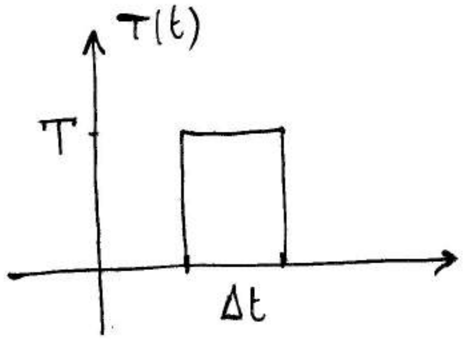
IMPULSIVE ORBIT TRANSFERS

IMPULSIVE THRUST APPROXIMATION

If high thrust is applied for a short time interval, then its effect can be approximated as an istantaneous velocity variation, while the position does not change



From the mathematical point of view $T \Delta t = \Delta v$ and the impulsive thrust approximation is the limiting case when $T \to \infty$ while $\Delta t \to 0$ and $T \Delta t = \Delta v$ remains finite and equal to the velocity variation

This velocity change is often called also velocity impulse. As previously remarked, through a velocity impulse:

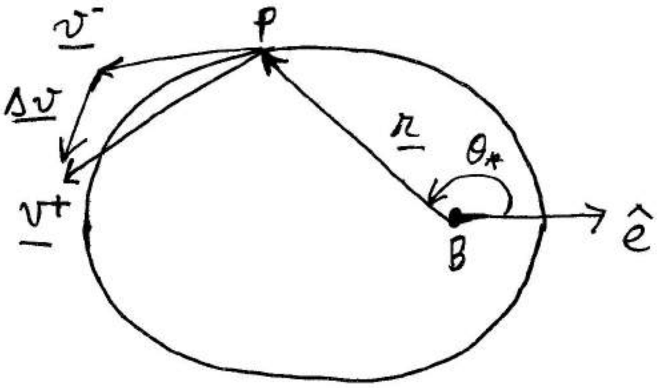
(a) Velocity changes (in direction and magnitude)
(b) Position does not change

The impulsive thrust approximation holds specifically for high thrust orbit maneuvers, when Δt is small.

The impulsive thrust approximation does not hold for

- ascent trajectory of launch vehicles
- low-thrust trajectories (where continuos thrust is applied for long time intervals)

- EFFECT OF VELOCITY IMPULSES



A velocity change in P does not modify $\underline{r}$, but only $\underline{v}$

$$\underline{v}^+ = \underline{v}^- + \Delta\underline{v}$$

At P the position and velocity are identified through θ_*^-

$$r = \frac{p^-}{1+e^- c_{\theta_*^-}} \qquad v_r^- = \sqrt{\frac{\mu}{p^-}}\, e^- \cos\theta_*^- \qquad v_\theta^- = \sqrt{\frac{\mu}{p^-}}\left(1+e^-\cos\theta_*^-\right)$$

The velocity impulse is $\Delta\underline{v} = \Delta v_\theta\, \hat{\theta} + \Delta v_r\, \hat{r}$.

Therefore
$$\begin{cases} v_r^+ = \sqrt{\dfrac{\mu}{p^-}}\, e^- s_{\theta_*^-} + \Delta v_r = \sqrt{\dfrac{\mu}{p^+}}\, e^+ s_{\theta_*^+} \\[2ex] v_\theta^+ = \sqrt{\dfrac{\mu}{p^-}}\left(1+e^- c_{\theta_*^-}\right) + \Delta v_\theta = \sqrt{\dfrac{\mu}{p^+}}\left(1+e^+ c_{\theta_*^+}\right) \end{cases}$$

As a first step, a^+ can be found through energy,

$$(v^+)^2 = (v_\theta^+)^2 + (v_r^+)^2 \qquad \mathcal{E}^+ = -\frac{\mu}{2a^+} = \frac{(v^+)^2}{2} - \frac{\mu}{r}$$

$$\longrightarrow \quad a^+ = \frac{\mu}{\dfrac{2\mu}{r} - (v^+)^2}$$

As a second step, e^+ can be found through angular momentum

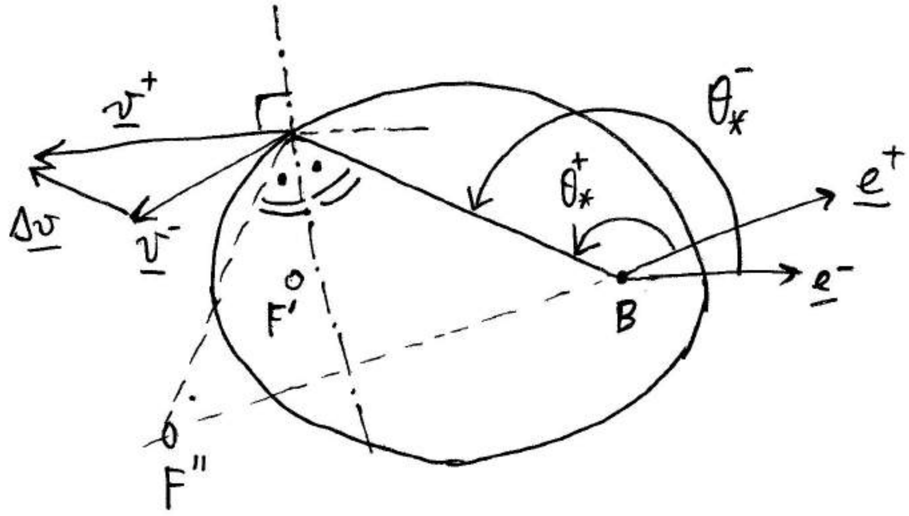
$$h^+ = r v_\theta^+ \qquad \text{and} \qquad h^+ = \sqrt{\mu a^+\left[1-(e^+)^2\right]}$$

$$\longrightarrow \quad e^+ = \sqrt{1 - \frac{r(v_\theta^+)^2}{\mu a^+}} \quad \longrightarrow \quad p^+ = a^+\left[1-(e^+)^2\right]$$

As a third step, the true anomaly θ_*^+ is found using the equations for r and v_r^+

