

$r = \text{const.}$ and $\underline{h} = \text{const.}$

Therefore, the radial acceleration $\ddot{r}\hat{r}$ is

$$\ddot{r}\hat{r} = 0 = \frac{d^2 \underline{r}}{dt^2} - \underline{\omega} \times (\underline{\omega} \times \underline{r})$$

i.e.

$$\underbrace{-\frac{\mu}{r^2}\hat{r}}_{\substack{\text{gravitational} \\ \text{acceleration} \\ \text{(real force)}}} - \underbrace{\underline{\omega} \times (\underline{\omega} \times \underline{r})}_{\substack{\text{centrifugal} \\ \text{acceleration} \\ \text{(fictious force)}}} = 0$$

These two accelerations balance at all times along a circular orbit, in the rotating frame attached to the vehicle

- Geosynchronous orbits

An elliptic orbit is termed geosynchronous if the orbit period is the same as the sidereal day.

A sidereal day is the time interval needed for the Earth to complete a rotation with respect to an inertial frame

$$T_{sid} = \frac{2\pi}{\omega_E} = 86164 \text{ sec}$$

Hence, a geosynchronous orbit has semimajor axis such that

$$2\pi\sqrt{\frac{a^3}{\mu_E}} \equiv T_{sid} \rightarrow \frac{a^3}{\mu_E} = \left(\frac{T_{sid}}{2\pi}\right)^2 \rightarrow a_{gs} = \left[\mu_E \left(\frac{T_{sid}}{2\pi}\right)^2\right]^{1/3}$$

The value of $a_{gs}$ is 42164 sec.

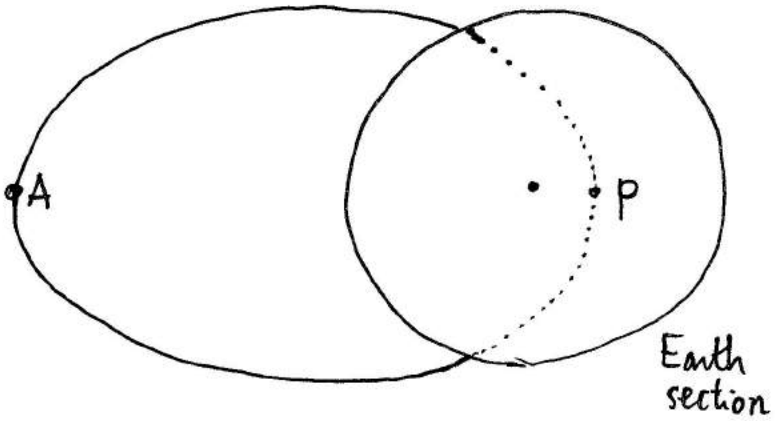
No assumption was done on $i$ and $e$ (apart the fact that the orbit must have periapse of sufficient altitude).

- Ballistic trajectories

These paths are arcs of elliptic trajectories. These have periapse radius less than the Earth radius, therefore the ballistic trajectory intersects the Earth surface twice

Periapse is inside the Earth

Apoapse is outside the Earth




These trajectories are used by ballistic missiles (ICBM)

- Rectilinear trajectories

These paths are such that $\underline{v} \parallel \underline{r} \Rightarrow \underline{h} = 0$

Moreover, from its definition $e = |\underline{e}| = |-\hat{r}| = 1$

This implies that $p = \frac{h^2}{\mu} = 0$ and $r_p = \frac{p}{1+e} = 0$

Three types of rectilinear paths exist:

(1) $a > 0$ and $\mathcal{E} < 0$ : rectilinear path of elliptic type (concretely, a segment)

(2) $a \to \infty$ and $\mathcal{E} = 0$: rectilinear path of parabolic type (line departing from attracting body, velocity vanishes as $r \to \infty$)

(3) $a < 0$ and $\mathcal{E} > 0$: rectilinear path of hyperbolic type (line from attracting body; $v$ does not vanish as $r \to \infty$)

## GENERAL CLASSIFICATION OF KEPLERIAN ORBITS

Keplerian trajectories
- conic sections (ellipse, parabola, hyperbola) $h \neq 0$, $e \geq 0$
- rectilinear trajectories (degenerate paths) $h = 0$, $e = 1$

Along rectilinear trajectories the motion is non-uniform and

$$r(1 + e\cos\theta_*) = p = \frac{h^2}{\mu} = 0 \quad \Rightarrow \quad 1 + \cos\theta_* = 0 \Rightarrow \theta_* = \pm\pi$$

because $\quad e = |\underline{e}| = |-\hat{r}| = 1$

In the end, for identifying a Keplerian path, one needs

(1) SEMIMAJOR AXIS $a$ or ENERGY $\mathcal{E}$

(2) ECCENTRICITY $e$

| $\mathcal{E}$ | $0 \le e < 1$ | $e = 1$ | $e > 1$ |
|---|---|---|---|
| $>0$ | | rectilinear hyperbola | hyperbola |
| $0$ | | rectilinear parabola<br>parabola | |
| $<0$ | ellipse | rectilinear ellipse | |

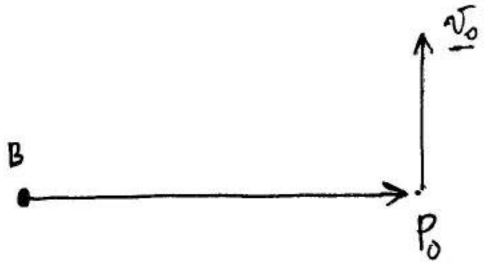
- Different Keplerian trajectories depending on $\underline{r}$ and $\underline{v}$

If $\underline{r}$ and $\underline{v}$ are given, then $\mathcal{E}$ and $e$ can be found

(a) $\mathcal{E} = \dfrac{v^2}{2} - \dfrac{\mu}{r} = -\dfrac{\mu}{2a} \longrightarrow a = \dfrac{\mu}{\frac{2\mu}{r} - v^2}$ (VIS VIVA equation)

(b) $h = |\underline{h}| = |\underline{r} \times \underline{v}| = \sqrt{\mu p} = \sqrt{\mu a(1-e^2)} \rightarrow$

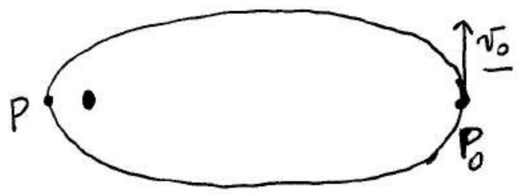
$$\rightarrow e = \sqrt{1 - \frac{h^2}{\mu a}}$$



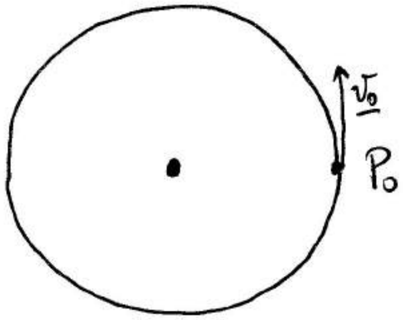

At point $P_0$ a velocity orthogonal to $\underline{r}$ is given. The resulting path depends on the magnitude of $\underline{v_0}$

(a)

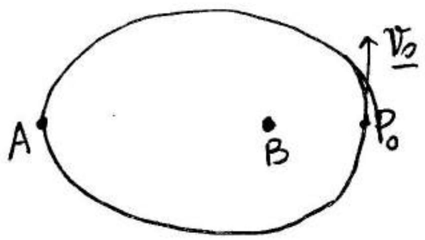

(b)



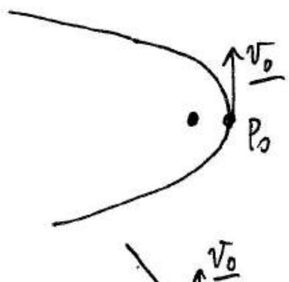

(c)




(d)




(e)




(f)


(a) $v_0 = 0 \rightarrow$ RECTILINEAR ELLIPSE (straight line converging to B)

(b) $0 < v_0 < v_I \rightarrow$ ELLIPSE with point $P_0$ as apoapse. Periapse radius increases as $v_0$ increases

(c) $v_0 = v_I \rightarrow$ CIRCULAR orbit with radius $r$ ; $v_I$ is the first cosmic velocity

(d) $v_I < v_0 < v_{II} \rightarrow$ ELLIPSE with point $P_0$ as periapse. Apoapse radius increases as $v_0$ increases

(e) $v_0 = v_{II} \rightarrow$ PARABOLA

$v_{II}$ is the second cosmic velocity

(f) $v_0 > v_{II} \rightarrow$ HYPERBOLA

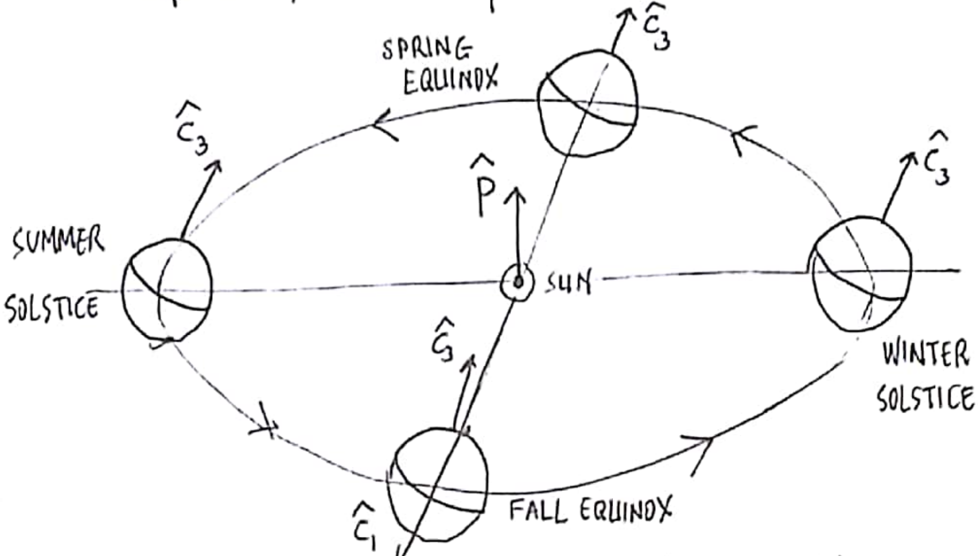
eccentricity $e$ increases as $v_0$ increases.

(g) $v_0 \rightarrow \infty \rightarrow$ straight line orthogonal to $\underline{r}$

## REPRESENTATION IN THREE DIMENSIONS

Usually, Earth orbits are represented in a suitable Earth Inertial Frame (ECI), centered in the center of the Earth.

