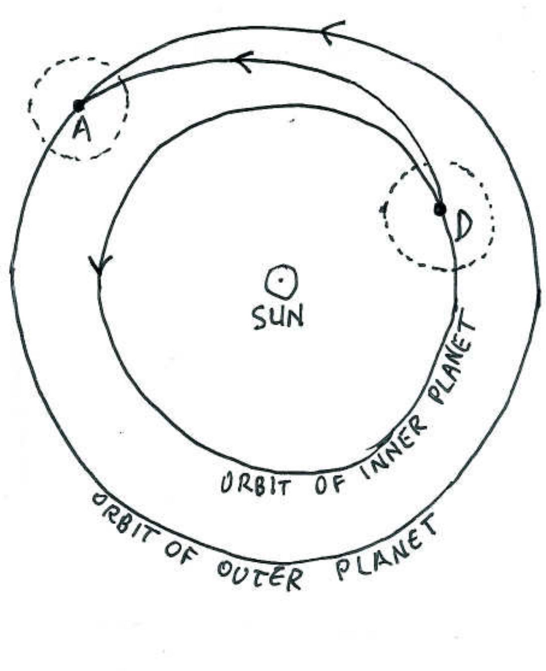
INTERPLANETARY TRAJECTORIES

INTRODUCTION

In this chapter interplanetary trajectories are investigated, using the PATCHED CONIC APPROXIMATION.

This is based on assuming that the spacecraft is subject to a single attracting body at a time. In other words, while a spacecraft is near the departure planet, it is affected only by its gravitational attraction. However, when it travels far apart from the departure planet, the gravitational attraction of the Sun becomes more and more important. In the patched-conics approximation the spacecraft is affected by the only gravitational acceleration of the Sun during the interplanetary (heliocentric) arc, from D to A in the figure.



At arrival at A, when the spacecraft is sufficiently close to the arrival planet only its gravitational attraction is taken into account.

In the end, in each trajectory arc only a single attracting body is taken into account, and the trajectories are KEPLERIAN (piecewise).

SPHERE OF INFLUENCE

In the previous discussion, a typical interplanetary mission was outlined. It is composed of three arcs:

(1) Departure from planet D : trajectory is Keplerian near D and is a HYPERBOLA

(2) Heliocentric transfer arc : trajectory is Keplerian and is an ELLIPSE (with Sun as the attracting body)

(3) Arrival at planet A: trajectory is Keplerian near A and is a HYPERBOLA

A question arises: which is the bound such that within it the spacecraft is subject to the attracting force of the planet (D or A) only?

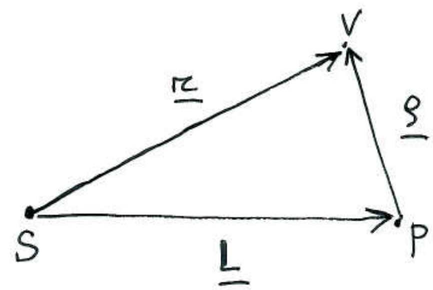
This bound is a surface termed SPHERE OF INFLUENCE. This is of course an approximating definition, because in all phases of spaceflight, a space vehicle is subject to the gravitational attraction of all bodies.

Nevertheless, the concept of SPHERE OF INFLUENCE helps understanding what is the limiting distance from a planet such that the planet gravitational attraction is dominating with respect to that of the Sun.

The formal derivation of a reasonable radius of the sphere of influence is given in the following.

As a first step, let $\underline{R_s}$, $\underline{R_p}$, $\underline{R_v}$ denote the position vector of the Sun, the Planet, and the space Vehicle (S, P, V). These three bodies, modeled as point masses, are subject to the universal law of gravitation and the 2nd Newton law

$$m_v \frac{d^2 \underline{R_v}}{dt^2} = -G \frac{m_v m_s}{r^3} \underline{r} - G \frac{m_v m_p}{\rho^3} \underline{\rho}$$



$$m_p \frac{d^2 \underline{R_p}}{dt^2} = + G \frac{m_v m_p}{\rho^3} \underline{\rho} - G \frac{m_p m_s}{L^3} \underline{L}$$

$$m_s \frac{d^2 \underline{R_s}}{dt^2} = G \frac{m_p m_s}{L^3} \underline{L} + G \frac{m_s m_v}{r^3} \underline{r}$$

$(\underline{R_s}, \underline{R_p}, \underline{R_v}$ not portrayed$)$

The vehicle motion relative to P is described by $\underline{\rho} = \underline{R_v} - \underline{R_p}$

$$(1) \quad \frac{d^2 \underline{\rho}}{dt^2} = -G \frac{m_p + m_v}{\rho^3} \underline{\rho} - G m_s \left[\frac{\underline{r}}{r^3} - \frac{\underline{L}}{L^3} \right]$$

The vehicle motion relative to S is described by $\underline{r} = \underline{R_v} - \underline{R_s}$

$$(2) \quad \frac{d^2 \underline{r}}{dt^2} = -G \frac{m_s + m_v}{r^3} \underline{r} - G m_p \left[\frac{\underline{\rho}}{\rho^3} + \frac{\underline{L}}{L^3} \right]$$

Because $m_s \gg m_p \gg m_v$ and $\left| \frac{\underline{\rho}}{\rho^3} \right| \gg \left| \frac{\underline{L}}{L^3} \right|$ Eqs. (1) and (2) simplify to

$$(1) \quad \frac{d^2 \underline{\rho}}{dt^2} = -G \frac{m_p}{\rho^3} \underline{\rho} - G m_s \left[\frac{\underline{r}}{r^3} - \frac{\underline{L}}{L^3} \right]$$

Motion relative to Planet

$$(2) \quad \frac{d^2 \underline{r}}{dt^2} = -G \frac{m_s}{r^3} \underline{r} - G m_p \frac{\underline{\rho}}{\rho^3}$$

Motion relative to Sun

In the previous expressions the first term represents the gravitational acceleration due to the dominating body, i.e.

for (1) the planet, within the sphere of influence

for (2) the sun, off the sphere of influence

Therefore, for each equation the remaining term is the perturbing acceleration due to the other body, i.e. :

(1) inside the sphere

$$-G\frac{m_p}{\rho^3}\underline{\rho} \quad \text{is due to the planet and dominates}$$

$$-Gm_s\left[\frac{\underline{r}}{r^3}-\frac{\underline{L}}{L^3}\right] \quad \text{is due to the Sun and is a perturbation}$$

(2) outside the sphere

$$-G\frac{m_s}{r^3}\underline{r} \quad \text{is due to the Sun and dominates}$$

$$-Gm_p\frac{\underline{\rho}}{\rho^3} \quad \text{is due to the planet and is a perturbation}$$

The sphere of influence is the surface where the ratios of perturbing and dominating terms (in (1) and (2)) equal, i.e.

$$\frac{m_s\left|\frac{\underline{r}}{r^3}-\frac{\underline{L}}{L^3}\right|}{m_p\left|\frac{\underline{\rho}}{\rho^3}\right|} = \frac{m_p\left|\frac{\underline{\rho}}{\rho^3}\right|}{m_s\left|\frac{\underline{r}}{r^3}\right|}$$

Ratio inside the sphere

Ratio outside the sphere

Moreover, in the left-hand-side, the following approximation is used:

$$\frac{\underline{r}}{r^3} - \frac{\underline{L}}{L^3} \simeq \frac{\underline{r} - \underline{L}}{L^3} = \frac{\underline{\rho}}{L^3} \quad \text{employing} \quad r^3 = \left[L^2+\rho^2+2\underline{L}\cdot\underline{\rho}\right]^{3/2} \simeq L^3$$

In the right-hand side $\underline{r} \simeq \underline{L}$, and these two approximations lead to

$$\frac{m_s \frac{\rho}{L^3}}{m_p \frac{1}{\rho^2}} = \frac{m_p \frac{1}{\rho^2}}{m_s \frac{1}{L^2}} \longrightarrow \left(\frac{\rho}{L}\right)^5 = \left(\frac{m_p}{m_s}\right)^2$$

$$\longrightarrow \quad \rho = L\left(\frac{m_p}{m_s}\right)^{\frac{2}{5}}$$ RADIUS of the SPHERE OF INFLUENCE

It is worth emphasizing again that this is not an exact quantity. Instead, it is an estimate of the distance beyond which the Sun gravitational attraction dominates that of a planet.