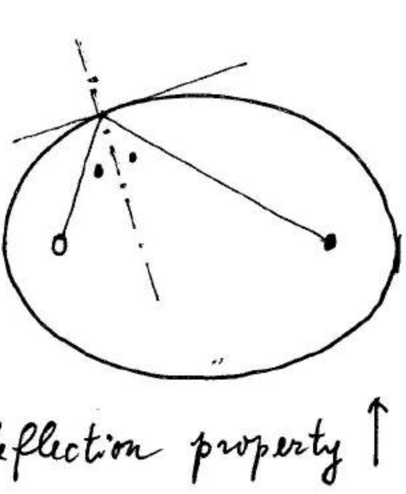
$$
\begin{cases}
r = \dfrac{p^+}{1+e^+ c_{\theta_*^+}} \longrightarrow c_{\theta_*^+} = \dfrac{p^+ - r}{r e^+} \\[2ex]
v_r^+ = \sqrt{\dfrac{\mu}{p^+}}\, e^+ s_{\theta_*^+} \longrightarrow s_{\theta_*^+} = \dfrac{v_r^+}{e^+}\sqrt{\dfrac{p^+}{\mu}}
\end{cases}
\qquad
\theta_*^+ = 2\,\mathrm{atan}\,\frac{s_{\theta_*^+}}{1+c_{\theta_*^+}}
$$

in this way θ_*^+ is correctly found in the interval θ_*^+



The apsidal line ($\parallel \underline{e}$) is rotated by angle $\theta_*^- - \theta_*^+$

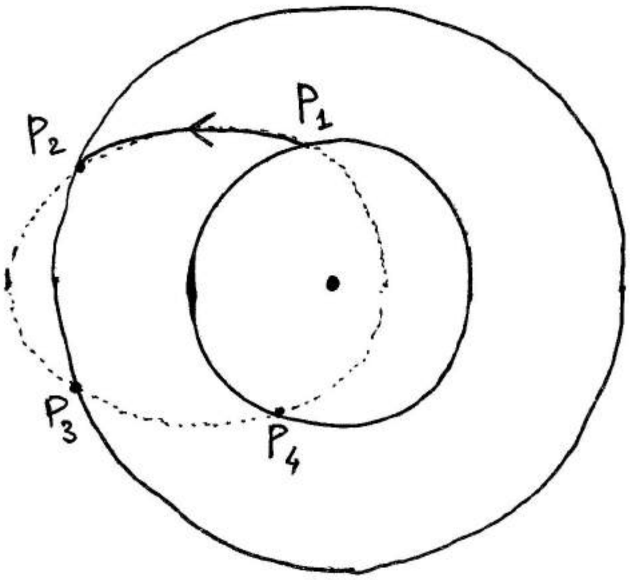
In the previous figure the vacant focus has moved from F' to F'' as a consequence of the velocity change. The line along which F'' is located fulfills the reflection property that holds for all points along an ellipse. After the velocity change the new velocity is $\underline{v}^+$ and is tangential to the new trajectory. The two foci B (where the center of the attracting body is located) and F'' must satisfy the reflection property. The distance from B to F'' is



Reflection property ↑

$$d(B, F'') = a^+ e^+$$

GLOBALLY OPTIMAL TRANSFERS BETWEEN COPLANAR CIRCULAR ORBITS



Initial and final orbits have specified radii R_i and R_f, respectively

The transfer arc (from P_1 to P_2) is elliptic and associated with (a_T, e_T)

semimajor axis and eccentricity of the transfer arc

Four possible transfers exist, i.e.

(P_1, P_2), (P_1, P_3), (P_4, P_2), (P_4, P_3)

However, it is easy to verify that all of them are equivalent in terms of total velocity change Δv_{tot}

The objective here is to identify (e_T, a_T) that minimize

$$\Delta v_{tot} = \Delta v_1 + \Delta v_2$$

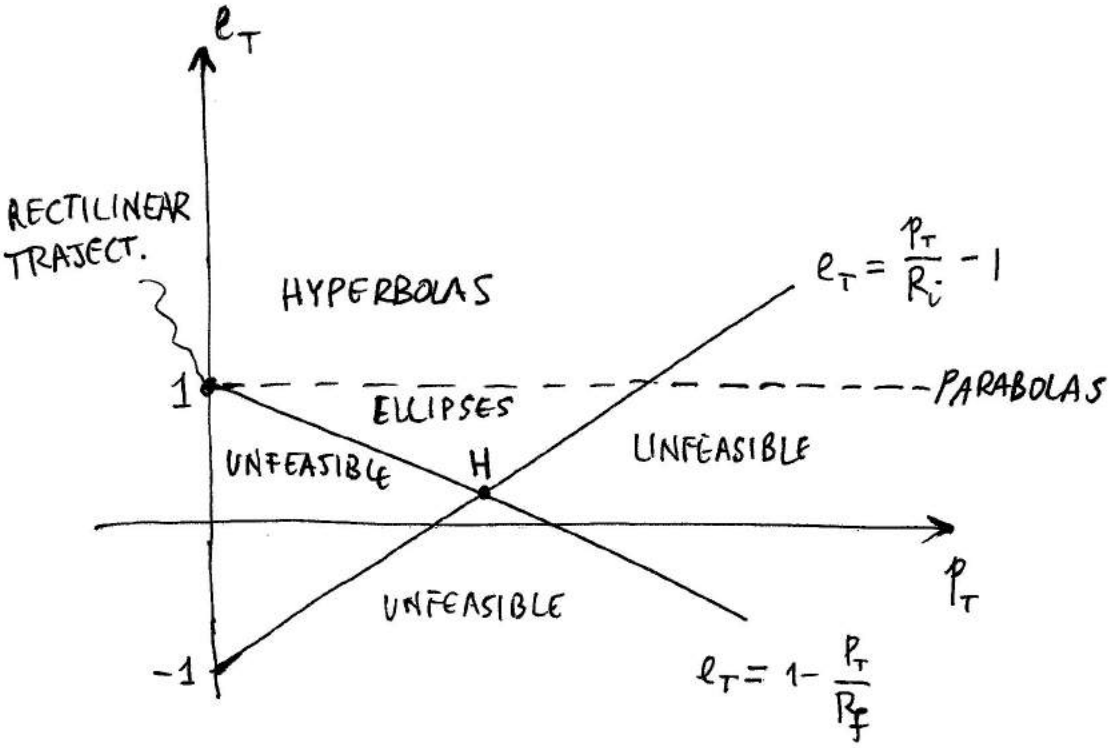
Minimizing Δv_{tot} implies minimizing the propellant consumption

Feasible transfers are such that

$$r_{PT} = \frac{p_T}{1+e_T} \leq R_i \quad \longrightarrow \quad e_T \geq \frac{p_T}{R_i} - 1$$

$$p_T = a_T\left(1-e_T^2\right)$$

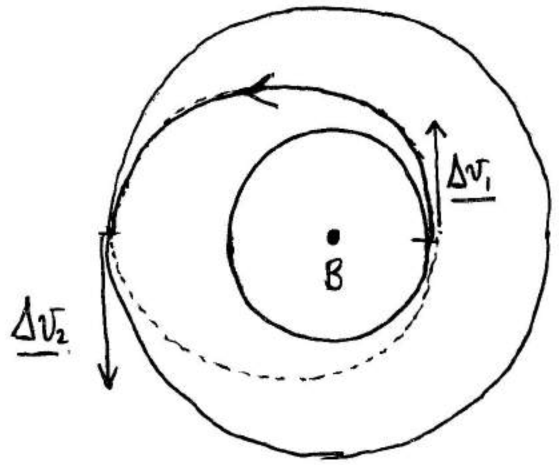
$$r_{AT} = \frac{p_T}{1-e_T} \geq R_f \quad \longrightarrow \quad e_T \geq 1 - \frac{p_T}{R_f}$$