As a preliminary step, the inertial direction $\hat{c}_1$ is to be defined, with reference to the Earth motion around the Sun




$\hat{c}_3$ is the Earth rotation axis, directed toward the North pole of the Earth

$\hat{P}$ is the ecliptic pole, orthogonal to the ecliptic plane

ECLIPTIC PLANE = plane of Earth orbit around the Sun

$\hat{c}_1$ = vernal axis, is the intersection of the ecliptic plane and the Earth equatorial plane

Both $\hat{c}_1$ and $\hat{c}_3$ are inertial axes, although, strictly speaking they are subject to the precession of equinoxes

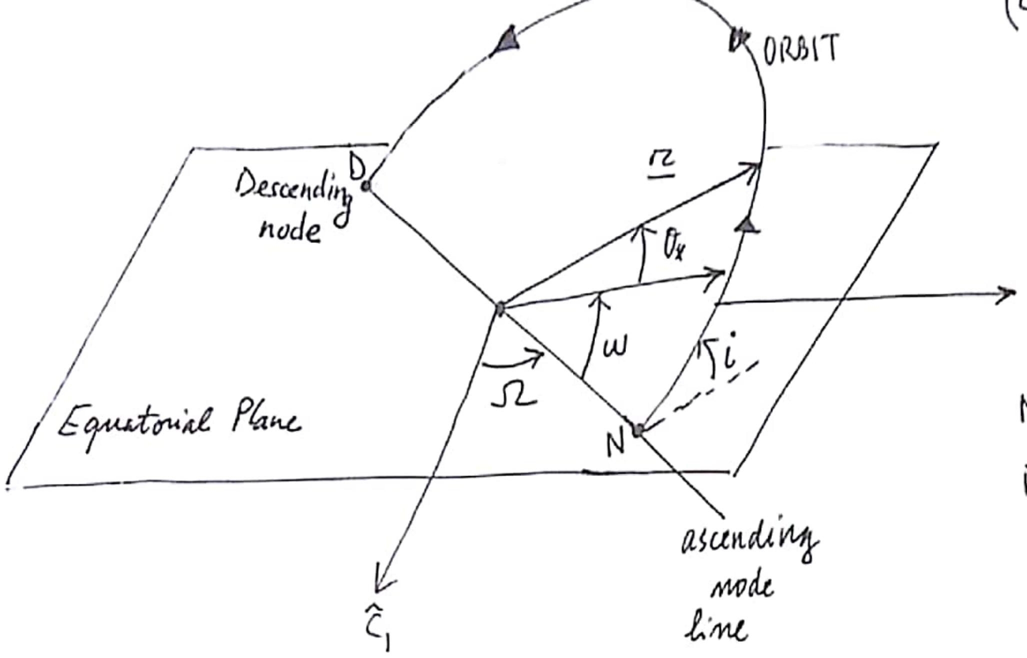

$\hat{c}_3$ describes a cone with axis $\hat{P}$ (ecliptic pole), in clockwise sense with a period of about 25700 yrs

However, for satellite motion, both $\hat{c}_1$ and $\hat{c}_3$ are assumed inertial

Remark. The seasons indicated in the previous figure are for the North emisphere

- Earth · Centered · Inertial frame (ECI)

This frame is associated with $(\hat{c}_1, \hat{c}_2, \hat{c}_3)$

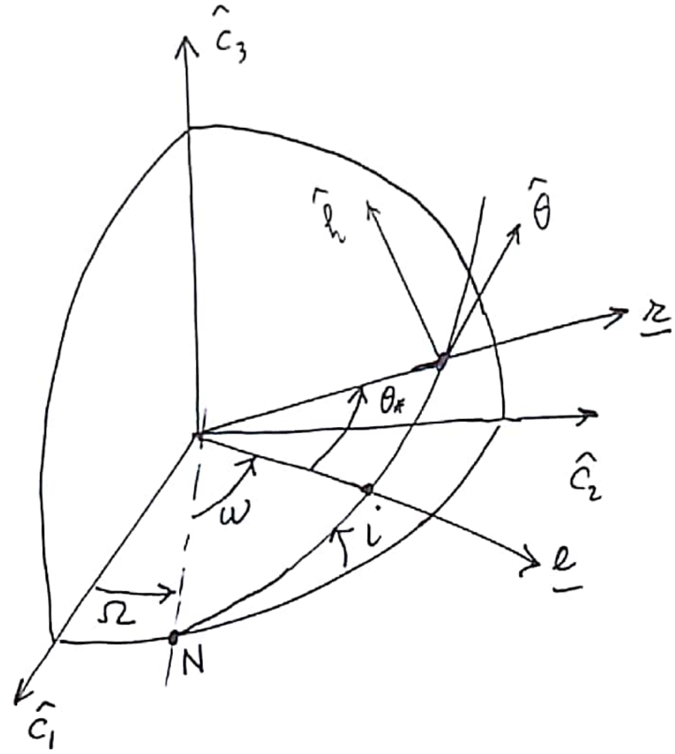



$(\hat{c}_1, \hat{c}_2) \leftrightarrow$ Earth equatorial plane

$\hat{c}_3 \parallel$ Earth rotation axis

N = ascending node

D = descending node




Orbit plane is identified by means of two angles:

(1) $\Omega$ (RAAN) $[-\pi, \pi[$
right ascension of the ascending node

(2) $i$ inclination $[0, \pi]$

At N (ascending node) the spacecraft crosses the equatorial plane from South to North

$w$ is the argument of perigee, and is taken counterclockwise from the nodal line. It identifies the position of perigee in the orbit plane. $w$ varies in $[-\pi, \pi[$

Because $\underline{h}$, $\underline{e}$ are first integrals, the three angles $(\Omega, i, \omega)$ do not vary for Keplerian orbits

Moreover, $\theta_t = \omega + \theta_*$ is termed argument of latitude

For circular orbits, no perigee is defined, therefore only $\theta_t$ is meaningful for the purpose of identifying $\underline{r}$

- Rotating frame $(\hat{r}, \hat{\theta}, \hat{h})$

This frame is portrayed in the previous figure and rotates together with the spacecraft. Thus, it is non-inertial

This rotating frame can be obtained from $(\hat{c}_1, \hat{c}_2, \hat{c}_3)$ through a sequence of 3 elementary rotations, i.e.

(i) rotation about axis 3 by angle $\Omega$ (counterclockwise)

(ii) rotation about axis 1 by angle $i$ (counterclockwise)

(iii) rotation about axis 3 by angle $\theta_t$ (counterclockwise)

Thus
$$\begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{h} \end{bmatrix} = \underbrace{R_3(\theta_t) R_1(i) R_3(\Omega)}_{R_A} \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{bmatrix}$$

and, after some steps,

$$R_A = \begin{bmatrix} c_{\theta_t} c_\Omega - s_{\theta_t} c_i s_\Omega & c_{\theta_t} s_\Omega + s_{\theta_t} c_i c_\Omega & s_{\theta_t} s_i \\ -s_{\theta_t} c_\Omega - c_{\theta_t} c_i s_\Omega & -s_{\theta_t} s_\Omega + c_{\theta_t} c_i c_\Omega & c_{\theta_t} s_i \\ s_i s_\Omega & -s_i c_\Omega & c_i \end{bmatrix}$$

- Orbit elements

For a Keplerian orbit, the following set of quantities

$(a, e, i, \Omega, \omega)$ is referred to as ORBIT ELEMENTS

they are 5 constant quantities (the number corresponds to 5 independent components of $\underline{h}$ and $\underline{e}$)

The sixth element is the instantaneous

true anomaly $\theta_*$ OR
eccentric anomaly $E$ OR
mean anomaly $M$

Specifying any of the latter three angles is equivalent to specifying the time of periapse passage (which corresponds to $\theta_* = E = M = 0$).

Definitely, a Keplerian orbit can be described through 6 quantities, and 5 out of 6 are constant in time.

$$\begin{cases} a = \text{semimajor axis, indicates the orbit size} \\ e = \text{eccentricity, indicates the orbit shape} \\ (\Omega, i) \text{ define the orbit plane orientation in space} \\ \omega \text{ identifies the periapse direction (with respect to the ascending node) in the orbit plane} \end{cases}$$

In a few cases, some of these orbit elements can be not defined

(a) CIRCULAR ORBITS: no perigee is defined, therefore only $\theta_t = \omega + \theta_*$ is defined ($\omega$ and $\theta_*$ not defined separately)

(b) EQUATORIAL ORBITS ($i = 0$ or $i = \pi$): no ascending node is defined, therefore only

$$\tilde{\omega} := \Omega + \omega \quad (\text{if } i = 0)$$

are defined

or $$\tilde{\omega} := \Omega - \omega \quad (\text{if } i = \pi)$$

($\omega$ and $\Omega$ not defined separately)

(c) CIRCULAR, EQUATORIAL ORBITS ($e = 0$ and $i = 0, \pi$): no perigee nor ascending node are defined, therefore only

$$\tilde{\lambda} = \theta_t + \Omega \quad (\text{if } i = 0)$$

are defined

or $$\tilde{\lambda} = \Omega - \theta_t \quad (\text{if } i = \pi)$$

($\Omega, \omega, \theta_*$ not defined separately)

$\tilde{\lambda}$ is termed true longitude

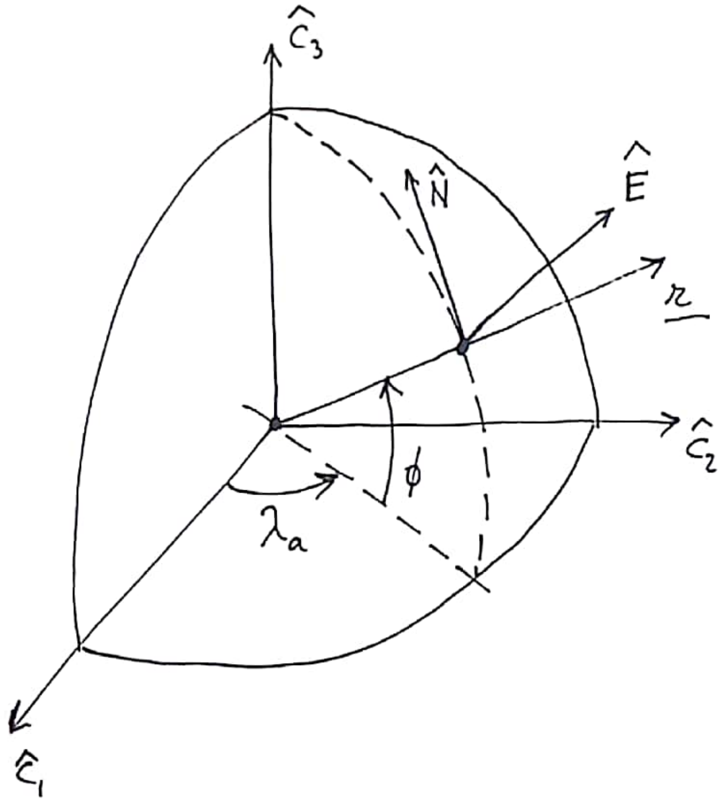
- **Cartesian coordinates**

In the ECI-frame, the position and velocity Cartesian coordinates are the components of $\underline{r}$ and $\underline{v}$ along $(\hat{c}_1, \hat{c}_2, \hat{c}_3)$, i.e.

$$\underline{r} = \begin{bmatrix} X & Y & Z \end{bmatrix} \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{bmatrix} \quad \text{and} \quad \underline{v} = \begin{bmatrix} V_x & V_y & V_z \end{bmatrix} \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{bmatrix}$$

Their derivation from the orbit elements is being addressed in the next subsections.

- **Spherical coordinates**




The position vector of a space vehicle can be identified through the following spherical coordinates:

$r$ = radius $(= |\underline{r}|)$

$\lambda_a$ = absolute longitude

$\phi$ = latitude

In order to have a unique representation for any $\underline{r}$ the following intervals are defined for $\lambda_a$ and $\phi$:

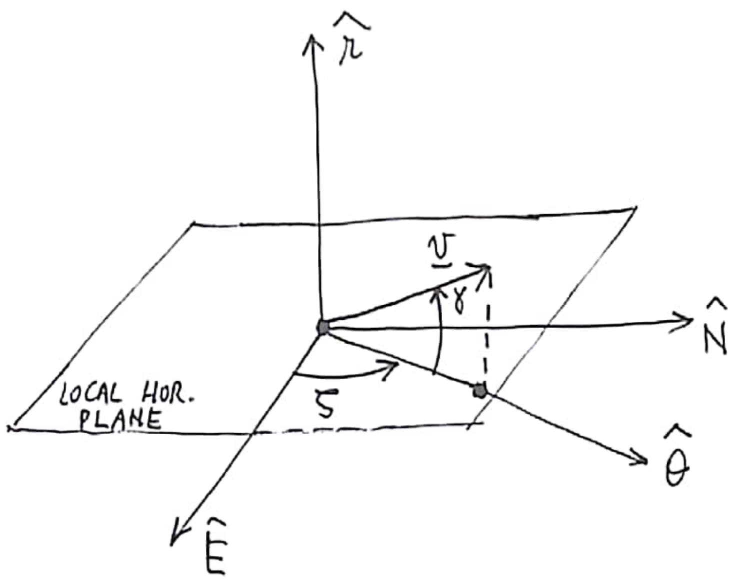
$$-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2} \quad \text{and} \quad -\pi \leq \lambda_a < \pi$$

In the previous figure, two unit vectors are portrayed

$\hat{E}$ = local East direction

$\hat{N}$ = local North direction

Thus, $(\hat{E}, \hat{N})$ represents the local horizontal plane (orthogonal to $\underline{r}$)




In the local horizontal plane, $\hat{\theta}$ is identified by projecting the instantaneous velocity onto the $(\hat{E}, \hat{N})$-plane

Two angles define the direction of $\underline{v}$ :

(i) $\gamma$ = flight path angle $\left(-\frac{\pi}{2} \le \gamma \le \frac{\pi}{2}\right)$

$\gamma > 0$ : flight in the upward direction
$\gamma < 0$ : flight in the downward direction
$\gamma = 0$ : flight in the local horizontal plane

(ii) $\zeta$ = heading angle $(-\pi \le \zeta < \pi)$
indicates the direction of flight with respect to $\hat{E}$

The intervals where $\gamma$ and $\zeta$ are defined are such that any velocity direction corresponds to a unique pair of values of $(\zeta, \gamma)$; only exception is when $\gamma = \pm\frac{\pi}{2}$ : in this case the value of $\zeta$ is not relevant.