	ρ (km)
EARTH	$9.24 \cdot 10^5$
VENUS	$6.17 \cdot 10^5$
MERCURY	$1.13 \cdot 10^5$
MARS	$5.74 \cdot 10^5$
JUPITER	$4.83 \cdot 10^7$
NEPTUNE	$8.67 \cdot 10^7$

The radius depends on L, which is the planet distance from the Sun

This explains why Neptune has ρ greater than that of Jupiter, despite the larger mass of Jupiter

• METHOD OF PATCHED CONICS

This approximate approach allows designing interplanetary missions, from a departing planet D to an arrival planet A by splitting the overall trajectory in three arcs:

(1) Planetocentric hyperbola at departure from D

(2) Heliocentric ellipse

(3) Planetocentric hyperbola at arrival at A

In each arc only the dominating body is considered

Because interplanetary distances are vast, the spheres of influence of planets are considered like points in arc (2). Instead, in arcs (1) and (3) the spheres of influence are assumed to end at infinite distance because their radius is very large when compared to the planet radius.

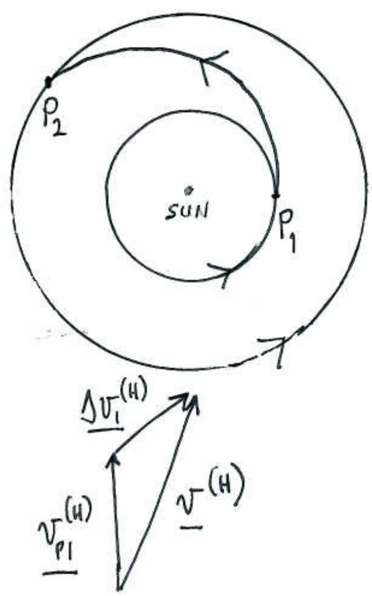
While this approach is sufficiently accurate for interplanetary missions in the Solar system, it is relatively inaccurate for Earth-Moon missions. In fact, the Moon turns out to have a sphere of influence of radius equal to 66200 km, which is non-negligible with respect to the Moon orbit semimajor axis (= 384400 km).

As a result, Earth-Moon missions are usually investigated using the dynamical framework of the restricted problem of three bodies.

- <u>Overview of the interplanetary arc</u>

In the interplanetary transfer arc (heliocentric phase) the spacecraft can travel

(a) toward an OUTER PLANET P_2



The transfer ellipse has semimajor axis greater than that of the circular orbit of the departing planet P_1

Therefore

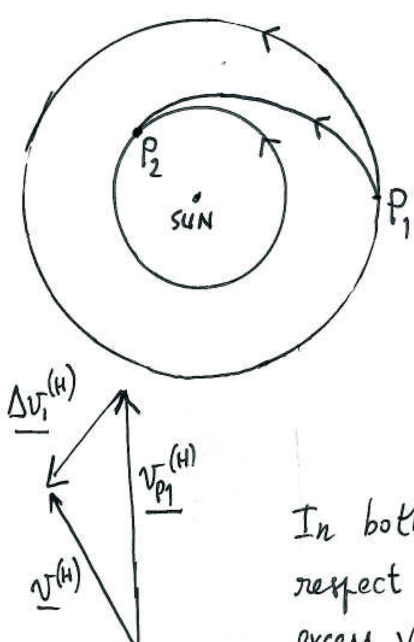
$$\left|\underline{v}^{(H)}\right| > \left|v_{P1}^{(H)}\right|$$

$\Rightarrow$ $\underline{\Delta v_1}^{(H)}$ must have positive component along $\underline{v}_{P1}^{(H)}$

where $\underline{v}^{(H)}$ = space vehicle heliocentric velocity

$\underline{v_{P1}}^{(H)}$ = heliocentric velocity of P_1

(b) toward an INNER PLANET



The transfer ellipse has semimajor axis smaller than that of the circular orbit of the departing planet P_1

Therefore, $\left|\underline{v}^{(H)}\right| < \left|\underline{v_{P1}}^{(H)}\right|$

$\Rightarrow$ $\underline{\Delta v_1}^{(H)}$ must have negative compon. along $\underline{v_{P1}}^{(H)}$

In both cases $\underline{\Delta v_1}^{(H)}$ is the vehicle velocity with respect to P_1, and can be regarded as the hyperbolic excess velocity along the planetocentric hyperbola.

- <u>Planetary departure</u>

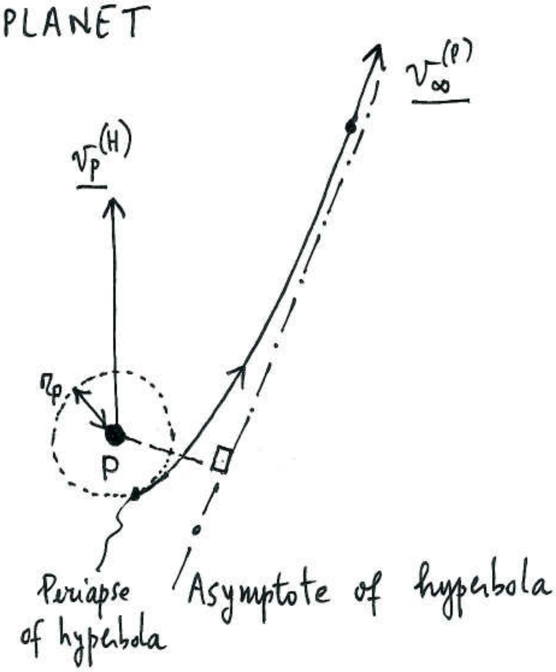
In order to inject into an interplanetary transfer arc, a spacecraft must have a relative velocity with respect to the departing planet while leaving its sphere of influence.

This velocity is the planetocentric hyperbolic excess velocity $\underline{v}_\infty^{(P)}$, which coincides with $\underline{\Delta v}_1^{(H)}$: $\underline{\Delta v}_1^{(H)} \equiv \underline{v}_\infty^{(P)}$

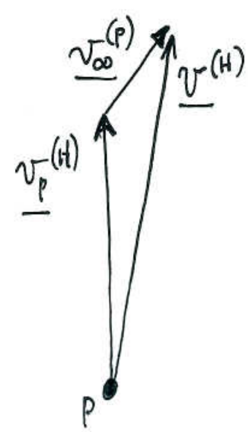
Departure can occur toward an inner or an outer planet

(A) DEPARTURE TOWARD AN OUTER PLANET

At infinite distance (i.e. at the sphere of influence) the planetocentric velocity is $\underline{v}_\infty^{(P)}$ and is parallel to the asymptote



$\underline{v}_P^{(H)}$ = orbital velocity of planet P around the Sun



$$\underline{v}^{(H)} = \underline{v}_P^{(H)} + \underline{v}_\infty^{(P)}$$

$\underline{v}^{(H)}$ = orbital velocity of spacecraft in the heliocentric frame (around the Sun)

r_p = periapse radius of the planetocentric hyperbola

Due to conservation of energy

$$\mathcal{E} = \frac{\left[v_\infty^{(P)}\right]^2}{2} = \frac{v_p^2}{2} - \frac{\mu_p}{r_p} = -\frac{\mu}{2a^{(P)}}$$

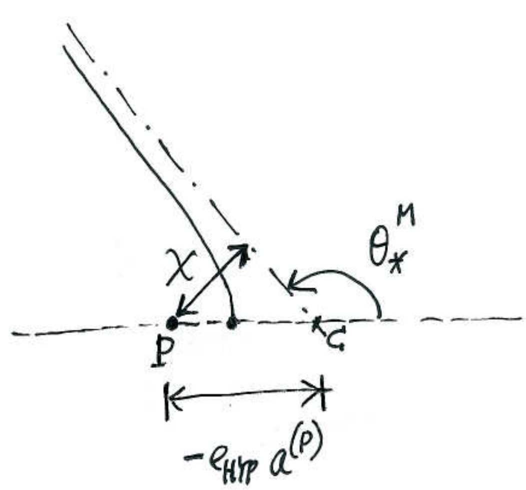
Thus, if $v_\infty^{(P)}$ (or $a^{(P)}$) and r_p are given, the planetocentric periapse velocity is

$$v_p = \sqrt{\frac{2\mu_p}{r_p} + \left[v_\infty^{(P)}\right]^2}$$ where μ_p = gravitational parameter of the planet

whereas the hyperbola eccentricity is

$$e_{HYP} = 1 - \frac{r_p}{a^{(P)}} = 1 + \frac{r_p}{\mu_p}\left[v_\infty^{(P)}\right]^2$$