At H one obtains

$$e_T = \frac{R_f - R_i}{R_f + R_i}$$

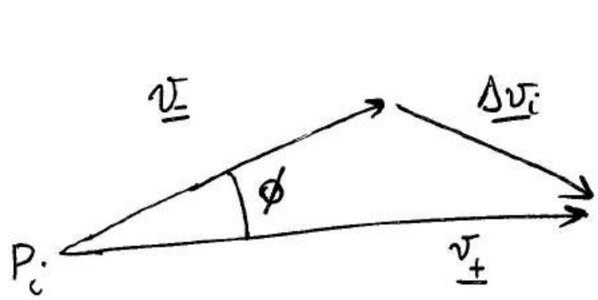
$$a_T = \frac{R_f + R_i}{2}$$



Point H corresponds to the bitangent transfer

$\underline{\Delta v_1}$ and $\underline{\Delta v_2}$ are tangent to the terminal orbits and in the direction of motion

Both $\underline{\Delta v_1}$ and $\Delta \underline{v_2}$ have, in fact, the effect of increasing the semimajor axis



At P_i: $\quad (\Delta v_i)^2 = (v_-)^2 + (v_+)^2 - 2 v_- v_+ \cos\phi$

where

$$\begin{cases} v_- = \sqrt{\dfrac{\mu}{R_k}} & (k = i \text{ or } f) \\ v_+ \cos\phi = v_\theta^+ = \sqrt{\dfrac{\mu}{p_T}} \left(1 + e^+ c_{\theta_x^+}\right) \end{cases}$$

$$= \frac{h^+}{r} = \frac{\sqrt{\mu p^+}}{r} = \frac{\sqrt{\mu p^+}}{R_k}$$

Moreover, $\quad (v_+)^2 = -\frac{\mu}{a^+} + \frac{2\mu}{r} = -\frac{\mu\left[1-(e^+)^2\right]}{p^+} + \frac{2\mu}{R_K}$

Hence, one obtains

$$(\Delta v_K)^2 = \frac{\mu}{R_K} + \frac{2\mu}{R_K} - \frac{\mu\left[1-(e^+)^2\right]}{p^+} - 2\sqrt{\frac{\mu}{R_K}}\,\frac{\sqrt{\mu p^+}}{R_K} \qquad \begin{pmatrix} p^+ = p_T \\ e^+ = e_T \end{pmatrix}$$

$$= \frac{3\mu}{R_K} - 2\frac{\mu}{R_K}\sqrt{\frac{p_T}{R_K}} - \frac{\mu}{p_T}\left(1-e_T^2\right)$$

and the partial derivative with respect to e_T is

$$\frac{\partial (\Delta v_K)^2}{\partial e_T} = \frac{2 e_T \mu}{p_T} > 0 \quad \Rightarrow \quad \Delta v_K \text{ increases as } e_T \text{ increases.}$$

But also the two boundaries

$$p_T = R_i\left(1+e_T\right) \quad \text{and} \quad p_T = R_f\left(1-e_T\right)$$

must be checked

(A) Along $p_T = R_i\left(1+e_T\right)$, after some algebra

$$(\Delta v_1)^2 = \frac{3\mu}{R_i} - 2\frac{\mu}{R_i}\sqrt{1+e_T} - \frac{\mu}{R_i}\left(1-e_T\right)$$

$$\frac{d(\Delta v_1)^2}{de_T} = \frac{\mu}{R_i}\,\frac{\sqrt{1+e_T}-1}{\sqrt{1+e_T}} > 0$$

$$(\Delta v_2)^2 = \frac{3\mu}{R_f} - \frac{2\mu}{R_f}\sqrt{\frac{R_i}{R_f}\left(1+e_T\right)} - \frac{\mu}{R_i}\left(1-e_T\right)$$

$$\frac{d(\Delta v_2)^2}{de_T} = \frac{\mu}{R_i R_f^{3/2}}\left[R_f^{3/2}\sqrt{1+e_T} - R_i^{3/2}\right] > 0$$

But $\quad \dfrac{d(\Delta v_i)^2}{de_T} = 2\,\Delta v_i\,\dfrac{d\,\Delta v_i}{de_T} > 0 \Rightarrow \dfrac{d\,\Delta v_i}{de_T} > 0$

(B) Along $p_T = R_f(1-e_T)$, after some algebra

$$(\Delta v_1)^2 = \frac{3\mu}{R_i} - \frac{2\mu}{R_i}\sqrt{\frac{R_f}{R_i}(1-e_T)} - \frac{\mu}{R_f}(1+e_T)$$

$$\frac{d(\Delta v_1)^2}{de_T} = \frac{\mu}{R_i^{3/2} R_f}\left[R_f^{3/2} - R_i^{3/2}\sqrt{1-e_T}\right] > 0$$

$$(\Delta v_2)^2 = \frac{3\mu}{R_f} - \frac{2\mu}{R_f}\sqrt{1-e_T} - \frac{\mu}{R_f}(1+e_T)$$

$$\frac{d(\Delta v_2)^2}{de_T} = \frac{\mu}{R_f\sqrt{1-e_T}}\left[1-\sqrt{1-e_T}\right] > 0$$

But $\dfrac{d(\Delta v_i)^2}{de_T} = 2\,\Delta v_i\,\dfrac{d\Delta v_i}{de_T} > 0 \quad \Rightarrow \quad \dfrac{d\Delta v_i}{de_T} > 0$

In short, along those two lines

$$\frac{d}{de_T}(\Delta v_1 + \Delta v_2) > 0$$

This condition, in addition to the previously found relation

$$\frac{\partial}{\partial e_T}(\Delta v_K)^2 > 0$$

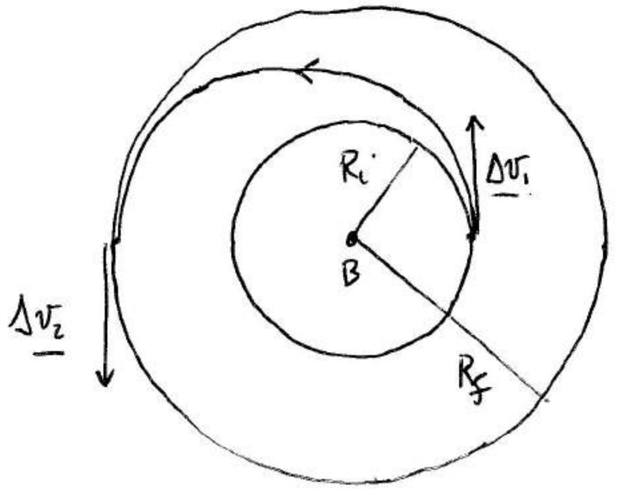
implies that H is associated with the GLOBALLY OPTIMAL two-impulse transfer between the initial and the final circular orbit, i.e. the HOHMANN TRANSFER.