By inspection of the previous two figures, one can obtain the sequence of elementary rotations that allow attaining $(\hat{r}, \hat{\theta}, \hat{h})$ (LVLH-frame) from $(\hat{c}_1, \hat{c}_2, \hat{c}_3)$ (ECI-frame)

This sequence is here written in terms of $(\lambda_a, \phi, \xi)$:

(i) rotation about axis 3 by angle $\lambda_a$ (counterclockwise)
(ii) rotation about axis 2 by angle $\phi$ (clockwise)
(iii) rotation about axis 1 by angle $\xi$ (counterclockwise)

Thus
$$\begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{h} \end{bmatrix} = \underbrace{R_1(\xi) R_2(-\phi) R_3(\lambda_a)}_{R_B} \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{bmatrix}$$

and, after some steps,

$$R_B = \begin{bmatrix} c_\phi c_{\lambda_a} & c_\phi s_{\lambda_a} & s_\phi \\ -s_\xi s_\phi c_{\lambda_a} - c_\xi s_{\lambda_a} & -s_\xi s_\phi s_{\lambda_a} + c_\xi c_{\lambda_a} & s_\xi c_\phi \\ -c_\xi s_\phi c_{\lambda_a} + s_\xi s_{\lambda_a} & -c_\xi s_\phi s_{\lambda_a} - s_\xi c_{\lambda_a} & c_\xi c_\phi \end{bmatrix}$$

This matrix must coincide with that obtained previously (named $R_A$), and this circumstance is being profitably used in the derivations that follow

- Given $(a, e, i, \Omega, \omega, \theta_*)$ find $(X, Y, Z, V_x, V_y, V_z)$

As a first step, the following relations were found

$$\underset{\sim}{r} = r\hat{r} = \frac{a(1-e^2)}{1+e\,c_{\theta_*}}\,\hat{r} \qquad (a)$$

$$\underset{\sim}{v} = v_r\hat{r} + v_\theta\hat{\theta} \quad \text{where} \quad \begin{cases} v_r = \sqrt{\dfrac{\mu}{a(1-e^2)}}\; e\, s_{\theta_*} & (b) \\ v_\theta = \sqrt{\dfrac{\mu}{a(1-e^2)}}\,\left(1+e\,c_{\theta_*}\right) & (c) \end{cases}$$

However, $(\hat{r}, \hat{\theta})$ can be expressed as functions of $(\hat{c}_1, \hat{c}_2, \hat{c}_3)$, by inspecting the first two rows of matrix $R_A$.
After some algebra, one obtains

$$\begin{cases} X = r\left[c_{\theta_t} c_\Omega - c_i\, s_{\theta_t} s_\Omega\right] & ✓ \\ Y = r\left[c_{\theta_t} s_\Omega + c_i\, c_{\theta_t} c_\Omega\right] & ✓ \\ Z = r\, s_i\, s_{\theta_t} & ✓ \end{cases} \qquad r = \frac{a(1-e^2)}{1+e\,c_{\theta_*}}$$

$$\begin{cases} V_x = \sqrt{\dfrac{\mu}{a(1-e^2)}}\left[-c_\Omega\left(s_{\theta_t} + e\, s_\omega\right) - s_\Omega\, c_i\left(c_{\theta_t} + e\, c_\omega\right)\right] & ✓ \\ V_y = \sqrt{\dfrac{\mu}{a(1-e^2)}}\left[c_\Omega\, c_i\left(c_{\theta_t} + e\, c_\omega\right) - s_\Omega\left(s_{\theta_t} + e\, s_\omega\right)\right] & ✓ \\ V_z = \sqrt{\dfrac{\mu}{a(1-e^2)}}\; s_i\left[c_{\theta_t} + e\, c_\omega\right] & ✓ \end{cases}$$

In the expressions for $(V_x, V_y, V_z)$ the terms that involved $\theta_*$ have been replaced with $(\theta_t - \omega)$, where $\omega$ is the argument of periapse, in order to obtain simpler expressions (those previously shown for $V_x, V_y, V_z$).

• Given $(a, e, i, \Omega, \omega, \theta_*)$, find $(r, \lambda_a, \phi, \gamma, v, \xi)$

As a first step, $r$ and $v$ can be found

$$r = \frac{a(1-e^2)}{1+e\,c_{\theta_*}} \quad ✓ \qquad v = \sqrt{\frac{\mu}{a(1-e^2)}}\sqrt{e^2+1+2e\,c_{\theta_*}} \quad ✓$$

Then, $\gamma$ is found using the radial component of $\underline{v}$,

$$v_r = v\,s_\gamma = \sqrt{\frac{\mu}{a(1-e^2)}}\,e\,s_{\theta_*}$$

$$\rightarrow \quad \gamma = \operatorname{asin}\frac{e\,s_{\theta_*}}{\sqrt{e^2+1+2e\,c_{\theta_*}}} \quad ✓$$

The remaining three angles $(\lambda_a, \phi, \xi)$ are found by identifying matrices $R_A$ and $R_B$, using their elements $(1,1)$, $(1,2)$, $(1,3)$, $(2,3)$, $(2,4)$, after calculating $\theta_t = \theta_* + \omega$

(a) Element $(1,3)$: $\quad s_\phi = s_{\theta_t} s_i \rightarrow \phi = \operatorname{asin}\left[s_{\theta_t} s_i\right]$ ✓

(b) Elements $(2,3)$ and $(3,3)$:

$$\left.\begin{aligned} s_\xi c_\phi &= c_{\theta_t} s_i \rightarrow s_\xi = \frac{c_{\theta_t} s_i}{c_\phi} \\ c_\xi c_\phi &= c_i \rightarrow c_\xi = \frac{c_i}{c_\phi} \end{aligned}\right\} \rightarrow \xi = 2\operatorname{atan}\frac{s_\xi}{1+c_\xi} \quad ✓$$

(c) Elements $(1,1)$ and $(1,2)$

$$\left.\begin{aligned} c_\phi c_{\lambda_a} &= c_{\theta_t} c_\Omega - s_{\theta_t} c_i s_\Omega \rightarrow c_{\lambda_a} = \frac{c_{\theta_t} c_\Omega - s_{\theta_t} c_i s_\Omega}{c_\phi} \\ c_\phi s_{\lambda_a} &= c_{\theta_t} s_\Omega + s_{\theta_t} c_i c_\Omega \rightarrow s_{\lambda_a} = \frac{c_{\theta_t} s_\Omega + s_{\theta_t} c_i c_\Omega}{c_\phi} \end{aligned}\right\} \rightarrow \lambda_a = 2\operatorname{atan}\frac{s_{\lambda_a}}{1+c_{\lambda_a}} \quad ✓$$

• Given $(X, Y, Z, V_x, V_y, V_z)$, find $(a, e, i, \Omega, \omega, \theta_*)$

As a first step, one can calculate

$$r = \sqrt{X^2+Y^2+Z^2} \qquad \text{and} \qquad v = \sqrt{V_x^2+V_y^2+V_z^2}$$

Through the vis viva equation that holds for energy,

$$\mathcal{E} = -\frac{\mu}{r} + \frac{v^2}{2} = -\frac{\mu}{2a} \rightarrow a = \frac{\mu}{\frac{2\mu}{r} - v^2}$$ ✓

Moreover, the radial component of $\underline{v}$ can be found,

$$v_r = \frac{\underline{v} \cdot \underline{r}}{r} = \frac{XV_x + YV_y + ZV_z}{\sqrt{X^2+Y^2+Z^2}} = \sqrt{\frac{\mu}{a(1-e^2)}}\, e S_{\theta_*}$$

as well as $\underline{h} = \underline{r} \times \underline{v} =$

$$= \begin{vmatrix} \hat{c}_1 & \hat{c}_2 & \hat{c}_3 \\ X & Y & Z \\ V_x & V_y & V_z \end{vmatrix} = \begin{bmatrix} YV_z - ZV_y \\ ZV_x - XV_z \\ XV_y - YV_x \end{bmatrix}^T \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{bmatrix}$$

whose magnitude $h$ is also given by

$$h = \sqrt{\mu a (1-e^2)} \rightarrow e = \sqrt{1 - \frac{h^2}{\mu a}}$$ ✓

Now, because $r$ and $v_r$ are known (in terms of $X, Y, Z, V_x, V_y, V_z$) one can find $\theta_*$

$$r = \frac{p}{1+eC_{\theta_*}} \longrightarrow C_{\theta_*} = \frac{1}{e}\left[\frac{p}{r} - 1\right]$$

$$v_r = \sqrt{\frac{\mu}{p}}\, e S_{\theta_*} \longrightarrow S_{\theta_*} = \frac{v_r}{e}\sqrt{\frac{p}{\mu}}$$

$$\rightarrow \theta_* = 2\,\text{atan}\frac{S_{\theta_*}}{1+C_{\theta_*}}$$ ✓

$p = a(1-e^2)$ is the semilatus rectum

Because $\underline{h}$ is known, one can find $\hat{h}$

$$\hat{h} = \frac{\underline{h}}{h} \quad , \quad \text{then also} \quad \hat{\theta} = \hat{h} \times \hat{r}$$

Both unit vectors $(\hat{\theta}, \hat{h})$, as well as $\hat{r}$ are known, i.e. their 9 components along $(\hat{c}_1, \hat{c}_2, \hat{c}_3)$ have been calculated through the previous steps. These 9 components correspond to the 9 elements of matrix $R_A$

$$\begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{h} \end{bmatrix} = \begin{bmatrix} r_1 & r_2 & r_3 \\ \theta_1 & \theta_2 & \theta_3 \\ h_1 & h_2 & h_3 \end{bmatrix} \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{bmatrix}$$

$r_k$ = component of $\hat{r}$ along $\hat{c}_k$

$\theta_k$ = component of $\hat{\theta}$ along $\hat{c}_k$

$h_k$ = component of $\hat{h}$ along $\hat{c}_k$

By inspecting the analytical expression of matrix $R_A$ :

(a) Element $(3,3)$: $\quad c_i = h_3 \longrightarrow i = \operatorname{acos} h_3 \quad$ ✓

(b) Elements $(1,3)$ and $(2,3)$:

$$\left.\begin{array}{l} s_{\theta_t} s_i = r_3 \longrightarrow s_{\theta_t} = \dfrac{r_3}{s_i} \\ s_i c_{\theta_t} = \theta_3 \longrightarrow c_{\theta_t} = \dfrac{\theta_3}{s_i} \end{array}\right\} \longrightarrow \theta_t = 2 \operatorname{atan} \frac{s_{\theta_t}}{1 + c_{\theta_t}} \quad ✓$$

(c) Elements $(3,1)$ and $(3,2)$

$$\left.\begin{array}{l} s_i s_\Omega = h_1 \longrightarrow s_\Omega = \dfrac{h_1}{s_i} \\ s_i c_\Omega = -h_2 \longrightarrow c_\Omega = \dfrac{-h_2}{s_i} \end{array}\right\} \longrightarrow \Omega = 2 \operatorname{atan} \frac{s_\Omega}{1 + c_\Omega} \quad ✓$$

Finally, since $\theta_t$ and $\theta_*$ are known, one obtains

$$\omega = \theta_t - \theta_*$$

- Given $(X, Y, Z, V_x, V_y, V_z)$, find $(r, \lambda_a, \phi, \gamma, v, \zeta)$

As a first step, $v$ and $r$ are found,

$$r = \sqrt{X^2 + Y^2 + Z^2} \quad \checkmark \quad \text{and} \quad v = \sqrt{V_x^2 + V_y^2 + V_z^2} \quad \checkmark$$

Moreover, the radial component of $\underline{v}$ is

$$v_r = \frac{\underline{v} \cdot \underline{r}}{r} = \frac{X V_x + Y V_y + Z V_z}{\sqrt{X^2 + Y^2 + Z^2}} = v S_\gamma$$

$$\rightarrow \quad \gamma = \operatorname{asin} \frac{v_r}{v} \quad \checkmark$$