The distance between the center of the planet and the outgoing asymptote is denoted with ϱ and is given by



$$\chi = -e_{HYP}\, a^{(P)} \sin\left(\pi - \theta_*^M\right) =$$

$$= -e_{HYP}\, a^{(P)} \sin\theta_*^M =$$

$$= -e_{HYP}\, a^{(P)} \sqrt{1 - \frac{1}{e_{HYP}^2}} =$$

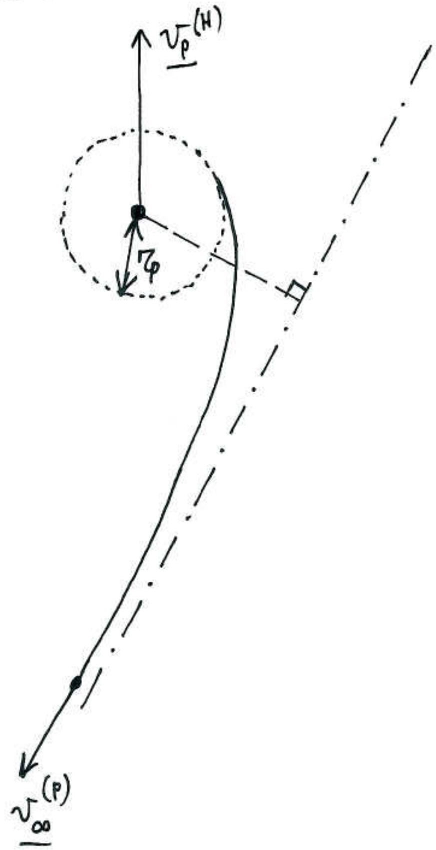
$$= -a^{(P)}\sqrt{e_{HYP}^2 - 1}$$

$$\chi = -e_{HYP}\, a^{(P)} \sin\left(\pi - \theta_*^M\right)$$

After leaving the planet sphere of influence, the spacecraft travels with a heliocentric velocity $\underline{v}^{(H)} = \underline{v}_P^{(H)} + \underline{v}_\infty^{(P)}$

(B) DEPARTURE TOWARD AN INNER PLANET

In this case the hyperbolic excess velocity relative to the planet has direction with negative projection (component) along $\underline{v}_P^{(H)}$

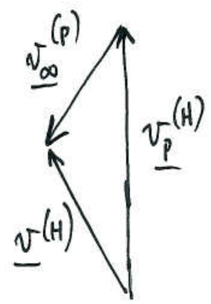


$\underline{v}_\infty^{(P)}$ again parallel to asymptote

$\underline{v}_P^{(H)}$ = orbital velocity of planet P around the Sun

$\underline{v}^{(H)}$ = orbital velocity of spacecraft in the heliocentric frame

r_p = periapse radius of the planetocentric hyperbola



$$\underline{v}^{(H)} = \underline{v}_P^{(H)} + \underline{v}_\infty^{(P)}$$

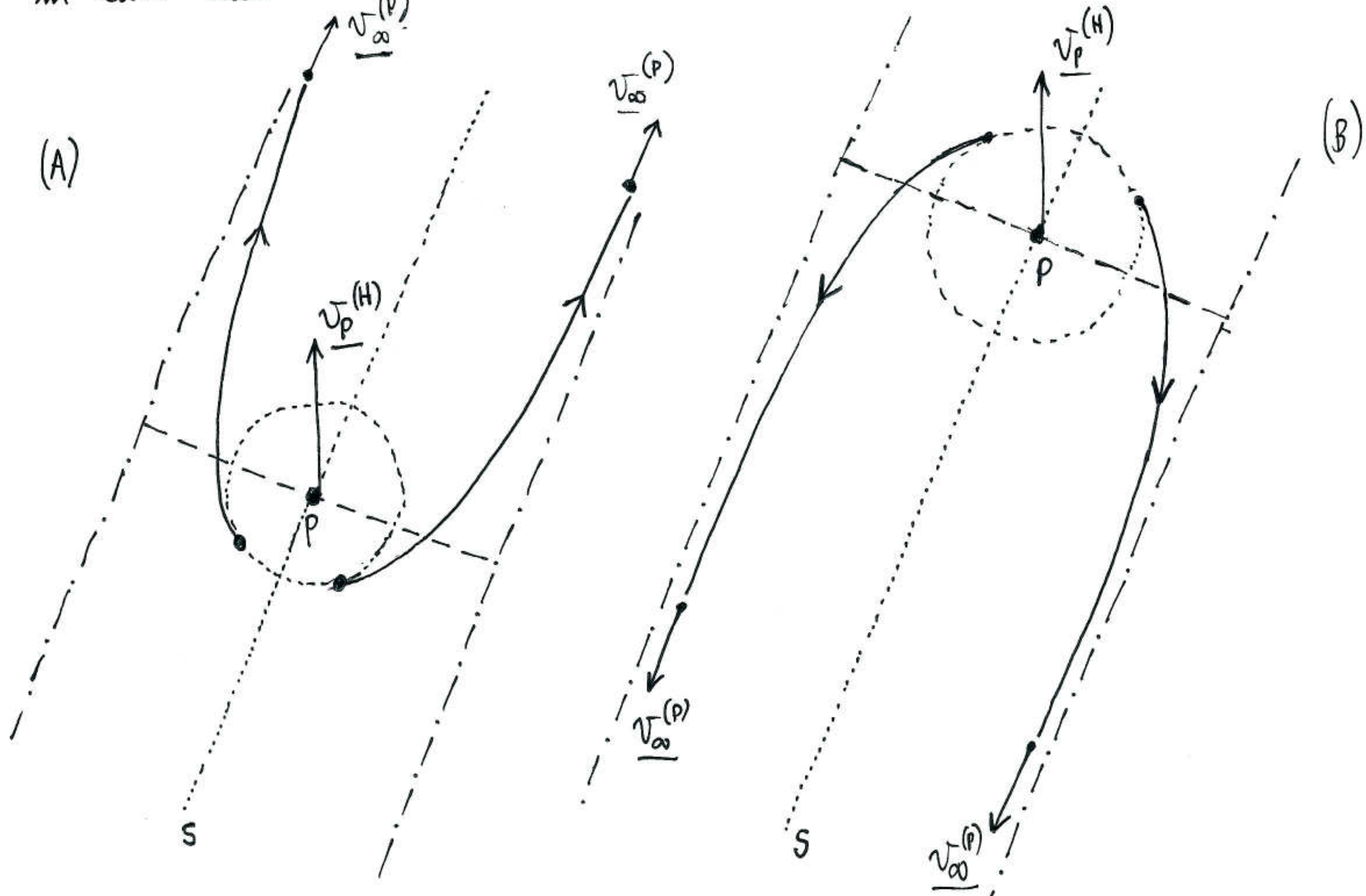
Also in this case, conservation of energy can be employed in order to find v_p once $a^{(P)}$ (or $v_\infty^{(P)}$) and r_p are known

Moreover, the following (already found) relations hold:

$$e_{HYP} = 1 + \frac{r_p}{\mu_P}\left[v_\infty^{(P)}\right]^2 \qquad \chi = -a^{(P)}\sqrt{e_{HYP}^2 - 1}$$

After leaving the planet sphere of influence, the spacecraft travels with a heliocentric velocity $\underline{v}^{(H)} = \underline{v}_P^{(H)} + \underline{v}_\infty^{(P)}$

While exiting the sphere of influence, two options exist in each case



(A) Departure toward an OUTER planet (B) Departure toward an INNER planet

At the sphere of influence these two options (for each case (A) or (B)) are indistinguishable, and associated with identical $\{e_{HYP}, r_P, a^{(P)}\}$

These considerations assume hyperbolas lying on the orbit plane of the planet.

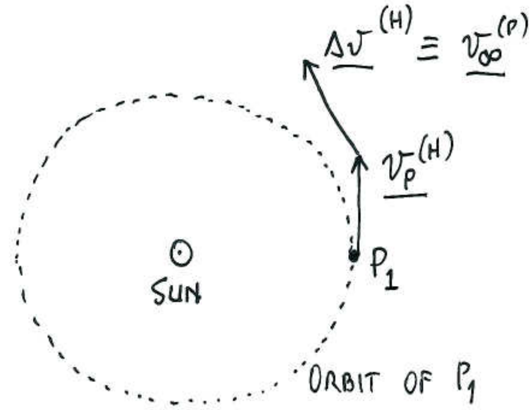
In three dimensions an infinite number of hyperbolas exist, belonging to the cylindrical surface with axis s (see figure).

The two hyperbolas shown in figure are the intersections of this cylindrical surface with the orbit plane of planet P.

- **Interplanetary transfer arc**

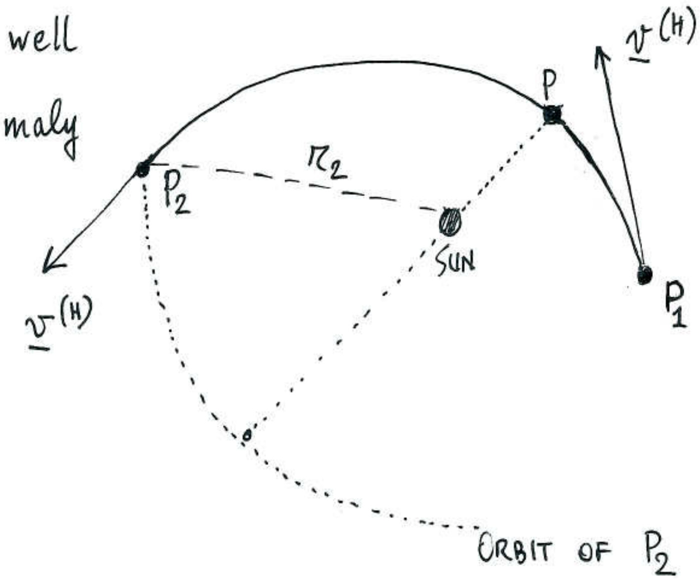
The heliocentric transfer arc is an elliptic arc, with the Sun at the focus.