The time of flight to perform this transfer is equal to half period of the elliptic transfer orbit, i.e.

$$\Delta t = \pi\sqrt{\frac{a_T^3}{\mu}}$$

- <u>Hohmann transfer</u>

In the previous section, the bitangent HOHMANN TRANSFER was proven to be the globally optimal circle-to-circle transfer, in the class of two-impulse transfers.
This means that it minimizes the overall $\Delta v = \Delta v_1 + \Delta v_2$ i.e. the overall propellant consumption



The transfer arc is elliptic with

$$a_T = \frac{R_i + R_f}{2}$$

$$e_T = \frac{R_f - R_i}{R_i + R_f}$$

Along this elliptic arc, the velocities at periapse and apoapse are

$$v_P = \sqrt{\frac{\mu}{a_T}\frac{1+e_T}{1-e_T}} = \sqrt{\frac{2\mu}{R_i + R_f}\frac{R_f}{R_i}}$$

$$v_A = \sqrt{\frac{\mu}{a_T}\frac{1-e_T}{1+e_T}} = \sqrt{\frac{2\mu}{R_i + R_f}\frac{R_i}{R_f}}$$

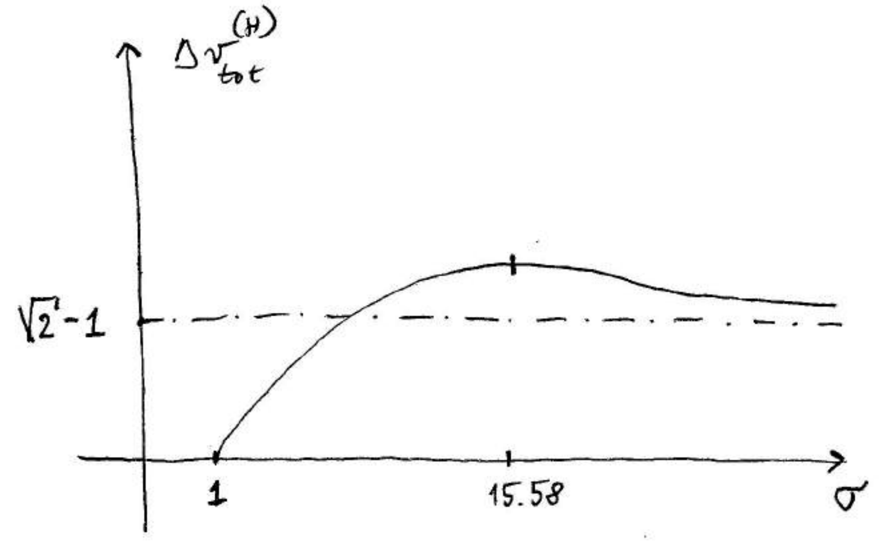
Therefore, the two velocity changes have magnitudes

$$\Delta v_1 = \sqrt{\frac{2\mu}{R_i + R_f}\frac{R_f}{R_i}} - \sqrt{\frac{\mu}{R_i}}$$

$$\Delta v_2 = \sqrt{\frac{\mu}{R_f}} - \sqrt{\frac{2\mu}{R_i + R_f}\frac{R_i}{R_f}}$$

Letting $\quad \sigma = \dfrac{R_f}{R_i} \quad$ one obtains

$$\frac{\Delta v_1}{\sqrt{\frac{\mu}{R_i}}} = \sqrt{\frac{2\sigma}{1+\sigma}} - 1 \qquad\qquad \frac{\Delta v_2}{\sqrt{\frac{\mu}{R_i}}} = \frac{1}{\sqrt{\sigma}} - \sqrt{\frac{2}{\sigma(\sigma+1)}}$$

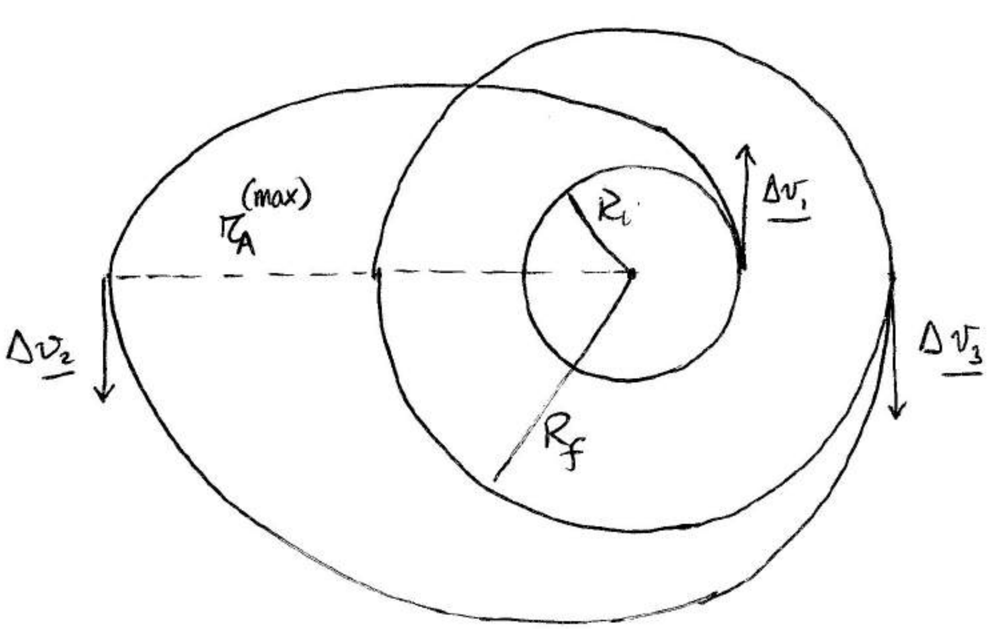


$$\Delta v_{tot}^{(H)} = \Delta v_1 + \Delta v_2$$

has maximum at $\sigma = 15.58$ and asymptote $\sqrt{2} - 1$. $\sigma \to \infty$ corresponds to the transfer toward parabolic path.