The (specific) angular momentum $\underline{h}$ is given by

$$\underline{h} = \underline{r} \times \underline{v} = \begin{vmatrix} \hat{c}_1 & \hat{c}_2 & \hat{c}_3 \\ X & Y & Z \\ V_x & V_y & V_z \end{vmatrix} = \begin{bmatrix} Y V_z - Z V_y \\ Z V_x - X V_z \\ X V_y - Y V_x \end{bmatrix}^T \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{bmatrix}$$

Moreover, once $\hat{h} = \frac{\underline{h}}{h}$ is found,

$$\hat{\theta} = \hat{h} \times \hat{r}$$

As a result, the components of $(\hat{r}, \hat{\theta}, \hat{h})$ along $(\hat{c}_1, \hat{c}_2, \hat{c}_3)$ are known,

$$\begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{h} \end{bmatrix} = \begin{bmatrix} r_1 & r_2 & r_3 \\ \theta_1 & \theta_2 & \theta_3 \\ h_1 & h_2 & h_3 \end{bmatrix} \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{bmatrix}$$

By inspecting the analytical expression of $R_B$ :

(a) Element (1,3): $s_\phi = r_3 \longrightarrow \phi = \operatorname{asin} r_3$ ✓

(b) Elements (1,1) and (1,2):

$$\left.\begin{aligned} c_\phi c_{\lambda_a} &= r_1 \longrightarrow c_{\lambda_a} = \frac{r_1}{c_\phi} \\ c_\phi s_{\lambda_a} &= r_2 \longrightarrow s_{\lambda_a} = \frac{r_2}{c_\phi} \end{aligned}\right\} \longrightarrow \lambda_a = 2\,\operatorname{atan}\frac{s_{\lambda_a}}{1+c_{\lambda_a}}$$ ✓

(c) Elements (2,3) and (3,3)

$$\left.\begin{aligned} s_\xi c_\phi &= \theta_3 \longrightarrow s_\xi = \frac{\theta_3}{c_\phi} \\ c_\xi c_\phi &= h_3 \longrightarrow c_\xi = \frac{h_3}{c_\phi} \end{aligned}\right\} \longrightarrow \xi = 2\,\operatorname{atan}\frac{s_\xi}{1+c_\xi}$$ ✓

- <u>Given $(r, \lambda_a, \phi, \gamma, v, \xi)$, find $(X, Y, Z, V_x, V_y, V_z)$</u>

From inspection of matrix $R_B$ ($1^{st}$ row) one obtains

$$X = r\, c_\phi c_{\lambda_a} \qquad Y = r\, c_\phi s_{\lambda_a} \qquad Z = r\, s_\phi$$ ✓

Moreover, due to the geometric definition of $\xi$ and $\gamma$,

$$\underline{v} = v\, c_\gamma\, \hat{\theta} + v\, s_\gamma\, \hat{r}$$

After inserting the expressions of $\hat{r}$ and $\hat{\theta}$ (from rows 1,2 of $R_B$), one obtains the three components of $\underline{v}$ along $(\hat{c}_1, \hat{c}_2, \hat{c}_3)$:

$$V_x = v\left[c_{\lambda_a} c_\phi s_\gamma - c_\gamma\left(c_\xi s_{\lambda_a} + c_{\lambda_a} s_\phi s_\xi\right)\right]$$ ✓

$$V_y = v\left[s_{\lambda_a} c_\phi s_\gamma + c_\gamma\left(c_\xi c_{\lambda_a} - s_{\lambda_a} s_\phi s_\xi\right)\right]$$ ✓

$$V_z = v\left[s_\gamma s_\phi + c_\gamma c_\phi s_\xi\right]$$ ✓

• Given $(r, \lambda_a, \phi, \gamma, v, \zeta)$, find $(a, e, i, \Omega, \omega, \theta_*)$

Through the vis viva equation one obtains $a$

$$\mathcal{E} = -\frac{\mu}{r} + \frac{v^2}{2} = -\frac{\mu}{2a} \rightarrow a = \frac{\mu}{\frac{2\mu}{r} - v^2}$$ ✓

The magnitude of $\underline{h}$ is

$$h = |\underline{r} \times \underline{v}| = r v c_\gamma \rightarrow e = \sqrt{1 - \frac{(r v c_\gamma)^2}{\mu a}}$$ ✓

The radial component $v_r$ is

$$v_r = v s_\gamma$$

Together with $r$, the value of $v_r$ leads to finding $\theta_*$

$$r = \frac{p}{1 + e c_{\theta_*}} \rightarrow c_{\theta_*} = \frac{1}{e}\left[\frac{p}{r} - 1\right]$$

$$v_r = \sqrt{\frac{\mu}{p}}\, e s_{\theta_*} \rightarrow s_{\theta_*} = \frac{v_r}{e}\sqrt{\frac{p}{\mu}}$$

$$\rightarrow \theta_* = 2\,\mathrm{atan}\frac{s_{\theta_*}}{1 + c_{\theta_*}}$$ ✓

Because $R_A \equiv R_B$ one can obtaining the following relations:

(a) Element $(3,3)$: $c_i = c_\zeta c_\phi \longrightarrow i = \mathrm{acos}(c_\zeta c_\phi)$ ✓

(b) Elements $(3,1)$ and $(3,2)$

$$s_i s_\Omega = -c_\zeta s_\phi c_{\lambda_a} + s_\zeta s_{\lambda_a} \rightarrow s_\Omega = \frac{-c_\zeta s_\phi c_{\lambda_a} + s_\zeta s_{\lambda_a}}{s_i}$$

$$-s_i c_\Omega = -c_\zeta s_\phi s_{\lambda_a} - s_\zeta c_{\lambda_a} \rightarrow c_\Omega = \frac{c_\zeta s_\phi s_{\lambda_a} + s_\zeta c_{\lambda_a}}{s_i}$$

$$\rightarrow \Omega = 2\,\mathrm{atan}\frac{s_\Omega}{1 + c_\Omega}$$ ✓

(c) Elements $(1,3)$ and $(2,3)$

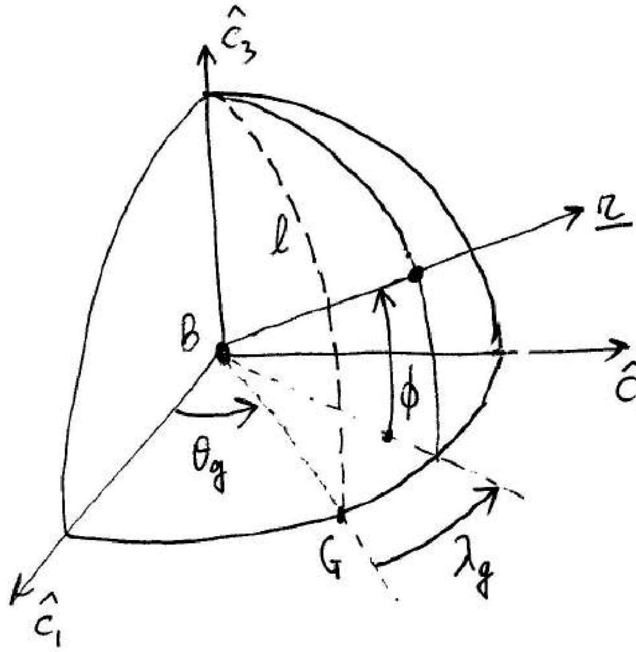
$$s_{\theta_t} s_i = s_\phi \rightarrow s_{\theta_t} = \frac{s_\phi}{s_i}$$

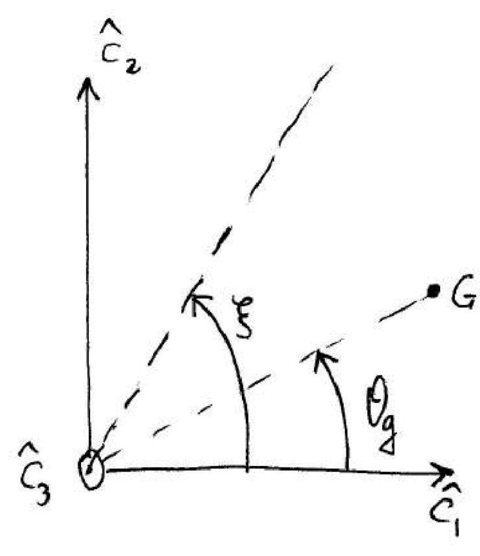
$$c_{\theta_t} s_i = s_\zeta c_\phi \rightarrow c_{\theta_t} = \frac{s_\zeta c_\phi}{s_i}$$

$$\rightarrow \theta_t = 2\,\mathrm{atan}\frac{s_{\theta_t}}{1 + c_{\theta_t}} \rightarrow \omega = \theta_t - \theta_*$$ ✓

- GROUNDTRACK

The satellite groundtrack is the plot of all the subsatellite positions on the Earth surface (considering its rotation).
The subsatellite is the projection of the satellite position on the Earth surface. If an observer is located where the subsatellite lies (at a certain time), then the satellite is exactly above the observer.
The ground track is represented on a Mercator map and indicates the regions flown over by the satellite.
In order to plot the ground track, the satellite latitude $\phi(t)$ and geographical longitude $\lambda_g(t)$ (as functions of time) are needed




$\ell$ is the Greenwich meridian identified by its abolute longitude $\theta_g = \theta_{g0} + \omega_E (t - t_0)$
($\omega_E$ = Earth rotation rate)

$\xi$ = absolute longitude

$\xi = \theta_g + \lambda_g$

$\lambda_g$ = geographical longitude

The satellite position can be written in $(\hat{c}_1, \hat{c}_2, \hat{c}_3)$ if the orbit elements $(a, e, i, \Omega, \omega)$ and the instantaneous true anomaly are Known

$$\underline{r} = \frac{a(1-e^2)}{1+e\,c_{\theta_*}} \begin{bmatrix} c_{\theta_t} c_\Omega - c_i s_{\theta_t} s_\Omega & c_{\theta_t} s_\Omega + c_i s_{\theta_t} c_\Omega & s_{\theta_t} s_i \end{bmatrix} \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{bmatrix}$$

where $\theta_t = \theta_* + \omega$ is the argument of latitude

However, the position vector $\underline{r}$ can be written also in terms of $\xi$ (absolute longitude) and $\phi$ (latitude), where their bounds are $-\pi \le \xi < \pi$ and $-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}$,

$$\underline{r} = \frac{a(1-e^2)}{1+e\,c_{\theta_*}} \begin{bmatrix} c_\phi c_\xi & c_\phi s_\xi & s_\phi \end{bmatrix} \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{bmatrix}$$

These two expressions for $\underline{r}$ must coincide, therefore

$$s_\phi = s_{\theta_t} s_i \longrightarrow \phi = a\sin(s_{\theta_t} s_i)$$

$$c_\phi c_\xi = c_{\theta_t} c_\Omega - c_i s_{\theta_t} s_\Omega \longrightarrow \left\{ c_\xi = \frac{c_{\theta_t} c_\Omega - c_i s_{\theta_t} s_\Omega}{\sqrt{1-(s_{\theta_t} s_i)^2}} \right.$$

$$c_\phi s_\xi = c_{\theta_t} s_\Omega + c_i s_{\theta_t} c_\Omega \longrightarrow \left\{ s_\xi = \frac{c_{\theta_t} s_\Omega + c_i s_{\theta_t} c_\Omega}{\sqrt{1-(s_{\theta_t} s_i)^2}} \right.$$

From the last two relations one can find

$$\xi = 2\,\mathrm{atan}\,\frac{s_\xi}{1+c_\xi} \quad \text{and then} \quad \lambda_g = \xi - \theta_g$$

- ## Latitude limits

The latitude is given by $\phi = \mathrm{asin}\left(s_{\theta_t}\, s_i\right)$

(a) If $0 < i < \frac{\pi}{2}$ (direct orbits)

$$\theta_t = \frac{\pi}{2} \qquad \sin\phi = \sin i \longrightarrow \phi_{max} = i$$

$$\theta_t = -\frac{\pi}{2} \qquad \sin\phi = -\sin i \longrightarrow \phi_{min} = -i$$

(b) If $\frac{\pi}{2} < i < \pi$ (retrograde orbits)