The hyperbolic excess velocity at planetary departure becomes a heliocentric Δv with respect to the planet orbital velocity



$$\underline{v_\infty}^{(P)} \equiv \underline{\Delta v}^{(H)}$$

Once $\underline{v}^{(H)}$ at P_1 is Known, the heliocentric orbit elements $a^{(H)}$ and $e^{(H)}$ can be found, as well as θ_{*1}, true anomaly at P_1.



At arrival at P_2, the true anomaly θ_{*2} can be found as well, because

$$r_2 = \frac{a^{(H)}\left[1-\left(e^{(H)}\right)^2\right]}{1+e^{(H)}\cos\theta_{*2}}$$

and $0 < \theta_{*2} \leq \pi$

at arrival at an outer planet

P_1 = departure planet

P_2 = arrival planet

P = periapse of heliocentric transfer arc $P_1 P_2$

If the transfer is toward an inner planet, then the latter is reached before arriving at periapse of the transfer elliptic arc.

This circumstance implies that

$$-\pi < \theta_{*2} \leq 0 \quad \text{at arrival at an inner planet}$$

In summary, once the spacecraft heliocentric position and velocity are known at P_1

(a) $a^{(H)}$, $e^{(H)}$ are found, as well as θ_{*1}

(b) θ_{*2} is found from the polar equation of ellipses, and between the two solutions the following one is chosen:

$$\begin{cases} 0 < \theta_{*2} \leq \pi & \text{Transfer toward an outer planet} \\ -\pi < \theta_{*2} \leq 0 & \text{transfer toward an inner planet} \end{cases}$$

If $a^{(H)}$, $e^{(H)}$, and θ_{*2} are known, then $\underline{v}^{(H)}$ at P_2 can be found, and has two components

$$\underline{v}^{(H)} = v_r^{(H)} \hat{r} + v_\theta^{(H)} \hat{\theta}$$

Planet P_2 has its own orbital velocity around the Sun, $\underline{v}_{P2}^{(H)}$

$$\underline{v}_{P2}^{(H)} = v_{P2,r}^{(H)} \hat{r} + v_{P2,\theta}^{(H)} \hat{\theta}$$

The spacecraft velocity relative to P_2 at arrival is

$$\underline{v}_\infty^{(P2)} = \underline{v}^{(H)} - \underline{v}_{P2}^{(H)} = \left[v_r^{(H)} - v_{P2,r}^{(H)} \right] \hat{r} + \left[v_\theta^{(H)} - v_{P2,\theta}^{(H)} \right] \hat{\theta}$$

- <u>Planetary arrival</u>

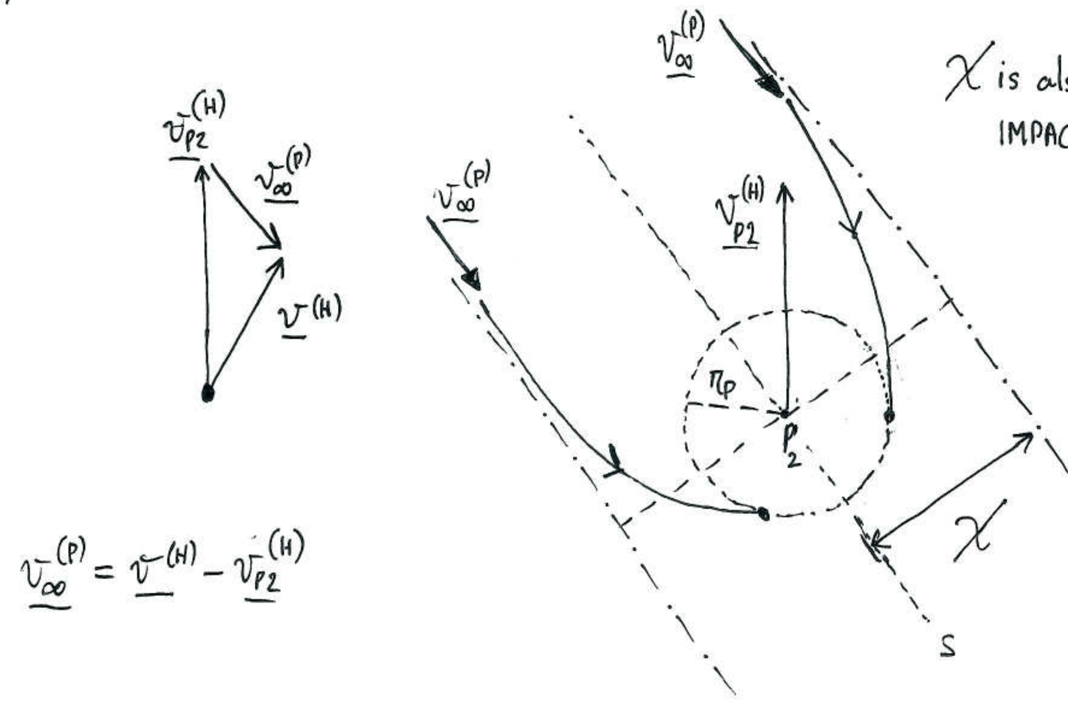
At arrival at the target planet P_2, the spacecraft has heliocentric velocity $\underline{v}^{(H)}$ and the planet has heliocentric velocity $\underline{v}_{P2}^{(H)}$.

Therefore, the spacecraft velocity relative to P_2 is

$$\underline{v}^{(H)} - \underline{v}_{P2}^{(H)} \equiv \underline{v}_{\infty}^{(P)}$$

and is now regarded as the hyperbolic excess velocity of the incoming hyperbola relative to the planet.

(A) ARRIVAL AT AN OUTER PLANET



χ is also termed IMPACT PARAMETER

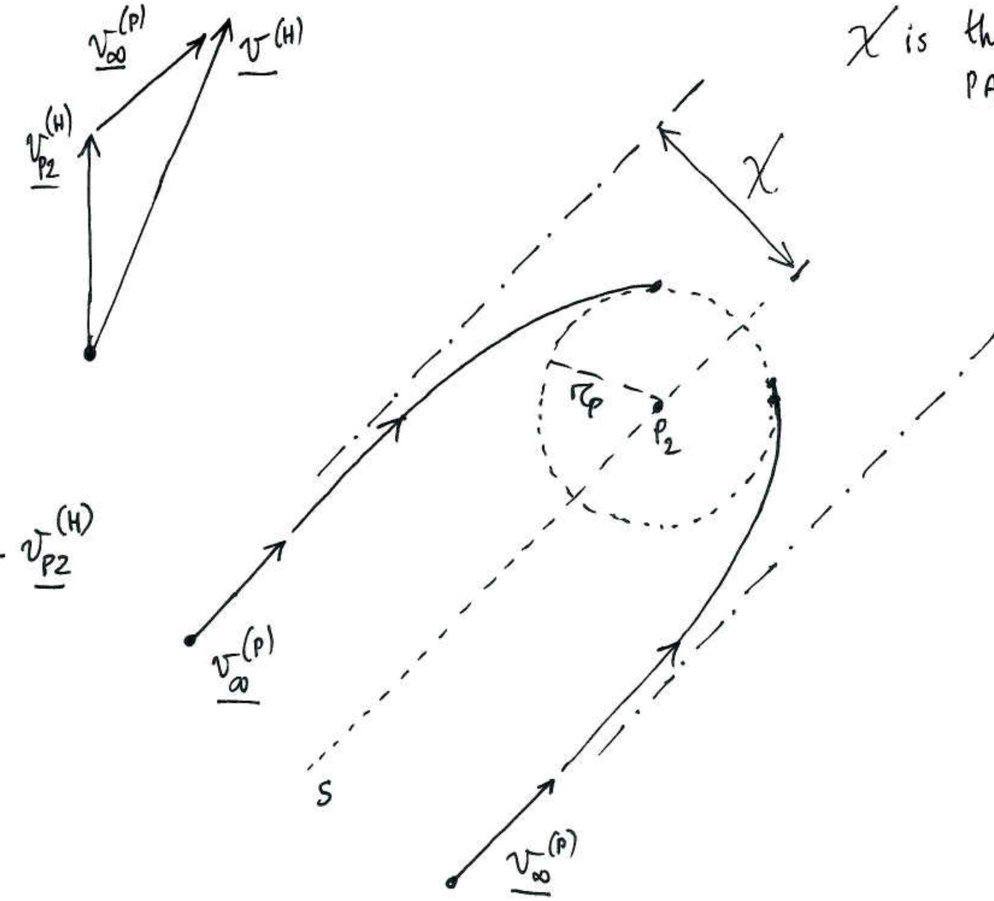
If $v_\infty^{(P)}$ (or $a^{(P)}$) and r_p are given, then the planetocentric periapse velocity is

$$v_p = \sqrt{\frac{2\mu_{P2}}{r_p} + \left[v_\infty^{(P)}\right]^2}$$

while

$$\begin{cases} e_{HYP} = 1 + \dfrac{r_p}{\mu_{P2}}\left[v_\infty^{(P)}\right]^2 \\ \chi = -\,a^{(P)}\sqrt{e_{HYP}^2 - 1} \end{cases}$$