- <u>Bielliptic transfer</u>

A different, three-impulse transfer can be proven to outperform the Hohmann transfer, for certain radii of the terminal orbits, R_i and R_f



BIELLIPTIC TRANSFER, involving two elliptic intermediate arcs, with apoapse radius $r_A^{(max)}$

<u>Δv_1</u> and <u>Δv_2</u> in the direction of motion (they yield an orbit with a greater semimajor axis)

<u>Δv_3</u> in the direction opposite to the motion (it yields an orbit with a smaller semimajor axis)

Along the two elliptic arcs the velocities at apoapse and periapse are

T1
$$\begin{cases} v_P^{(T1)} = \sqrt{\dfrac{2\mu}{R_i + r_A^{(max)}} \dfrac{r_A^{(max)}}{R_i}} \\[2ex] v_A^{(T1)} = \sqrt{\dfrac{2\mu}{R_i + r_A^{(max)}} \dfrac{R_i}{r_A^{(max)}}} \end{cases} \qquad a_{T1} = \frac{R_i + r_A^{(max)}}{2}$$

T2
$$\begin{cases} v_P^{(T2)} = \sqrt{\dfrac{2\mu}{R_f + r_A^{(max)}} \dfrac{r_A^{(max)}}{R_f}} \\[2ex] v_A^{(T2)} = \sqrt{\dfrac{2\mu}{R_f + r_A^{(max)}} \dfrac{R_f}{r_A^{(max)}}} \end{cases} \qquad a_{T2} = \frac{R_f + r_A^{(max)}}{2}$$

Therefore, the three velocity changes are

$$\Delta v_1^{(BE)} = v_P^{(T1)} - \sqrt{\frac{\mu}{R_i}} = \sqrt{\frac{2\mu}{R_i + r_A^{(max)}} \frac{r_A^{(max)}}{R_i}} - \sqrt{\frac{\mu}{R_i}}$$

$$\Delta v_2^{(BE)} = v_A^{(T2)} - v_A^{(T1)} = \sqrt{\frac{2\mu}{R_f + r_A^{(max)}} \frac{R_f}{r_A^{(max)}}} - \sqrt{\frac{2\mu}{R_i + r_A^{(max)}} \frac{R_i}{r_A^{(max)}}}$$

$$\Delta v_3^{(BE)} = v_P^{(T2)} - \sqrt{\frac{\mu}{R_f}} = \sqrt{\frac{2\mu}{R_f + r_A^{(max)}} \frac{r_A^{(max)}}{R_f}} - \sqrt{\frac{\mu}{R_f}}$$

The bielliptic transfer requires a total transfer time

$$\Delta t = \underbrace{\pi \sqrt{\frac{a_{T1}^3}{\mu}}}_{\substack{\text{half-period} \\ \text{along ellipse 1}}} + \underbrace{\pi \sqrt{\frac{a_{T2}^3}{\mu}}}_{\substack{\text{half-period} \\ \text{along ellipse 2}}}$$

Concretely, as $r_A^{(max)}$ increases, Δv_{tot} reduces and Δt increases

In the limit as $r_A^{(max)} \longrightarrow \infty$, $\Delta t \to \infty$: this is the BIPARABOLIC TRANSFER (a limiting case), associated with

$$\Delta v_1^{(BP)} = (\sqrt{2} - 1)\sqrt{\frac{\mu}{R_i}}$$

$$\Delta v_2^{(BP)} = 0$$

$$\Delta v_3^{(BP)} = (\sqrt{2} - 1)\sqrt{\frac{\mu}{R_f}}$$

The biparabolic transfer can be proven to outperform any bielliptic transfer (for given values of R_i and R_f). The biparabolic transfer, however, is concretely infeasible (because $\Delta t \to \infty$) and can be regarded as a limiting case

Letting $\sigma = \frac{R_f}{R_i}$ and $\rho = \frac{r_A^{(max)}}{R_i} \geqslant \sigma$ one obtains

$$\frac{\Delta v_1^{(BE)}}{\sqrt{\frac{\mu}{R_i}}} = \sqrt{\frac{2\rho}{1+\rho}} - 1$$

$$\frac{\Delta v_2^{(BE)}}{\sqrt{\frac{\mu}{R_i}}} = \sqrt{\frac{2\sigma}{(\sigma+\rho)\rho}} - \sqrt{\frac{2}{\rho(\rho+1)}}$$

$$\frac{\Delta v_3^{(BE)}}{\sqrt{\frac{\mu}{R_i}}} = \sqrt{\frac{2\rho}{\sigma(\sigma+\rho)}} - \frac{1}{\sqrt{\sigma}}$$

while for the biparabolic transfer $(\rho \to \infty)$

$$\frac{\Delta v_1^{(BP)}}{\sqrt{\frac{\mu}{R_i}}} = \sqrt{2} - 1 \qquad \frac{\Delta v_2^{(BP)}}{\sqrt{\frac{\mu}{R_i}}} = 0 \qquad \frac{\Delta v_3^{(BP)}}{\sqrt{\frac{\mu}{R_i}}} = \frac{\sqrt{2} - 1}{\sqrt{\sigma}}$$

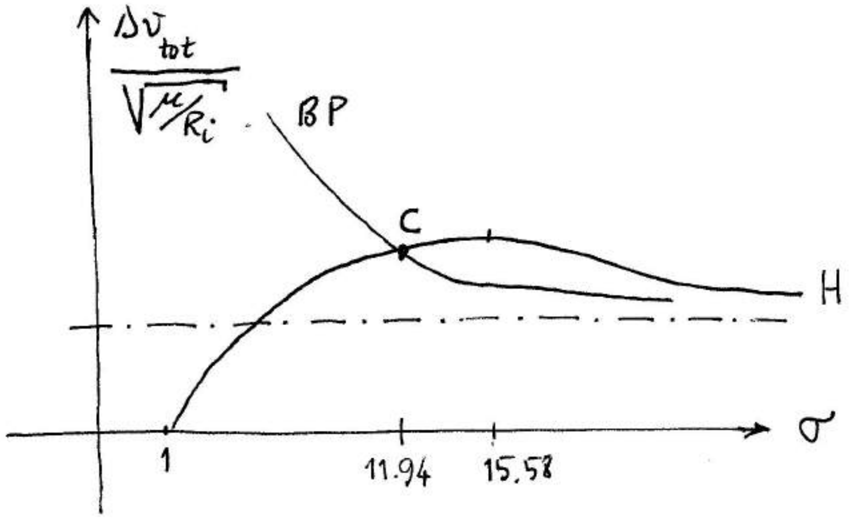
• <u>Hohmann transfer vs bielliptic transfer</u>

The two mentioned transfers can be compared, in order to ascertain which one corresponds to the smallest Δv_{tot}

The bielliptic transfer is defined in terms of two parameters i.e. $\sigma = \frac{R_f}{R_i}$ and $\rho = \frac{r_A^{(max)}}{R_i}$

For comparison of a bielliptic transfer with a Hohmann transfer the value of ρ is to be specified.