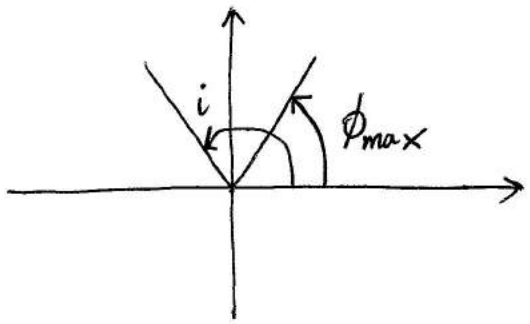
$$\theta_t = \frac{\pi}{2} \qquad \sin\phi = \sin i \longrightarrow \phi_{max} = \pi - i$$

$$\theta_t = -\frac{\pi}{2} \qquad \sin\phi = -\sin i \longrightarrow \phi_{min} = i - \pi$$

In this second case one must consider that

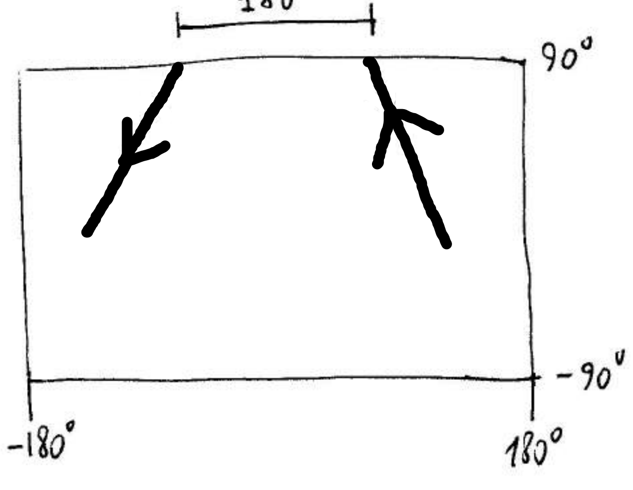
$$-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$$

unlike $i$ $(0 \leq i \leq \pi)$




Polar orbits have inclination $i = \frac{\pi}{2}$, therefore they cross all latitudes.

When the satellite flies over the North (or South) pole, the longitude changes by $\pi$ (see figure), which is a discontinuity related to the Mercator map

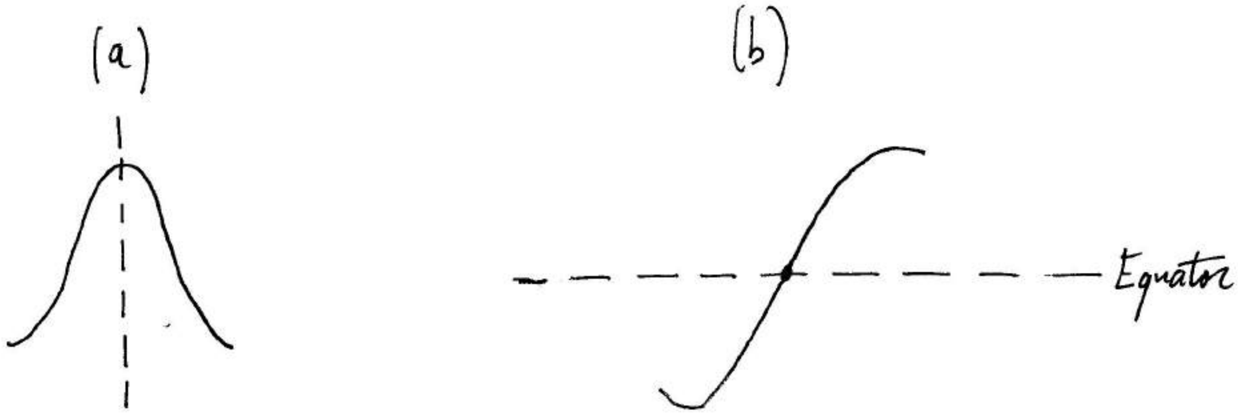


• <u>Symmetry properties</u>

In special cases the ground track exhibits symmetry properties, due to the symmetry of $\underline{v}$ with respect to the apsidal line and due to the fact that two points on the Earth with the same latitude have the same inertial velocity toward East

(a) If $e \neq 0$ and $\omega = \pm \frac{\pi}{2}$ $\longrightarrow$ ground track symmetrical with respect to the meridian that passes through the maximum and minimum latitude

(b) If $e \neq 0$ and $\omega = 0, \pi$ $\longrightarrow$ ground track symmetrical with respect to the points at which it crosses the equatorial line




(c) If $e = 0$ $\longrightarrow$ both symmetries (a) and (b)

These symmetry properties may be useful for identifying $e$ and $\omega$ by inspecting the satellite ground track

- Geosynchronous orbits

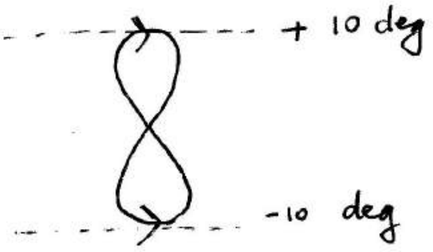
If the orbit period is 1 sidereal day, then the satellite motion is synchronous with the Earth rotation, and this implies that the ground track is repeated. More specifically,

(a) If $0 < i < \frac{\pi}{2} \rightarrow$ closed ground track: only a limited longitude range for the ground track (see next examples)

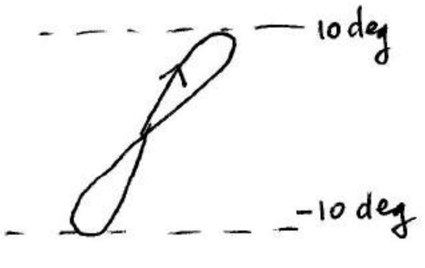
(b) If $\frac{\pi}{2} < i < \pi \rightarrow$ repeating ground track, which spans all the longitudes (see next examples)

Some special case:

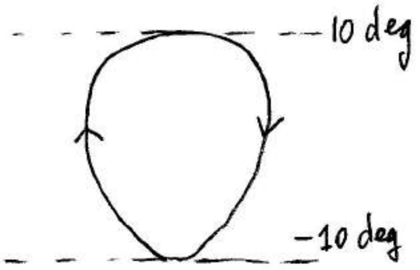
(1) $i = 10$ deg, $e = 0$




(2) $i = 10$ deg, $e \neq 0$, $\omega = 0$




(3) $i = 10$ deg, $e \neq 0$, $\omega = \frac{\pi}{2}$




The ground track evolves toward East if the subsatellite has velocity toward East greater than a point of the Earth at that point

- Geostationary orbits

These orbits have 3 characteristics

(a) geosynchronous ($a = 42164$ Km)

(b) circular ($e = 0$)

(c) equatorial ($i = 0$)

In this way the ground track is a point along the equator and does not move. As a result, the satellite has always the same position in the sky for any observer located on the Earth surface.

It is obvious that geostationary orbits are a subset of geosynchronous orbits.

- Motion along the ground track

Two cases must be distinguished:

(a) Retrograde orbits $\left(\frac{\pi}{2} < i < \pi\right)$: the satellite inertial velocity has negative component toward East (in fact the satellite flies toward West). As a result the ground track is always traveled from East to West, regardless of all the orbit elements

(b) Direct orbits $\left(0 < i < \frac{\pi}{2}\right)$: the satellite inertial velocity has positive component toward East. This means that also the subsatellite has inertial velocity toward East, and this component must be compared with the inertial velocity of the point on the Earth at same latitude

As a result, direct orbits can be traveled either

(i) from West to East or (ii) from East to West

depending on the semimajor axis, eccentricity, and argument of perigee

As a general rule, given $P_0$ (an arbitrary point on the ground track) and $P_1$ (the point after an orbital period, located at the same latitude as $P_0$; the ground track has the same slope at $P_0$ and $P_1$)

(1) $P_1$ has moved toward East if $T < T_{geosyn} = 1$ sidereal day

(2) $P_1$ has moved toward West if $T > T_{geosyn} = 1$ sidereal day

Qualitatively, high direct orbits (2) move globally toward West, whereas low direct orbits (1) move globally toward East

Geosynchronous direct orbits have closed ground tracks, which are traveled in part toward West, in part toward East.

# Examples of Satellite Ground Tracks

$$1\begin{cases} a=\left(\mu/\omega_E^2\right)^{1/3} \\ e=0 \\ i=20\text{ deg} \\ \Omega=10\text{ deg} \\ \theta_{t0}=5\text{ deg} \end{cases} \quad 2\begin{cases} a=\left(\mu/\omega_E^2\right)^{1/3} \\ e=0.4 \\ i=10\text{ deg} \\ \Omega=30\text{ deg} \\ \omega=0\text{ deg} \\ M_0=300\text{ deg} \end{cases} \quad 3\begin{cases} a=\left(\mu/\omega_E^2\right)^{1/3} \\ e=0.4 \\ i=10\text{ deg} \\ \Omega=30\text{ deg} \\ \omega=90\text{ deg} \\ M_0=300\text{ deg} \end{cases} \quad 4\begin{cases} a=\left(\mu/\omega_E^2\right)^{1/3} \\ e=0.6 \\ i=100\text{ deg} \\ \Omega=300\text{ deg} \\ \omega=-90\text{ deg} \\ M_0=300\text{ deg} \end{cases} \quad 5\begin{cases} a=\left(\mu/\omega_E^2\right)^{1/3} \\ e=0.6 \\ i=80\text{ deg} \\ \Omega=300\text{ deg} \\ \omega=-90\text{ deg} \\ M_0=300\text{ deg} \end{cases}$$

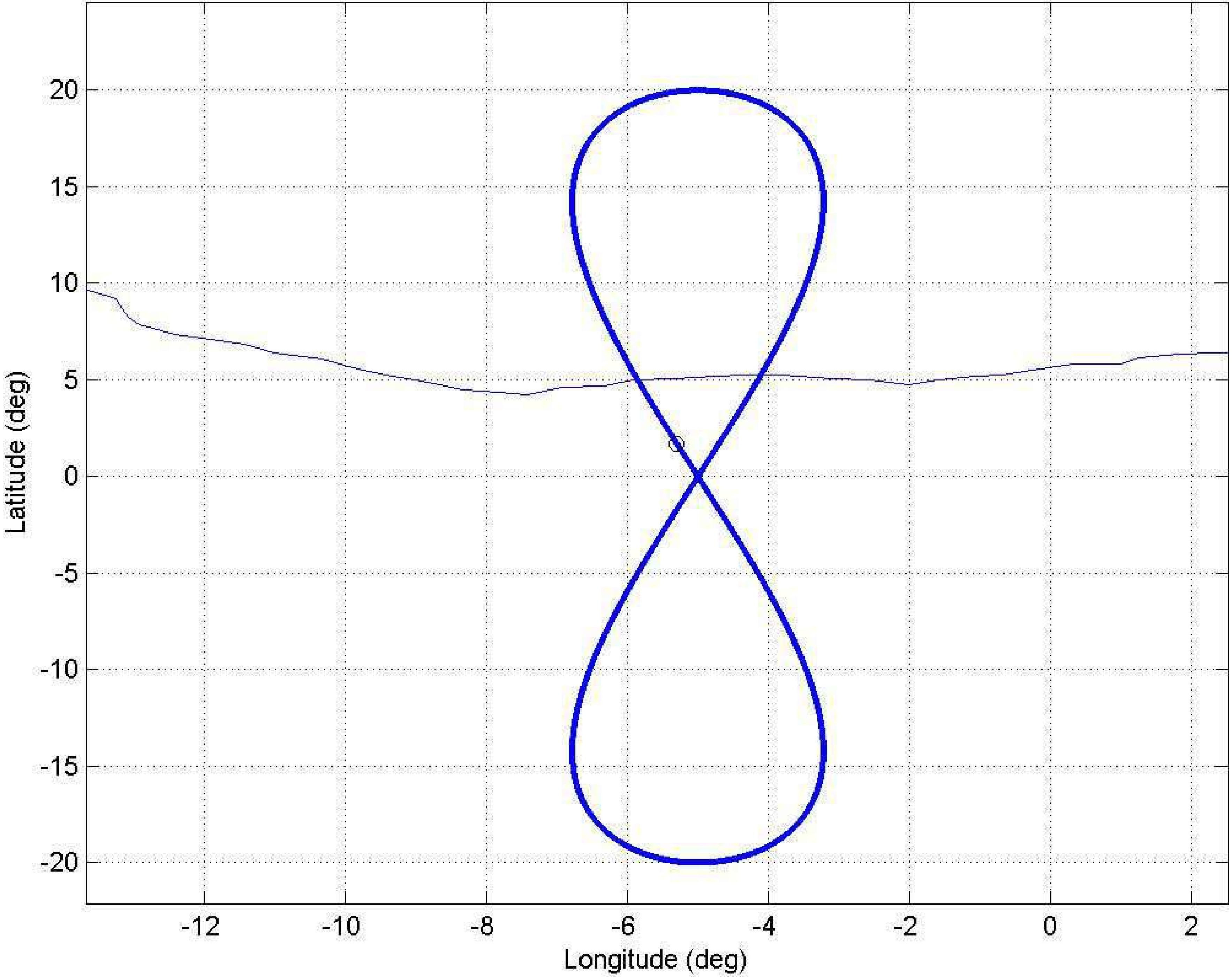
$$6\begin{cases} a=\left(\mu/\omega_E^2\right)^{1/3} \\ e=0 \\ i=90\text{ deg} \\ \Omega=10\text{ deg} \\ \theta_{t0}=5\text{ deg} \end{cases} \quad 7\begin{cases} a=100000\text{ km} \\ e=0 \\ i=50\text{ deg} \\ \Omega=300\text{ deg} \\ \theta_{t0}=195\text{ deg} \end{cases} \quad 8\begin{cases} a=7500\text{ km} \\ e=0.05 \\ i=60\text{ deg} \\ \Omega=10\text{ deg} \\ \omega=0\text{ deg} \\ M_0=300\text{ deg} \end{cases} \quad 9\begin{cases} a=50000\text{ km} \\ e=0.08 \\ i=80\text{ deg} \\ \Omega=10\text{ deg} \\ \omega=-90\text{ deg} \\ M_0=100\text{ deg} \end{cases} \quad 10\begin{cases} a=8500\text{ km} \\ e=0.1 \\ i=100\text{ deg} \\ \Omega=30\text{ deg} \\ \omega=-180\text{ deg} \\ M_0=10\text{ deg} \end{cases}$$