(B) ARRIVAL AT AN INNER PLANET



χ is the IMPACT PARAMETER

$$\underline{v}_\infty^{(P)} = \underline{v}^{(H)} - \underline{v}_{P2}^{(H)}$$

Again, if $v_\infty^{(P)}$ (or $a^{(P)}$) and r_p are given, then the planetocentric periapse velocity is

$$v_p = \sqrt{\frac{2\mu_{P2}}{r_p} + \left[v_\infty^{(P)}\right]^2} \quad \text{while} \quad \begin{cases} e_{HYP} = 1 + \dfrac{r_p}{\mu_{P2}}\left[v_\infty^{(P)}\right]^2 \\ \chi = -a^{(P)}\sqrt{e_{HYP}^2 - 1} \end{cases}$$

Similarly to what occurs at departure from a planet, the two options (for each case (A) and (B)) are the two hyperbolas belonging to the orbit plane of the planet.

In three dimensions, an infinite number of hyperbolas exist, belonging to the cylindrical surface with axis s (see figures).

- Overall interplanetary trajectory design

For a mission departing from planet P_1 and arriving at planet P_2, at least two velocity changes are needed:

(a) At periapse of the outgoing hyperbola around P_1

(b) At periapse of the incoming hyperbola around P_2

The optimal, alternative 3-impulse sequence with a maximum apoapse radius of intermediate arcs is here neglected, for the sake of simplicity.

The two previous impulses (a) and (b) have magnitude

$$\Delta v_1^{(P)} = \left| \underline{v}_P^{(P1)} - \underline{v}_i^{(P1)} \right| \qquad \text{and} \qquad \Delta v_2^{(P)} = \left| \underline{v}_P^{(P2)} - \underline{v}_f^{(P2)} \right|$$

where $\underline{v}_P^{(P1)}$ = periapse velocity along the outgoing hyperbola at P_1

$\underline{v}_i^{(P1)}$ = velocity right before the first impulse

$\underline{v}_P^{(P2)}$ = periapse velocity along the incoming hyperbola at P_2

$\underline{v}_f^{(P2)}$ = velocity after the second impulse.

It is apparent that the first step of mission design consists in defining the interplanetary transfer arc, which can be a HOHMANN-LIKE transfer (or any alternative transfer)

Once $\underline{\Delta v_1}^{(H)}$ and $\underline{\Delta v_2}^{(H)}$ are known, the respective planetocentric velocities are immediately found, because

$$\underline{\Delta v_1}^{(H)} \equiv \underline{v_\infty}^{(P1)} \qquad \text{and} \qquad \underline{\Delta v_2}^{(H)} \equiv \underline{v_\infty}^{(P2)}$$

Moreover, the periapse radius is usually specified, both at departure and at arrival.

If r_p is specified, and $\underline{v}_\infty^{(P)}$ is found at the previous steps, the semimajor axis $a^{(P)}$ and eccentricity e_{HYP} of the planetocentric hyperbolas are found through the following relations

$$a^{(P)} = -\frac{\mu_P}{\left[v_\infty^{(P)}\right]^2} \qquad \text{and} \qquad e_{HYP} = 1 + \frac{r_P}{\mu_P}\left[v_\infty^{(P)}\right]^2$$

The last step consists in choosing the departing hyperbola and the arrival hyperbola. If these hyperbolas are assumed to belong to the plane of the orbit of the planet, then two options exist. In three dimensions, an infinite number of outgoing or incoming hyperbolas are available, and the choice depends on some mission specification.

In summary, the following three steps are needed for preliminary interplanetary trajectory design:

(1) Define INTERPLANETARY ARC and obtain $\underline{\Delta v_1}^{(H)}$ and $\underline{\Delta v_2}^{(H)}$

(2) Define DEPARTING HYPERBOLA on the basis of

$\underline{v_\infty}^{(P1)} \equiv \underline{\Delta v_1}^{(H)}$ and r_p (periapse radius at departure)

(3) Define ARRIVAL HYPERBOLA on the basis of

$\underline{v_\infty}^{(P2)} \equiv \underline{\Delta v_2}^{(H)}$ and r_p (periapse radius at arrival)

PLANETARY ENCOUNTER OPPORTUNITIES

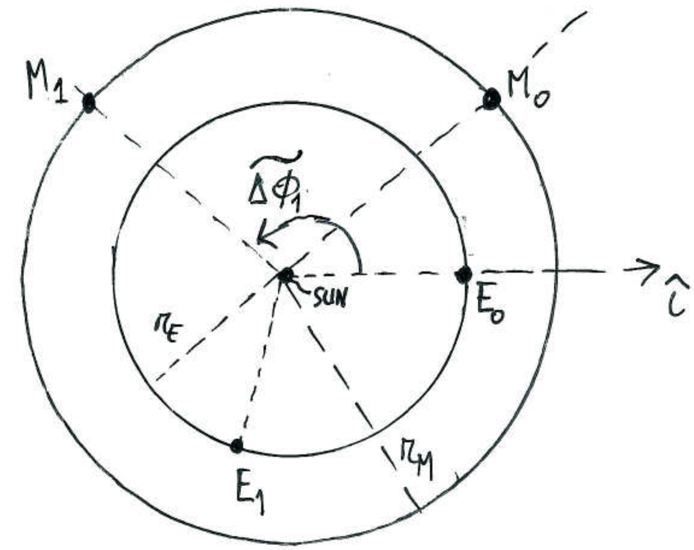
In order that planetary encounter occurs at the end of the heliocentric arc, suitable timing must be selected.

(A) This means that once the outgoing interplanetary transfer time is found, Δt_1, launch must occur at an appropriate time, such that the arrival planet is in the correct position at the arrival time

Let t_0 the time at which the interplanetary arc starts:

Earth is at E_0 at t_0

Mars is at M_0 at t_0



The Earth angle from $\hat{\imath}$ is

$$\phi_E = \phi_{E,0} + n_E(t-t_0)$$

($\hat{\imath}$ chosen such that $\phi_{E,0} = 0$)

The Mars angle from $\hat{\imath}$ is

$$\phi_M = \phi_{M,0} + n_M(t-t_0)$$

where $n_E = \sqrt{\dfrac{\mu_S}{r_E^3}}$ is the Earth orbit angular rate

$n_M = \sqrt{\dfrac{\mu_S}{r_M^3}}$ is the Mars orbit angular rate

At t_1, the spacecraft is located at $\Delta\tilde{\phi}_1$, and Mars must be located at the same point in order that encounter can occur. Therefore

$$\Delta\tilde{\phi}_1 = \phi_{M,0} + n_M \Delta t_1 \quad \longrightarrow \quad \phi_{M,0} = \Delta\tilde{\phi}_1 - n_M \Delta t_1$$