The most favorable bielliptic transfer is the biparabolic transfer ($\rho \to \infty$)



BIPARABOLIC (BP) vs HOHMANN (H)

Both H and BP have a horizontal asymptote

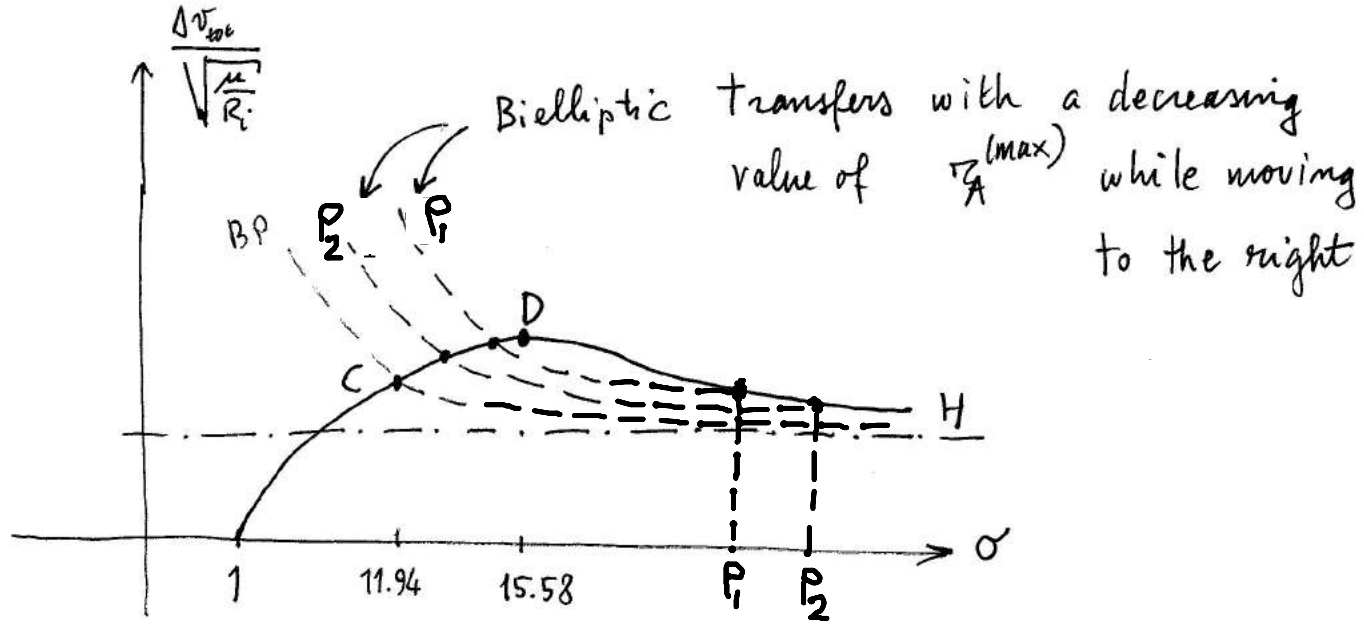
$$\frac{\Delta v_{tot}}{\sqrt{\mu/R_i}} \xrightarrow[\sigma \to \infty]{} \sqrt{2} - 1$$

From the figure

(a) $\sigma < 11.94 \longrightarrow$ Hohmann transfer is more convenient

(b) $\sigma > 11.94 \longrightarrow$ Biparabolic transfer is more convenient

However, if $r_A^{(max)}$ decreases (i.e. ρ decreases), then the intersection point C moves toward the value 15.58.



All these intersections are located in the interval $[11.94, 15.58]$ for σ. This circumstance implies that if $\sigma > 15.58$ the bielliptic transfer is unequivocally optimal.

In short, for a given ratio $\sigma = \frac{R_f}{R_i}$

(a) If $\sigma < 11.94 \longrightarrow$ HOHMANN TRANSFER is globally optimal

(b) If $11.94 < \sigma < 15.58 \longrightarrow$ HOHMANN or BIELLIPTIC transf. is globally optimal (depending on $\rho = \frac{r_A^{(max)}}{R_i} \geq \sigma$)

(c) If $\sigma > 15.58 \longrightarrow$ BIELLIPTIC TRANSFER is globally optimal (with $\rho \geq \sigma$)

• GLOBALLY OPTIMAL TRANSFER FROM ELLIPTIC ORBIT TO HYPERBOLIC PATH

– Planar transfers, with no constraint on the orientation of the terminal orbits, which are

(a) Initial ellipse, with specified r_A and r_P

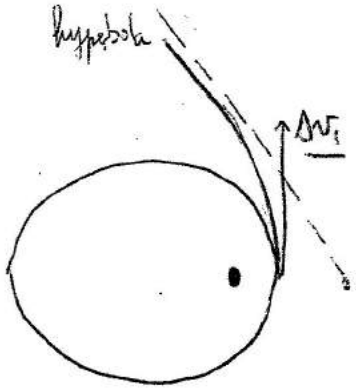
(b) Final hyperbola ($\mathcal{E}_f > 0$) or parabola ($\mathcal{E}_f = 0$), with specified $\mathcal{E}_f$, coplanar with the ellipse

– The max apoapse radius and min periapse radius of intermediate orbits are specified and denoted with r_A^{max} and r_P^{min}