$$\left(\mu=398604.3\ \text{km}^3/\text{sec}^2,\ \omega_E=7.292115\cdot10^{-5}\ \text{sec}^{-1},\ \theta_{g0}=20\text{ deg}\right)$$

P.S. For circular orbits only $\theta_t$ is meaningful

MP

# 1. Geosynchronous, circular, inclined orbit

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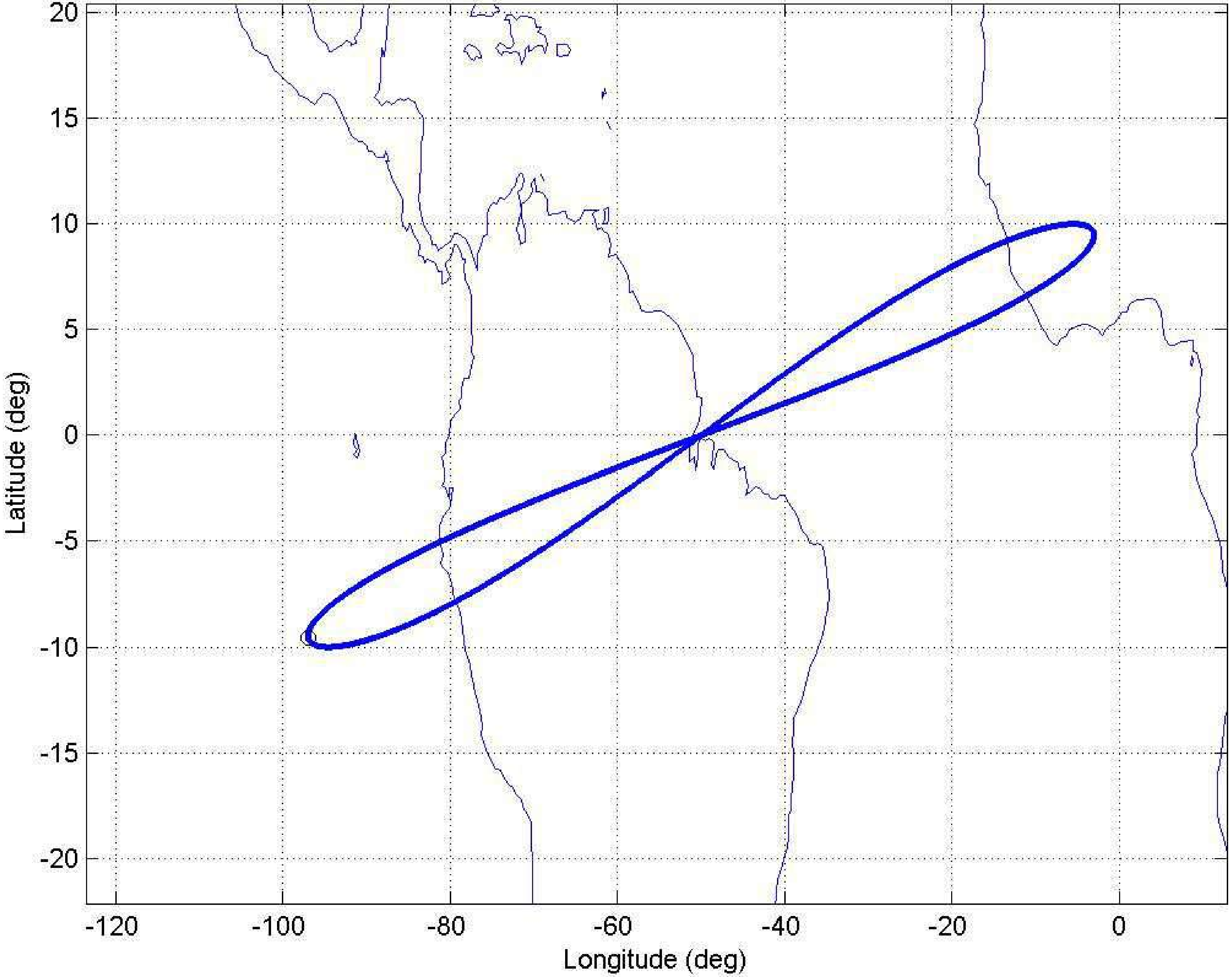
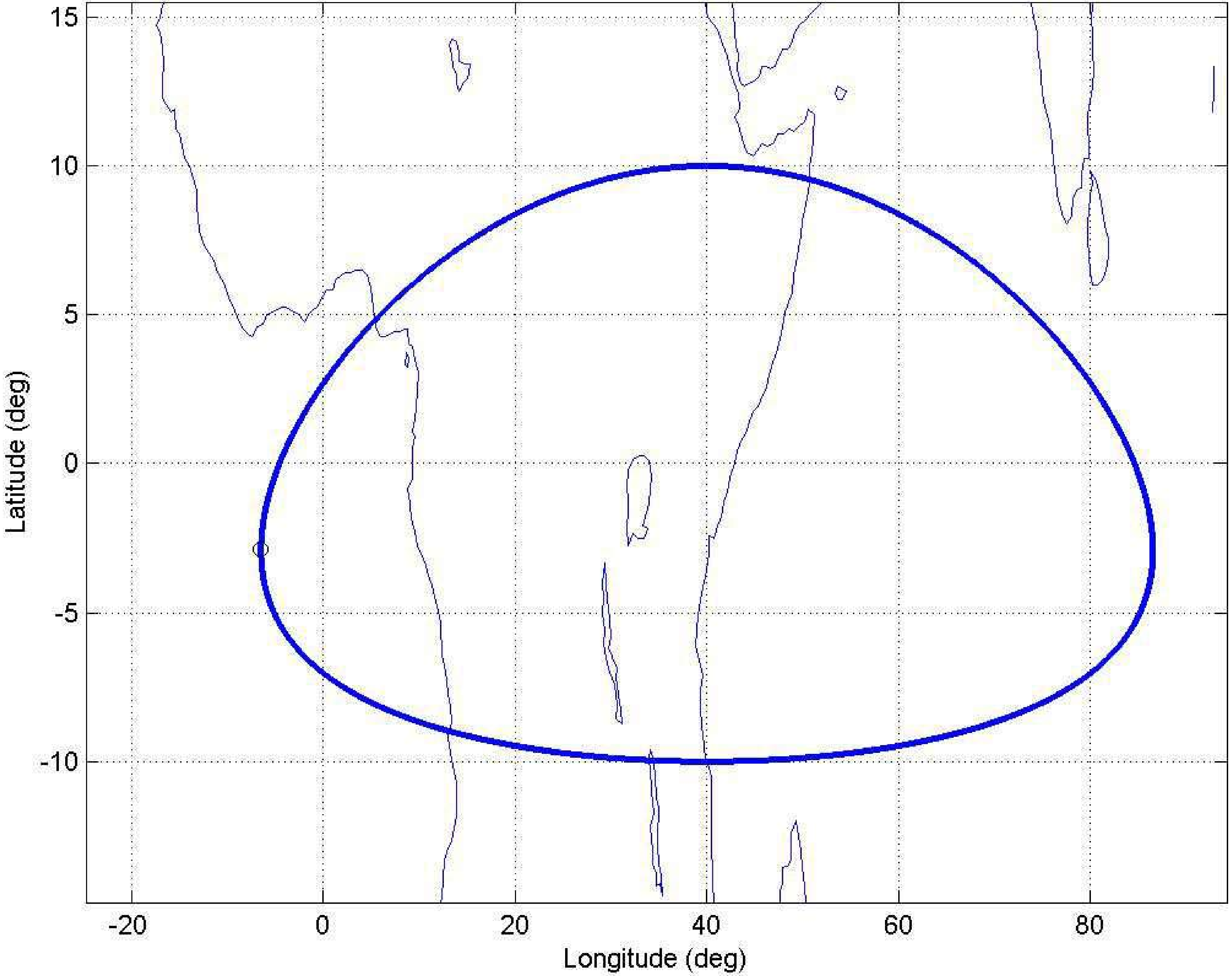
## 2. Geosynchronous, eccentric, inclined orbit (argument of perigee = 0)