As an example, if the interplanetary transfer arc is a Hohmann ellipse, then

$$\Delta\tilde{\phi}_1 = \pi \quad \text{and} \quad \Delta t_1 = \pi\sqrt{\frac{a_T^3}{\mu_S}} \quad , \quad a_T = \frac{r_M + r_E}{2}$$

and $\phi_{M,0} = 44.6$ deg

The Earth position at t_1 is E_1 and can be found through the previous relation,

$$\phi_{E,1} = \phi_{E,0} + n_E \Delta t_1$$

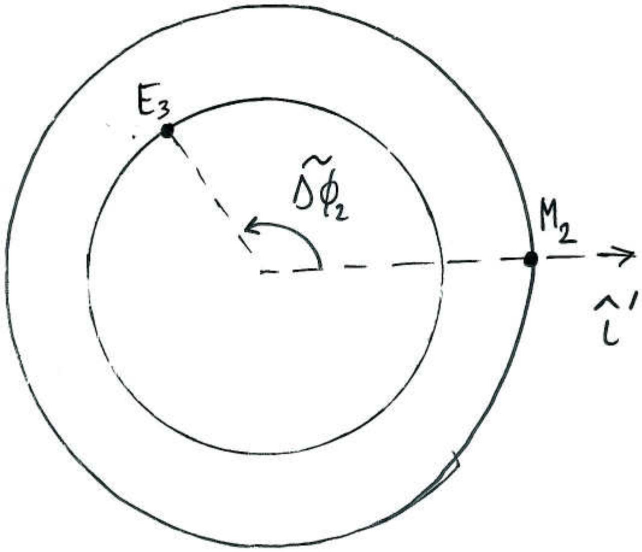
(B) The returning trip is assumed to take the time Δt_2, and correct phasing between the two planets (Earth and Mars) is again needed. Let t_2 be the departure time from Mars

$$\phi_{E,2} = \phi_{E,1} + n_E (t_2 - t_1)$$

$$\phi_{M,2} = \phi_{M,1} + n_M (t_2 - t_1)$$

$t_W = t_2 - t_1$ is the waiting time to spend at Mars

From inspection of this figure



$$\phi_{E,3} - \phi_{M,2} = \widetilde{\Delta\phi_2}$$

where $\widetilde{\Delta\phi_2}$ is known once the heliocentric transfer arc is specified

$\hat{i}'$ such that $\phi_{M,2} = 0$

(displaced by $\hat{i}$)

Because

$$\begin{cases} \phi_{E,3} = \phi_{E,1} + n_E t_W + n_E \Delta t_2 \\ \phi_{M,2} = \phi_{M,1} + n_M t_W \end{cases}$$

one obtains

$$\phi_{E,1} + n_E t_W + n_E \Delta t_2 - \phi_{M,1} - n_M t_W = \Delta\phi_2$$ leading to

$$t_W = \frac{\widetilde{\Delta\phi_2} + \phi_{M,1} - \phi_{E,1} - n_E \Delta t_2}{n_E - n_M}$$

In general, an infinite number of launch opportunities exist; in fact the relative positions of Earth and Mars repeat every synodic period Δt_{SYN}

Therefore, also the wait times are infinite

$$t_W = \frac{\widetilde{\Delta\phi_2} + \phi_{M,1} - \phi_{E,1} - n_E \Delta t_2}{n_E - n_M} + k \, \Delta t_{SYN} \qquad (k \text{ integer})$$

Δt_{SYN} is being derived in the following

- Synodic period

It is the time interval after which the relative configurations of two planets repeat

Therefore, let $\phi_{E,0}$ and $\phi_{M,0}$ denote the two initial displacements of Earth and Mars from $\hat{\imath}$. After a synodic period Δt_{SYN},

$$\phi_E - \phi_M = \left(\phi_{E,0} + n_E \Delta t_{SYN}\right) - \left(\phi_{M,0} + n_M \Delta t_{SYN}\right) = \phi_{E,0} - \phi_{M,0} \pm 2\pi$$

+ is selected for Earth and Mars because $n_E > n_M$

− is selected if Venus replaces Mars because $n_E < n_V$

(n_V is Venus orbital angular rate)

Therefore, in general

$$\Delta t_{SYN} = \frac{2\pi}{|n_E - n_M|} \quad \text{for Earth-Mars relative positions}$$

$$\Delta t_{SYN} = \frac{2\pi}{|n_E - n_V|} \quad \text{for Earth-Venus relative positions}$$

Of course, everything can be extended to any pair of planets in circular orbits about the same star.

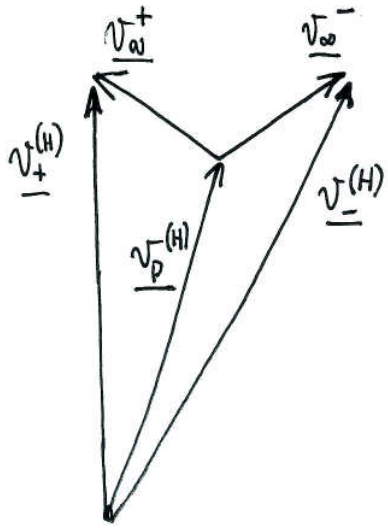
At previous page, the minimum wait time corresponds to choosing the integer k such that the expression

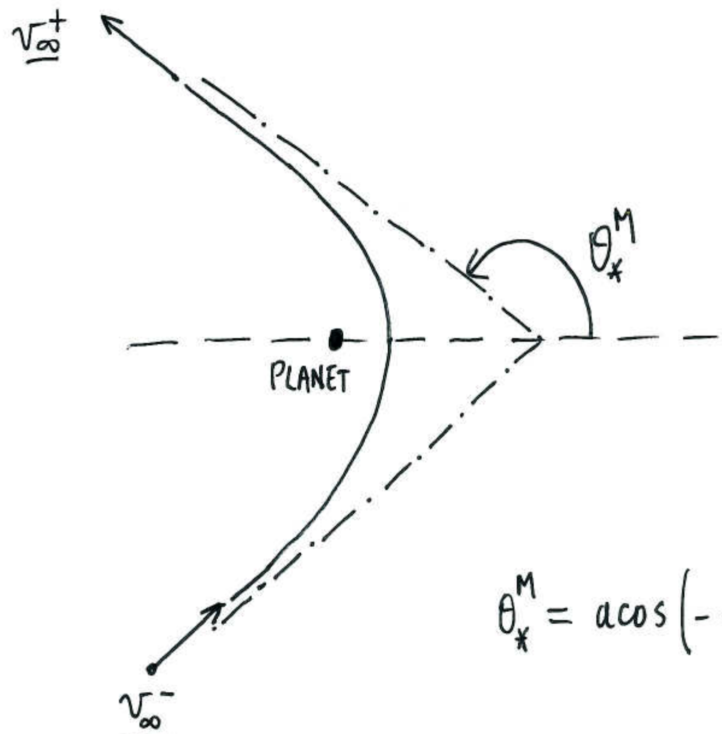
$t_W = \dfrac{\tilde{\Delta\phi}_2 + \phi_{M,1} - \phi_{E,1} - n_E \Delta t_2}{n_E - n_M} + k\,\Delta t_{SYN}$ yields the least value

PLANETARY FLYBY

If a spacecraft enters the sphere of influence of a planet and no maneuver is performed, then it travels along the hyperbola up to reaching again the sphere of influence while going far away from the planet.

This is termed planetary flyby





$$\theta_*^M = a\cos\left(-\frac{1}{e_{HYP}}\right)$$

$$\underline{v}_{\pm}^{(H)} = \underline{v}_P^{(H)} + \underline{v}_{\infty}^{\pm}$$

In the previous figures

$\underline{v}_P^{(H)}$ = heliocentric velocity of planet P

$\underline{v}_+^{(H)}$ = spacecraft heliocentric velocity after the flyby

$\underline{v}_-^{(H)}$ = spacecraft heliocentric velocity before the flyby

From inspection of the left figure $\quad \left|\underline{v}_+^{(H)}\right| < \left|\underline{v}_-^{(H)}\right|$

Then, planetary flyby has the effect of

(i) changing the magnitude of the spacecraft heliocentric velocity

(ii) rotating the direction of the spacecraft heliocentric velocity

This means that a spacecraft can emerge from the planet sphere of influence with a velocity changed in magnitude. In the heliocentric Keplerian path this means that

$$\mathcal{E}_+^{(H)} \neq \mathcal{E}_-^{(H)}$$

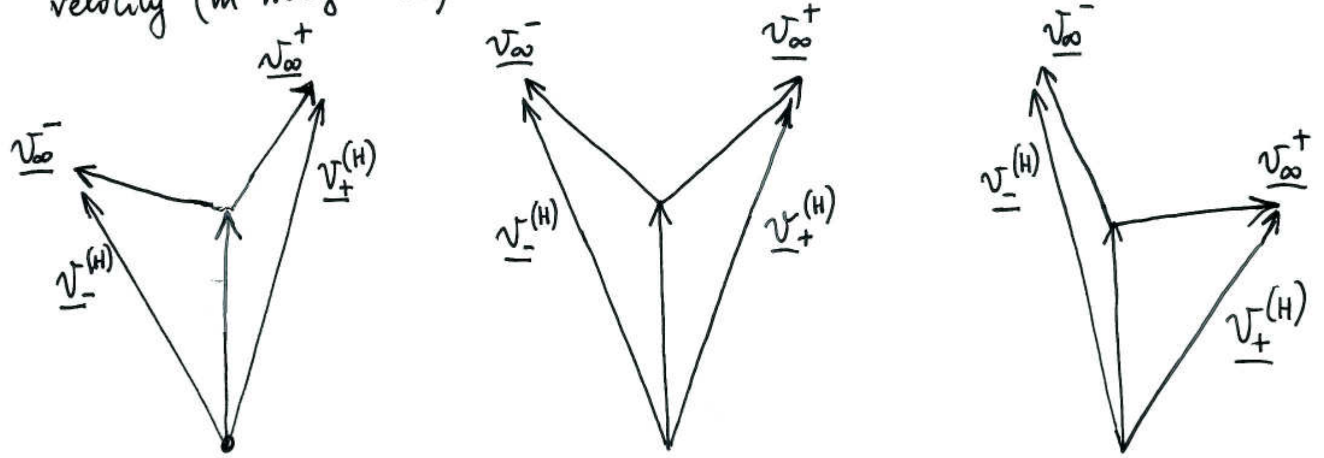
i.e. the Keplerian heliocentric energy of the spacecraft has changed. The energy exchange occurs with the planet. However, due to the large mass of the planet with respect to that of the spacecraft, the dynamical effect (velocity change) on the planet is completely negligible.