→ 2 cases can occur

(1) $\mathcal{E}_f < \dfrac{\mu}{r_A^{max}} \Rightarrow$ glob opt transfer is

(a) impulse at periapse, to increase the energy to $\mathcal{E}_f$

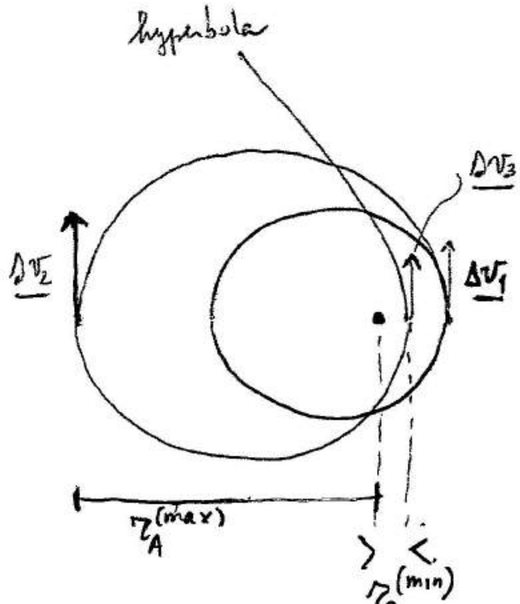


(2) $\mathcal{E}_f > \dfrac{\mu}{r_A^{max}}$

(a) impulse at periapse, to increase the apoapse radius to r_A^{max}

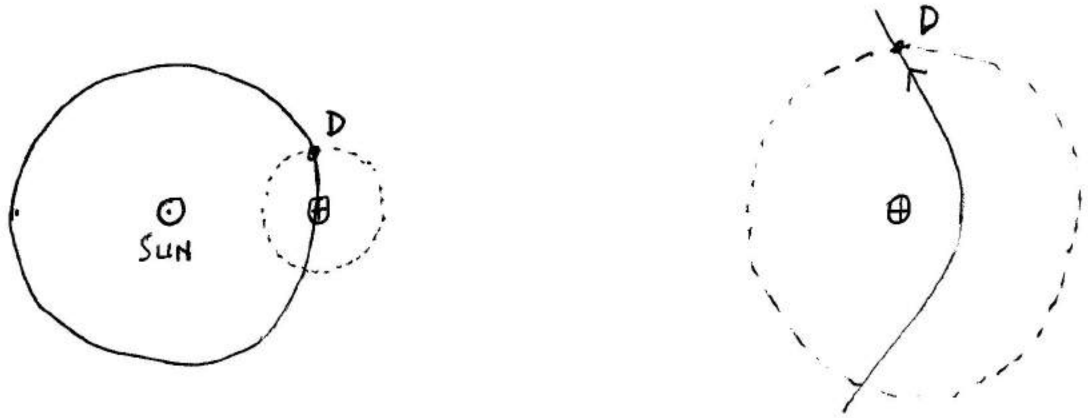
(b) impulse at apoapse, to decrease the periapse radius to r_P^{min}

(c) impulse at periapse, to increase the energy to $\mathcal{E}_f$



- Injection into hyperbolic path for interplanetary missions

If a spacecraft is injected into a hyperbolic path, then it leaves the main attracting body (e.g., the Earth). Its velocity at infinite distance is denoted with v_∞ and termed HYPERBOLIC EXCESS VELOCITY



Relative to Earth the trajectory is a hyperbola. However, at the sphere of influence (where Earth attraction "ceases", i.e. becomes less important than solar gravitational attraction), point D, the spacecraft has velocity v_∞ relative to the Earth. If this v_∞ has a proper direction, the spacecraft travels in a heliocentric elliptic trajectory toward the destination planet

As an example, consider a Hohmann-like transfer from Earth to Mars.

The heliocentric ellipse has the following semimajor axis and eccentricity:

$$a_T = \frac{R_e + R_m}{2} \qquad e_T = \frac{R_m - R_e}{R_m + R_e}$$

where R_e and R_m are the radii of Earth and Mars (circular) orbits around the Sun (the modest eccentricity of these two orbits is neglected)

The Hohmann transfer needs two velocity changes, i.e.

$$\Delta v_1^{(H)} = \sqrt{\frac{\mu_{SUN}}{a_T}\frac{1+e_T}{1-e_T}} - \sqrt{\frac{\mu_{SUN}}{R_e}}$$

$$\Delta v_2^{(H)} = \sqrt{\frac{\mu_{SUN}}{R_m}} - \sqrt{\frac{\mu_{SUN}}{a_T}\frac{1-e_T}{1+e_T}}$$

$\Delta v_1^{(H)}$ can be regarded as $v_\infty^{(E)}$, i.e. the hyperbolic excess velocity becomes a heliocentric additional velocity with respect to the orbital velocity of the Earth

Moreover $\Delta v_2^{(H)}$ can be regarded as the hyperbolic excess velocity $v_\infty^{(M)}$, while entering the Mars sphere of influence.

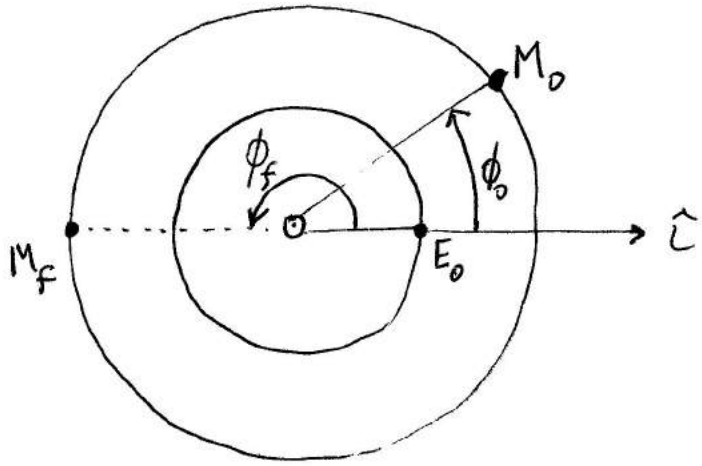
Because $\Delta v_1^{(H)}$ and $\Delta v_2^{(H)}$ can be calculated, then also $v_\infty^{(E)}$ and $v_\infty^{(M)}$ are found, and the velocity at periapse of an Earth orbit at departure toward Mars can be found through the equation of Energy

$$\mathcal{E}^{(E)} = \frac{\left[v_\infty^{(E)}\right]^2}{2} = \frac{\left[v_P^{(E)}\right]^2}{2} - \frac{\mu_E}{r_P^{(E)}} \longrightarrow v_P^{(E)}$$

In a similar way one can find the velocity at periapse of a Mars orbit

$$\mathcal{E}^{(M)} = \frac{\left[v_\infty^{(M)}\right]^2}{2} = \frac{\left[v_p^{(M)}\right]^2}{2} - \frac{\mu_M}{r_p^{(M)}} \longrightarrow v_p^{(M)}$$

An important aspect is orbit phasing at departure toward Mars



If the Earth is located as in the left figure (E_0), which is the correct phasing (angle ϕ_0) of Mars at t_0 such that planet encounter can occur?

After $\Delta t = \pi\sqrt{\frac{a_T^3}{\mu_{SUN}}}$ Mars must be located at M_f

i.e. $$\phi_f = \phi_0 + \sqrt{\frac{\mu_{SUN}}{R_m^3}}\,\Delta t = \pi$$

$$\rightarrow \quad \phi_0 = \pi - \pi\sqrt{\frac{a_T^3}{R_m^3}}$$

In this way, a Hohmann-like transfer can be used for an Earth-to-Mars mission.

THREE-DIMENSIONAL ORBIT TRANSFERS

Orbit transfers in three dimensions involve changes of velocity $\underline{\Delta v}$ which can lie on the orbital plane or out of it.