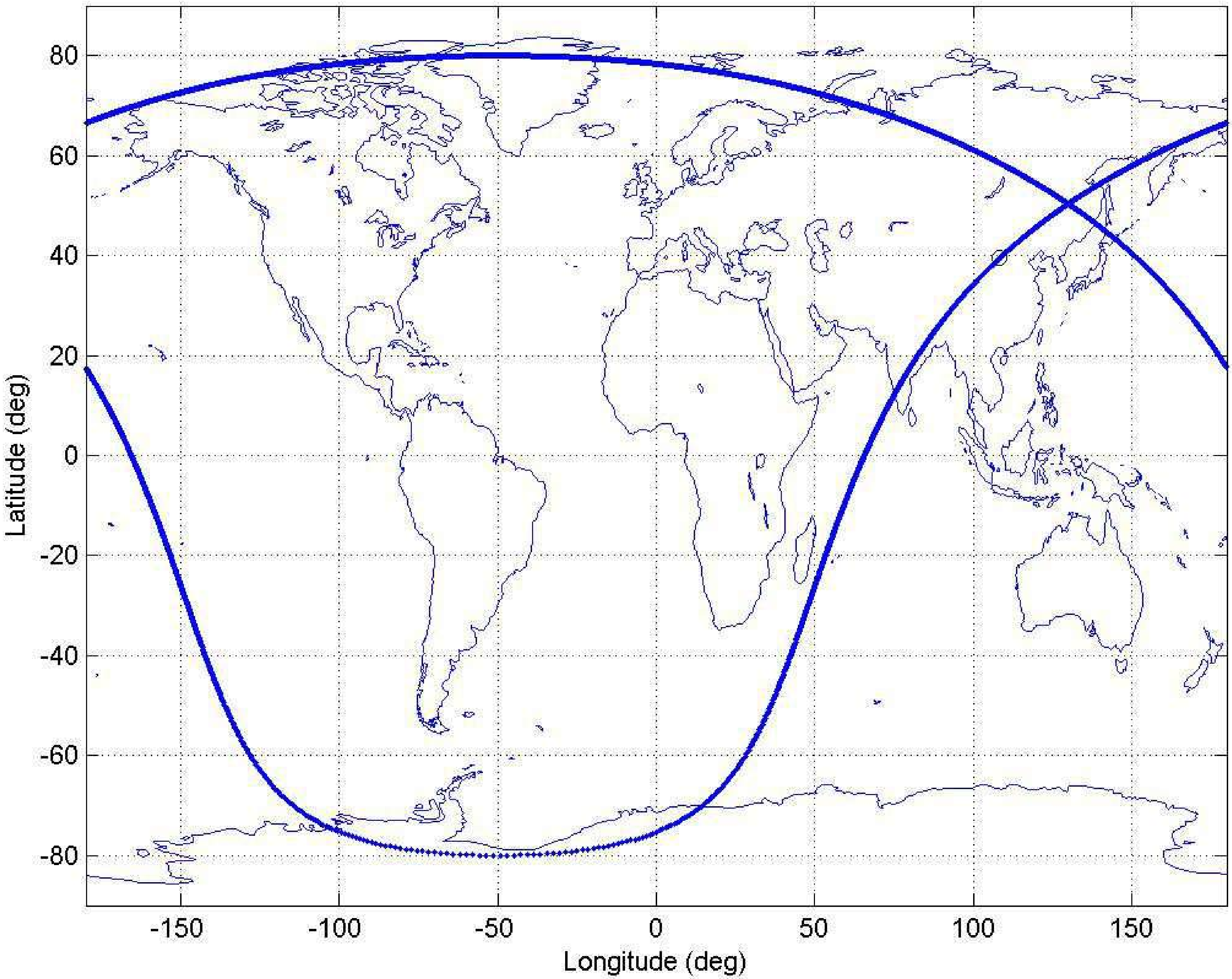


**3. Geosynchronous, eccentric, inclined orbit (argument of perigee = 90 deg)**



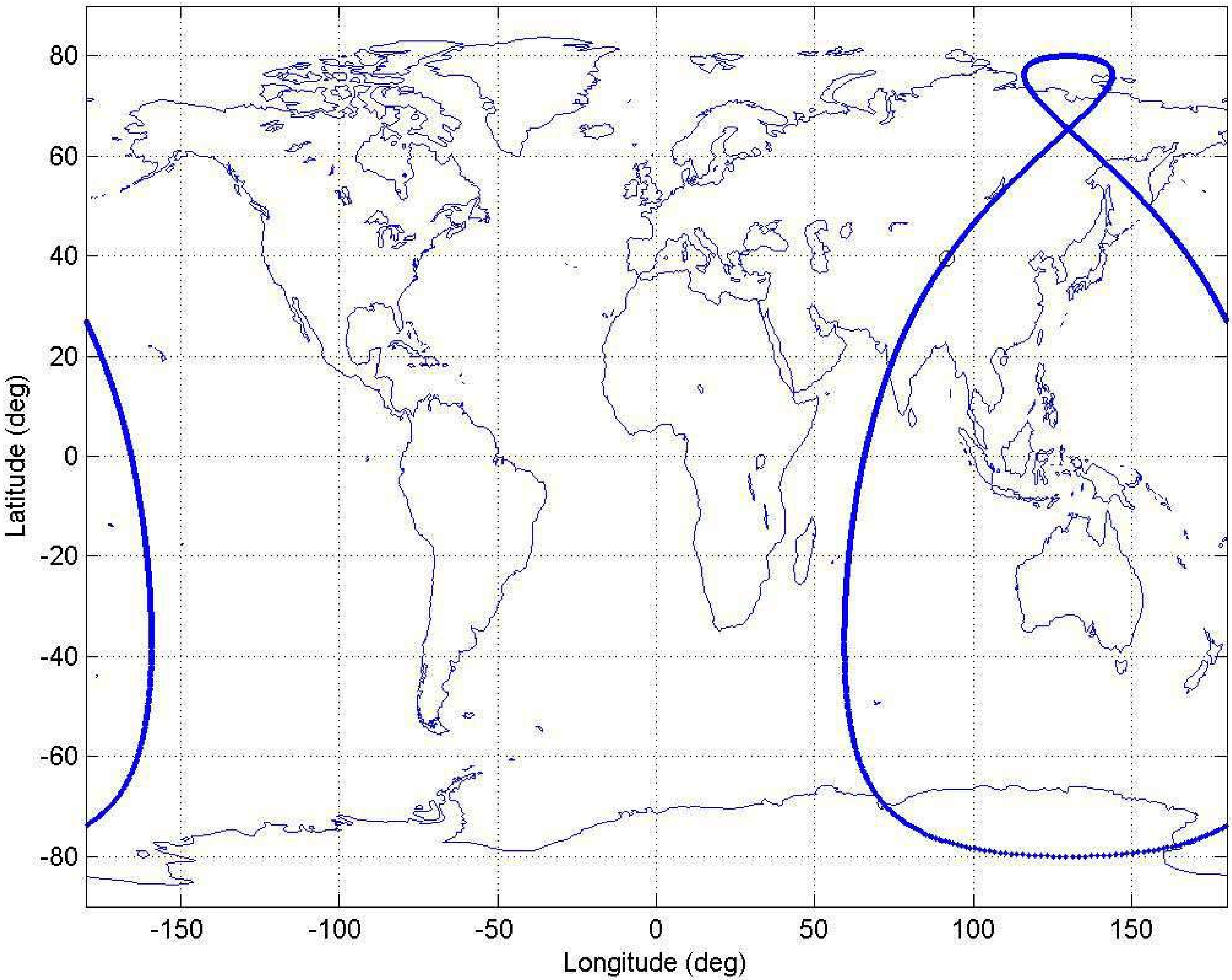

MP

# 4. Geosynchronous, eccentric, retrograde orbit



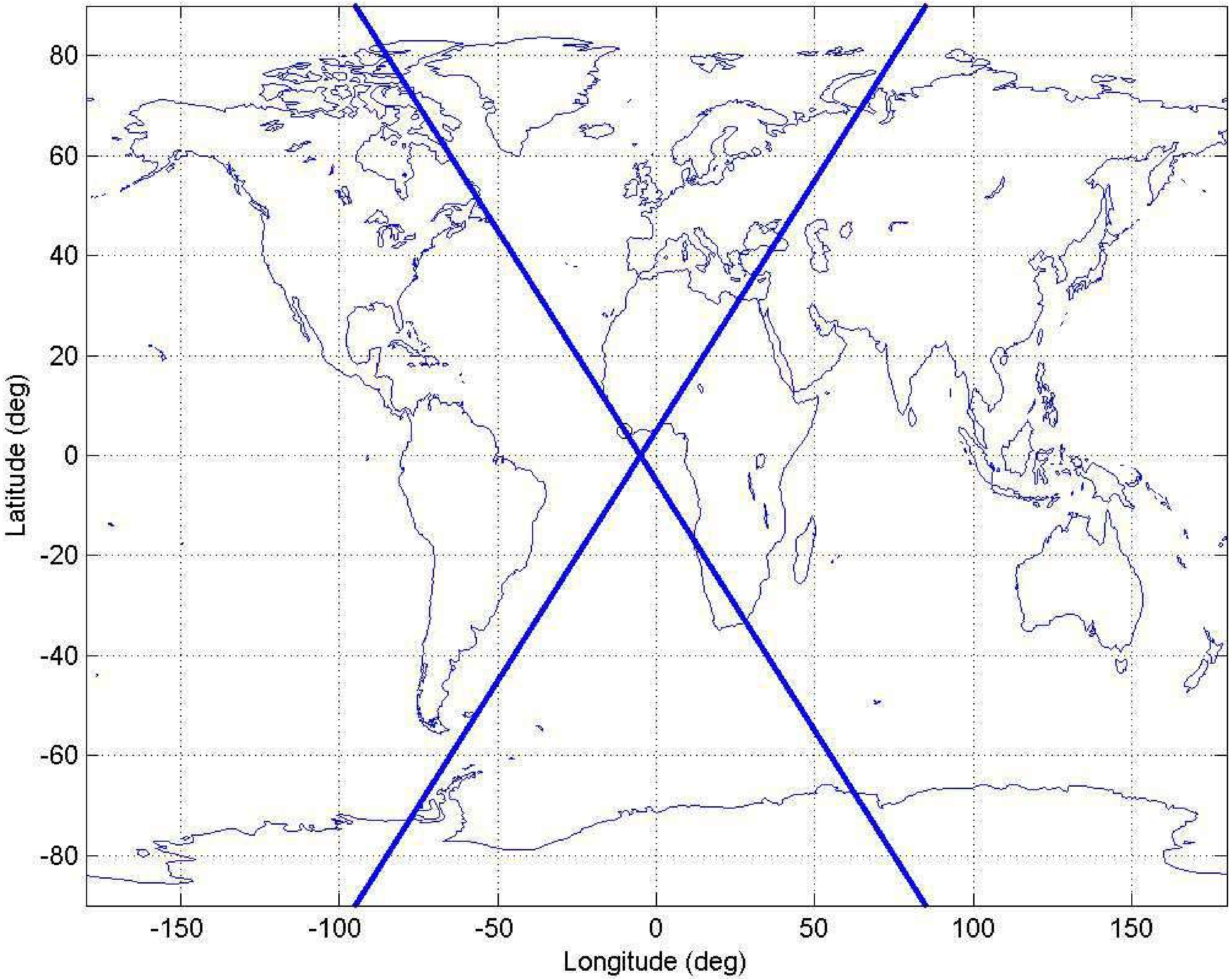

MP

# 5. Geosynchronous, eccentric, inclined orbit



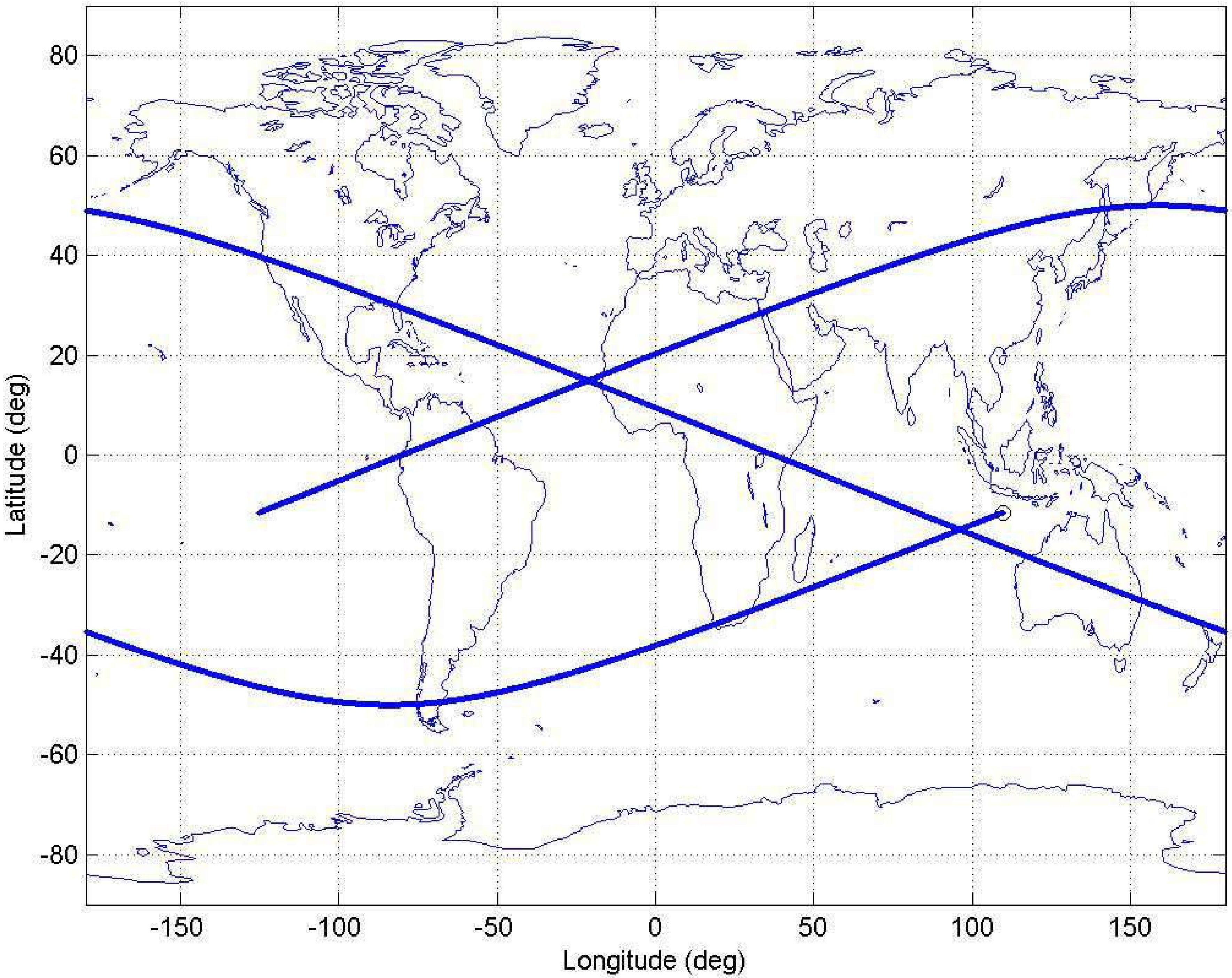

MP