Instead, as previously remarked, the spacecraft experiences velocity increase or reduction (in magnitude). This effect is sometimes termed also swingby, gravity assist, or slingshot, and allows (and allowed) performing many interplanetary trajectories and missions, which would be infeasible otherwise.

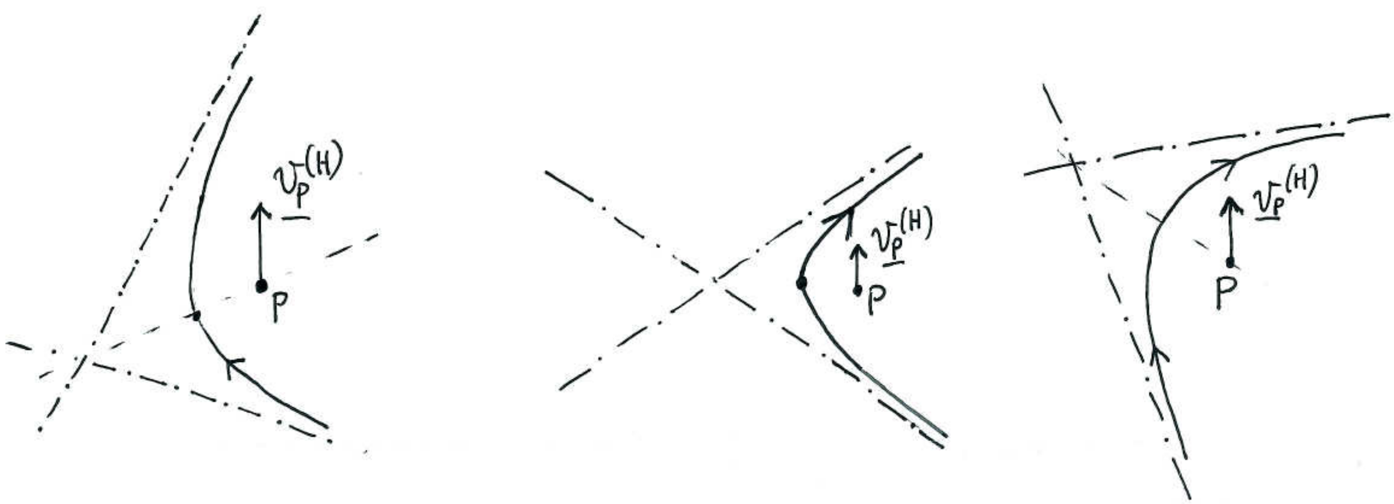
Examples are the Voyager missions, the Cassini mission to Saturn, and many other interplanetary trips, even out of ecliptic (e.g. Ulisses).

- <u>Qualitative effect of flyby</u>

Depending on the hyperbola geometry and orientation, flyby can lead to increasing or reducing the spacecraft velocity (in magnitude):

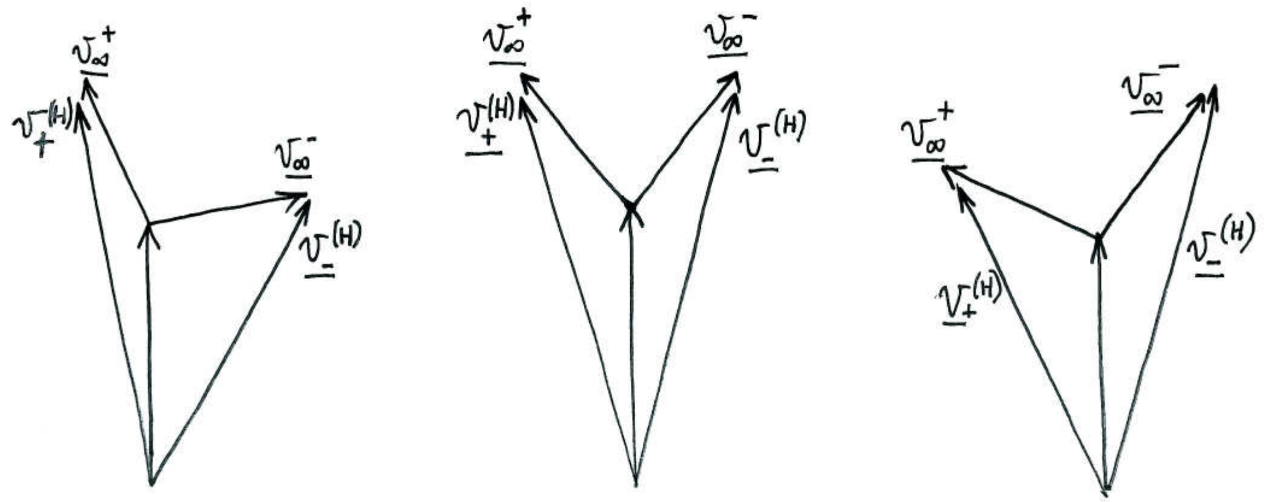


(a) Velocity INCREASE (b) Velocity UNCHANGED (c) Velocity REDUCTION

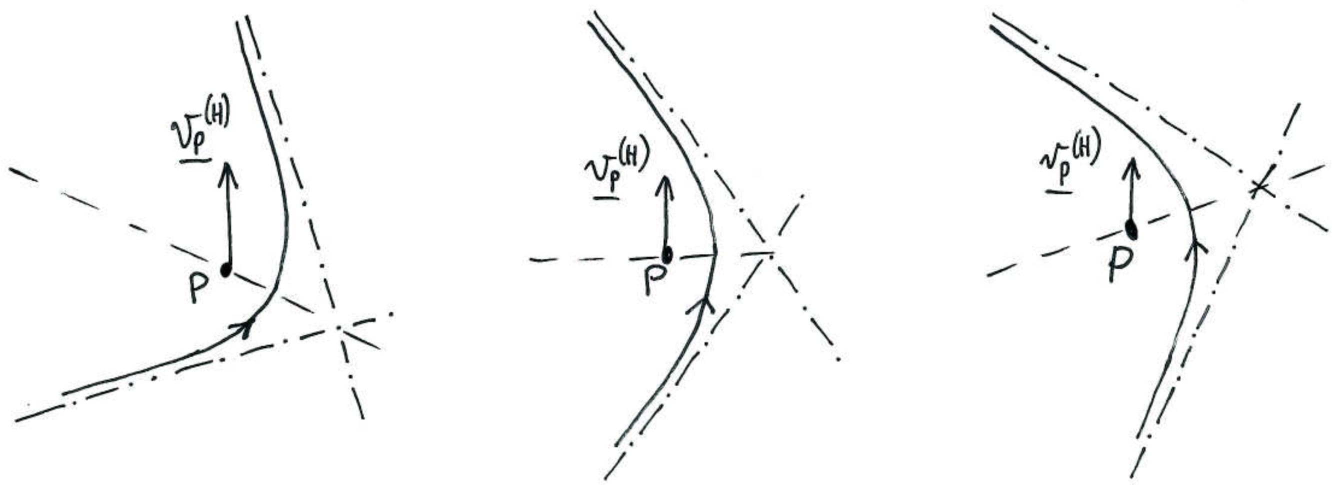


In the previous figures, CLOCKWISE flybys are portrayed, and $\underline{v}_\infty$ is rotated in clockwise sense (from $\underline{v}_\infty^-$ to $\underline{v}_\infty^+$). Three cases occur:

(a) Velocity INCREASE corresponds to periapse of hyperbola behind P
(b) Velocity UNCHANGED corresponds to periapse of hyperbola $\perp$ $\underline{v}_P^{(H)}$
(c) Velocity REDUCTION corresponds to periapse of hyperbola in front of P



(a) Velocity INCREASE (b) Velocity UNCHANGED (c) Velocity REDUCTION



In the previous figures, COUNTERCLOCKWISE flybys are portrayed, and $\underline{v}_\infty$ is rotated in counterclockwise sense (from $\underline{v}_\infty^-$ to $\underline{v}_\infty^+$).
Similarly to what occurs for clockwise flybys, three cases exist:

(a) Velocity INCREASE corresponds to periapse of hyperbola behind P

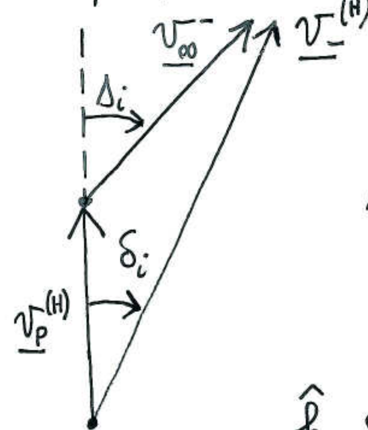
(b) Velocity UNCHANGED corresponds to periapse of hyperbola $\perp$ $\underline{v}_P^{(H)}$

(c) Velocity REDUCTION corresponds to periapse of hyperbola in front of P

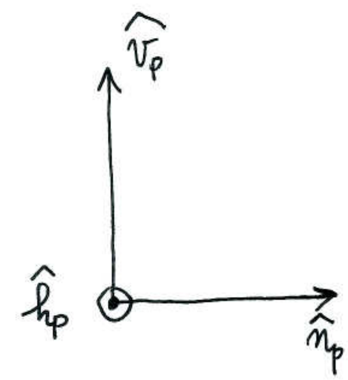
In the previous 6 figures of vector diagrams, the vertical vector (non-labeled) is $\underline{v}_P^{(H)}$

- <u>Quantitative analysis</u>

The objective is in finding $\underline{v}_+^{(H)}$ once $\underline{v}_-^{(H)}$ is given and $r_p^{(P)}$ (= periapse radius of hyperbola) is specified.



If $\underline{v}_-^{(H)}$ is specified, its angle δ_i with $\underline{v}_p^{(H)}$ can be found in a straightforward way. Let $\hat{v}_p$ be aligned with $\underline{v}_p^{(H)}$, whereas $\hat{n}_p$ is such that $\hat{n}_p \times \hat{v}_p = \hat{h}_p$ ($\hat{h}_p$ is the planet angular momentum around the Sun)



Due to these definitions

$$\underline{v}_p^{(H)} = \begin{bmatrix} 0 & v_p^{(H)} \end{bmatrix} \begin{bmatrix} \hat{n}_p \\ \hat{v}_p \end{bmatrix} \qquad \underline{v}_-^{(H)} = \begin{bmatrix} v_-^{(H)} \sin\delta_i & v_-^{(H)} \cos\delta_i \end{bmatrix} \begin{bmatrix} \hat{n}_p \\ \hat{v}_p \end{bmatrix}$$

$$\underline{v_\infty^-} = \underline{v}_-^{(H)} - \underline{v}_p^{(H)} = \begin{bmatrix} v_-^{(H)} s_{\delta_i} & v_-^{(H)} c_{\delta_i} - v_p^{(H)} \end{bmatrix} \begin{bmatrix} \hat{n}_p \\ \hat{v}_p \end{bmatrix}$$

$$= \begin{bmatrix} v_\infty^- s_{\Delta_i} & v_\infty^- c_{\Delta_i} \end{bmatrix} \begin{bmatrix} \hat{n}_p \\ \hat{v}_p \end{bmatrix}$$

Hence, one obtains:

$$\begin{cases} v_\infty^- s_{\Delta_i} = v_-^{(H)} s_{\delta_i} \\ v_\infty^- c_{\Delta_i} = v_-^{(H)} c_{\delta_i} - v_p^{(H)} \end{cases}$$

From these two equations, omitting the superscript (H) henceforth

$$v_\infty^- = \sqrt{(v_- c_{\delta_i} - v_P)^2 + v_-^2 s_{\delta_i}^2} = \sqrt{v_-^2 + v_P^2 - 2 v_P v_- c_{\delta_i}}$$

$$\begin{cases} c_{\Delta_i} = \dfrac{v_- c_{\delta_i} - v_P}{v_\infty^-} \\ s_{\Delta_i} = \dfrac{v_- s_{\delta_i}}{v_\infty^-} \end{cases} \longrightarrow \quad \Delta_i = 2 \operatorname{atan} \frac{s_{\Delta_i}}{1 + c_{\Delta_i}}$$

The hyperbolic excess velocity $\underline{v_\infty^-}$ is rotated by angle ξ,

given by $\quad \xi = 2\theta_*^M - \pi$

where $\theta_*^M = \operatorname{acos}\left(-\dfrac{1}{e_{HYP}}\right)$ and $e_{HYP} = 1 + \dfrac{r_P}{\mu_P} v_\infty^2$

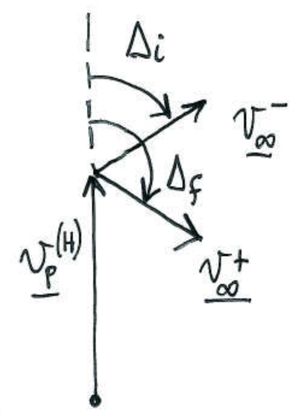
(and $v_\infty = |\underline{v_\infty^-}| = |\underline{v_\infty^+}|$ is used in the last equation)

Rotation of $\underline{v_\infty^-}$ occurs in :

(a) Clockwise sense if the hyperbola is flown in clockwise sense;

(b) Counterclockwise sense if the hyperbola is flown in counterclockwise sense;

$$\Delta_f = \begin{cases} \Delta_i + \xi & \text{case (a)} \\ \Delta_i - \xi & \text{case (b)} \end{cases}$$



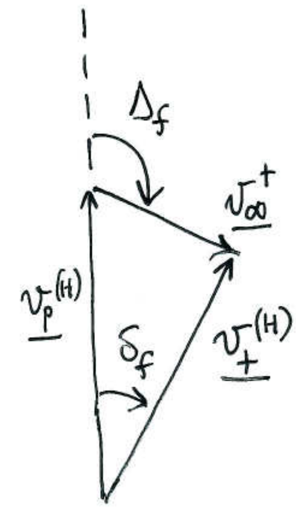
In the right figure, case (a) is portrayed

If Δ_f is obtained, then $\underline{v}_+^{(H)}$ can be calculated. In fact

$$\underline{v}_\infty^{+} = v_\infty \begin{bmatrix} s_{\Delta_f} & c_{\Delta_f} \end{bmatrix} \begin{bmatrix} \hat{n}_p \\ \hat{v}_p \end{bmatrix} \qquad \underline{v}_p^{(H)} = v_p^{(H)} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{n}_p \\ \hat{v}_p \end{bmatrix}$$

$$\underline{v}_+^{(H)} = \underline{v}_p^{(H)} + \underline{v}_\infty^{+} = \begin{bmatrix} v_\infty s_{\Delta_f} & v_\infty c_{\Delta_f} + v_p^{(H)} \end{bmatrix} \begin{bmatrix} \hat{n}_p \\ \hat{v}_p \end{bmatrix}$$

$$= \begin{bmatrix} v_+^{(H)} \sin \delta_f & v_+^{(H)} \cos \delta_f \end{bmatrix} \begin{bmatrix} \hat{n}_p \\ \hat{v}_p \end{bmatrix}$$



Hence, one obtains

$$\begin{cases} v_+^{(H)} s_{\delta_f} = v_\infty s_{\Delta_f} \\ v_+^{(H)} c_{\delta_f} = v_\infty c_{\Delta_f} + v_p^{(H)} \end{cases}$$

From these two equations, omitting the superscript (H) for v_p

$$v_+^{(H)} = \sqrt{v_\infty^2 + v_p^2 + 2 v_\infty v_p c_{\Delta_f}}$$

$$\begin{cases} c_{\delta_f} = \dfrac{v_p + v_\infty c_{\Delta_f}}{v_+^{(H)}} \\ s_{\delta_f} = \dfrac{v_\infty s_{\Delta_f}}{v_+^{(H)}} \end{cases} \longrightarrow \delta_f = 2 \operatorname{atan} \frac{s_{\delta_f}}{1 + c_{\delta_f}}$$

- Summary

For specified values of $(r_p, v_-^{(H)}, \delta_i)$; the following steps identify all the quantities involved in the planetary flyby:

(1) Calculate v_∞ ($\equiv v_\infty^- \equiv v_\infty^+$, i.e. magnitude of $\underline{v_\infty}$ preserves)

(2) Calculate Δ_i

(3) Calculate e_{HYP}

(4) Calculate ξ

(5) Calculate Δ_f for counterclockwise or clockwise pass (according to mission requirements)

(6) Calculate $v_+^{(H)}$

(7) Calculate δ_f

In this way the heliocentric velocity post-flyby, $\underline{v_+^{(H)}}$, is identified in magnitude and direction.