In-plane velocity changes

From the equation of energy,

$$\mathcal{E} = -\frac{\mu}{2a} = \frac{v^2}{2} - \frac{\mu}{r}$$

an in-plane change in velocity yields Δv and $\Delta r = 0$. To first order,

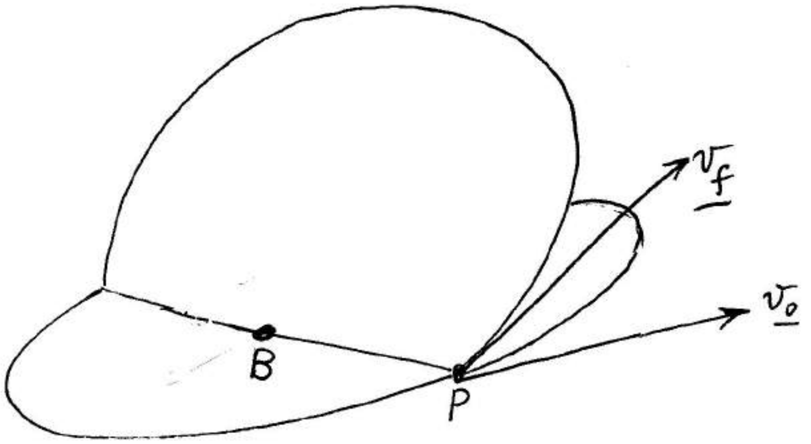
$$\frac{\mu}{2a^2}\Delta a = v\Delta v$$

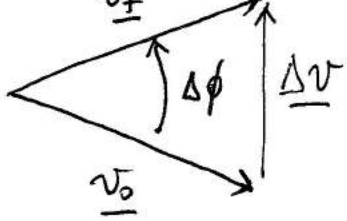
Therefore, if a given change Δa is requested, then

$$\Delta v = \frac{\mu}{2a^2}\frac{\Delta a}{v}$$

and it is apparent that Δv is minimized if the velocity change is applied where v is maximum, i.e. at periapse

Out-of-plane velocity changes





$$|\underline{v_0}| = |\underline{v_f}| = v$$

$$|\underline{\Delta v}| = 2v\sin\frac{\Delta\phi}{2}$$

The plane rotates about the line BP, where B is the center of the attracting body

It is apparent that out-of-plane velocity changes are more convenient where v is minimum, i.e. at apoapse. On the basis of these considerations, an example of three-dimensional transfer is considered.

- <u>Near-optimal LEO-GEO transfer</u>

Initial orbit is a Low Earth Orbit (LEO), circular (radius R_i) and with inclination $i_0 > 0$

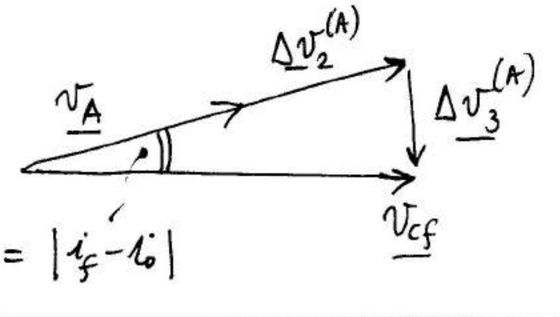
Final orbit is a Geosynchronous orbit, circular (radius R_f) and with inclination $i_f < i_0$

Two strategies are considered

(A) 3-impulse transfer:

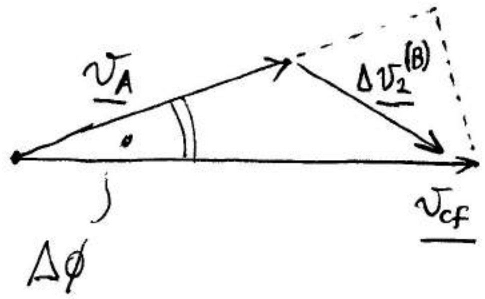
> first impulse to obtain $r_A = R_f$
> second impulse to obtain $r_P = R_f$, applied at apoapse
> third impulse to rotate the orbit, applied at the node

The last impulse must necessarily be applied at the (ascending or descending) node in order to rotate the orbit about the nodal line: in this way, the inclination is changed from i_0 to i_f, without altering Ω (right ascension of the ascending node)



$\underline{v}_A$ = velocity at apoapse (after first impulse)

(B) 2-impulse transfer : inspection of the previous figure suggests combining Δv_2 and Δv_3



$$\left|\Delta \underline{v}_2^{(B)}\right| < \left|\Delta \underline{v}_2^{(A)}\right| + \left|\Delta \underline{v}_3^{(A)}\right|$$

> first impulse at the node, to obtain $r_A = R_f$

> second impulse at the subsequent node, where apoapse is located, to do the combined maneuver shown in the last figure

Because the plane rotation occurs about the line of intersection of the two orbits, one must perform the second velocity change at the node (descending or ascending). This point is also the apoapse. This implies that the first maneuver must be done at the opposite node, so that the apoapse will be correctly located for the second velocity change.

First maneuver $\quad \Delta v_1^{(B)} = \sqrt{\dfrac{2\mu}{R_i+R_f}\dfrac{R_f}{R_i}} - \sqrt{\dfrac{\mu}{R_i}}$

Second maneuver $\quad \Delta v_2^{(B)} = \sqrt{v_A^2 + v_{cf}^2 - 2 v_A v_{cf} \cos\Delta\phi}$

where $\quad v_A = \sqrt{\dfrac{2\mu}{R_i+R_f}\dfrac{R_i}{R_f}} \quad$ and $\quad v_{cf} = \sqrt{\dfrac{\mu}{R_f}}$

Strategy (B) is not the "true" optimal (which comes from numerical optimization), but it is near-optimal (close to the true optimum)

CONCLUDING REMARK

All the orbit transfers considered so far involve an initial orbit A and a final orbit B

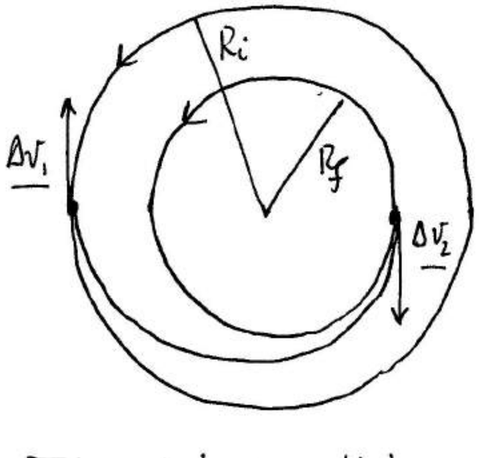
If the initial orbit is B and the final orbit is A the "returning" orbit transfer is performed through velocity changes that have

(i) same magnitude

(ii) opposite direction

If the transfer from A to B is optimal (i.e. it minimizes the overall Δv), then the returning transfer from B to A is optimal as well.

As an example, the interior Hohmann transfer starts from a circular orbit with radius R_i and ends at injection into the final orbit with radius R_f ($< R_i$)



$$\Delta v_1 = \sqrt{\frac{\mu}{R_i}} - \sqrt{\frac{\mu}{a_T}\frac{1-e_T}{1+e_T}}$$

$$\Delta v_2 = \sqrt{\frac{\mu}{a_T}\frac{1+e_T}{1-e_T}} - \sqrt{\frac{\mu}{R_f}}$$

where a_T and e_T are the semimajor axis and eccentricity of the elliptic arc.

The interior Hohmann transfer is the optimal two-impulse transfer between these two orbits.

Similar considerations hold for all the transfers investigated in these notes.