## 6. Geosynchronous, circular, polar orbit



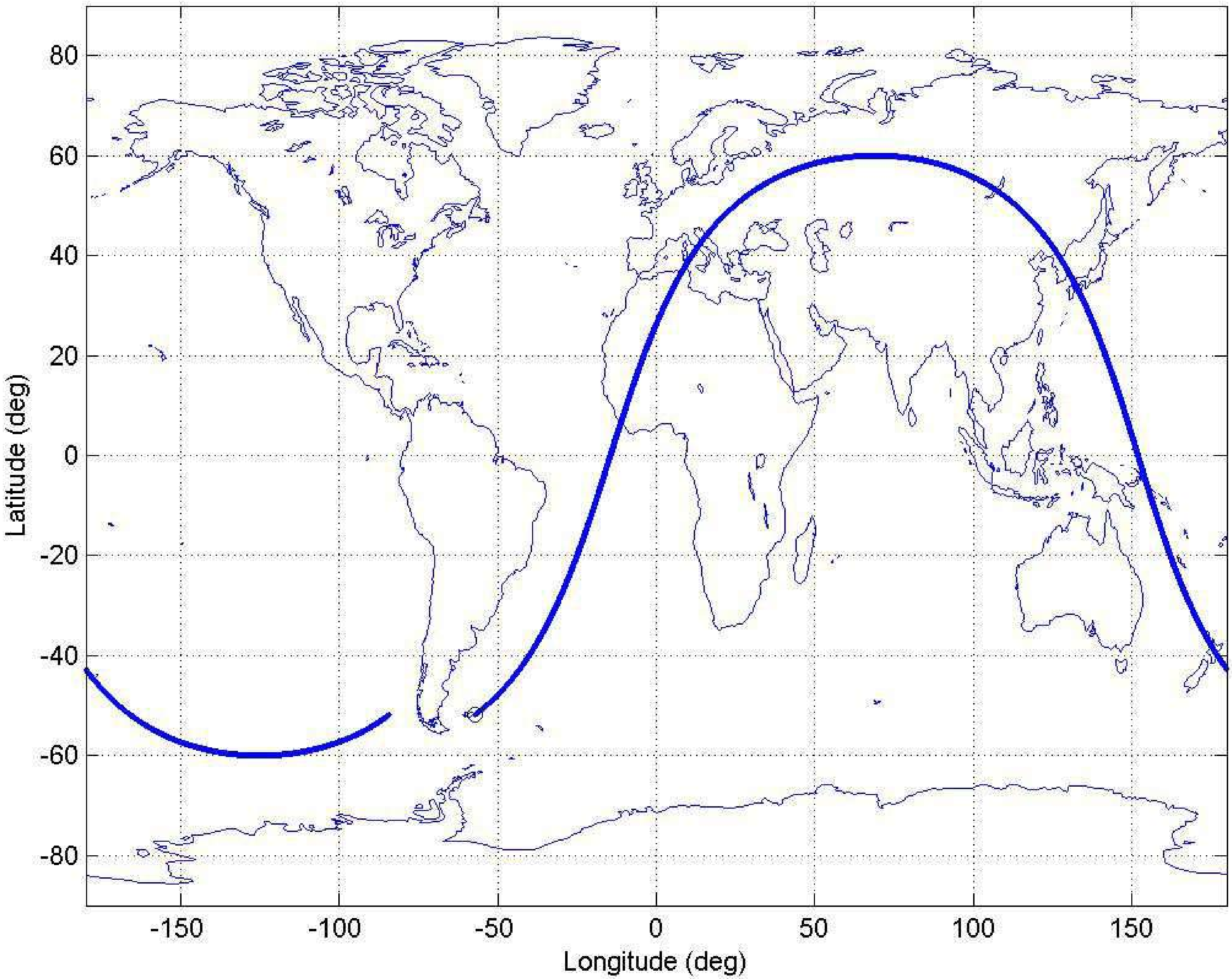

MP

# 7. Circular, direct high Earth orbit



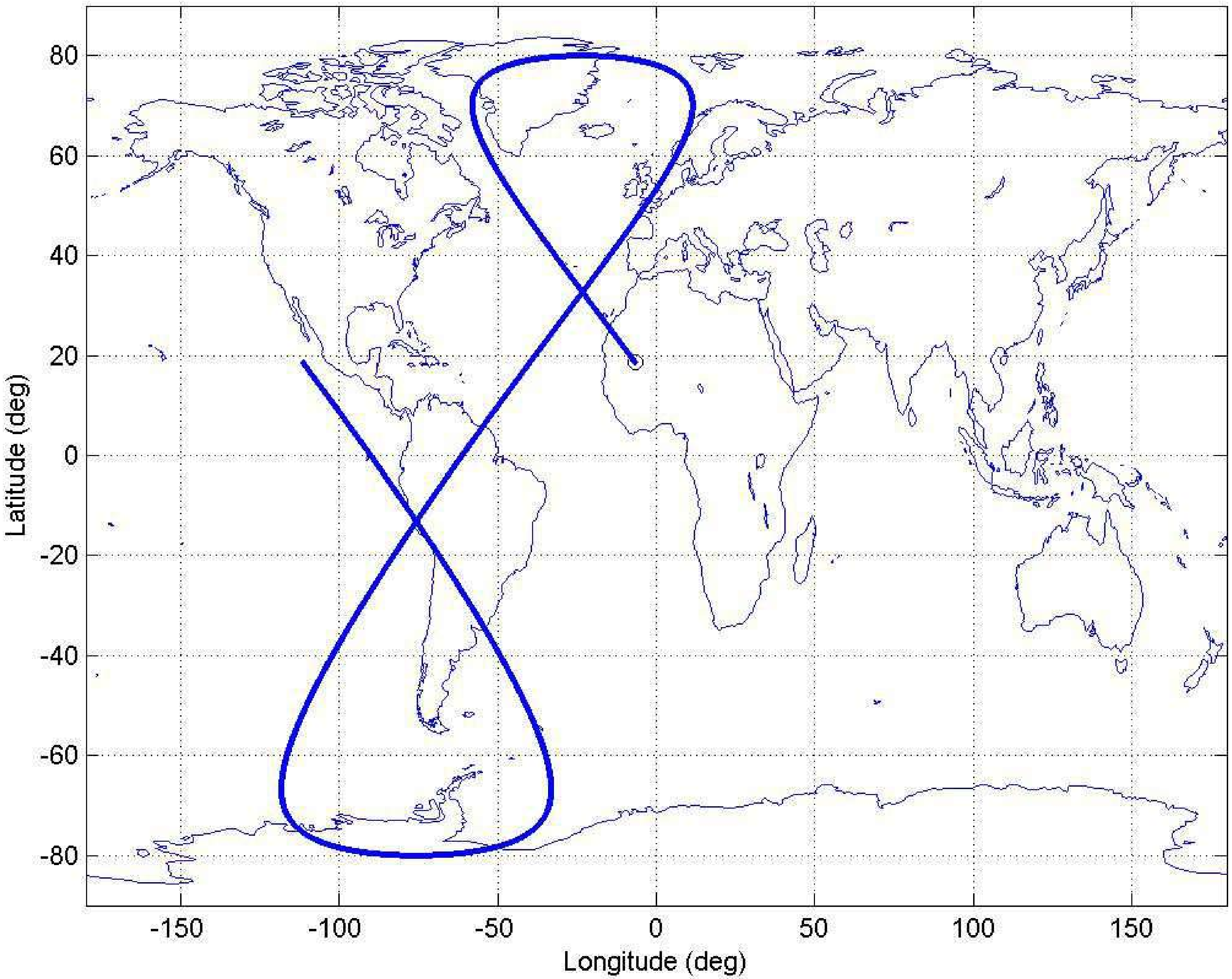

MP

# 8. Near-circular, direct low Earth orbit



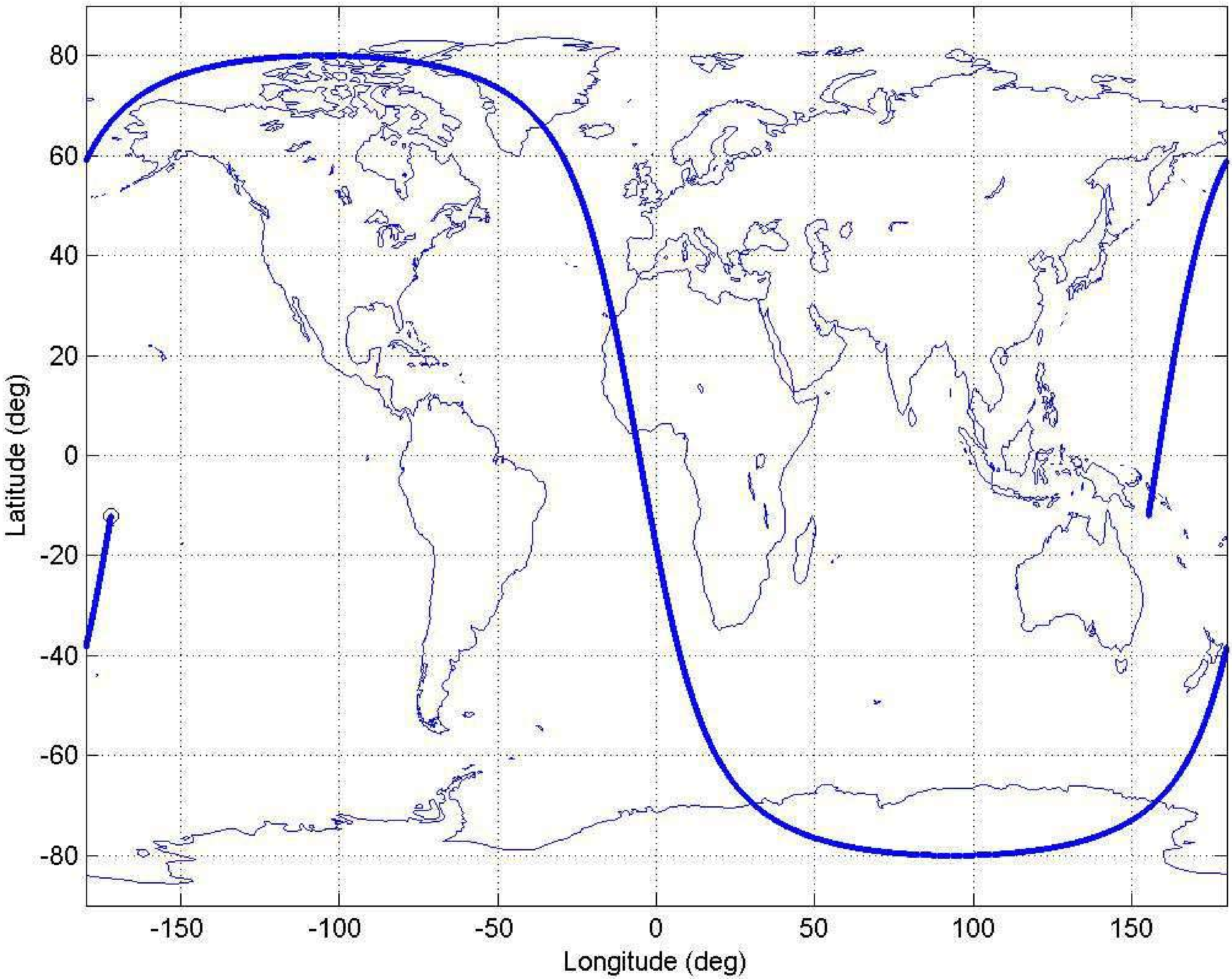

MP

# 9. Eccentric, direct high Earth orbit




MP

# 10. Eccentric, retrograde low Earth orbit




MP