ORBITAL MOTION IN MULTIBODY ENVIRONMENTS

INTRODUCTION

When two or more celestial bodies affect the spacecraft motion, the space vehicle dynamics is no longer Keplerian, and the simultaneous gravitational attraction of the relevant celestial bodies is to be taken into account.

As a first step, some basic properties of the motion of a system of N massive bodies are described.

Then, the motion of two massive bodies subject to their mutual attraction, is described.

Finally, the restricted three-body problem is introduced and analyzed in detail. In this dynamical framework, a single body (i.e., the spacecraft) has negligible mass with respect to the remaining two massive bodies, which are termed PRIMARIES. The problem is referred to as "restricted" because the gravitational action of the spacecraft on the massive bodies is neglected, due to its mass, much smaller than those of the two primaries.

The circular restricted three-body problem (CR3BP) is especially useful (and appropriate) for investigating the motion of a space vehicle in the Earth-Moon system, where the patched conic approximation is relatively inaccurate. Of course, the same framework may be useful also for alternative systems, provided that the motion of the two primaries takes places along circular orbits.

• PROBLEM OF N BODIES

If N massive bodies are subject to their mutual attraction, then orbital motion is the result of the simultaneous action of the gravitational forces.

In an inertial frame, for a point mass m_i, the Newton law holds:

$$m_i \ddot{\underline{R}}_i = G \sum_{j=1, j\neq i}^{N} \frac{m_i m_j}{r_{ij}^3} (\underline{R}_j - \underline{R}_i) \qquad r_{ij} = |\underline{r}_{ij}| = |\underline{R}_j - \underline{R}_i|$$

where $\underline{R}_i$ is the position vector of mass m_i in an inertial frame centered at 0. Gravitational forces are internal forces for this system, and their sum is 0. This circumstance implies that

$$\sum_{i=1}^{N} m_i \ddot{\underline{R}}_i = \sum_{\substack{i,j=1 \\ j\neq i}}^{N} G \frac{m_i m_j}{r_{ij}^3} \underline{r}_{ij} = \underline{0}$$

$$\Rightarrow \begin{cases} \sum m_i \dot{\underline{R}}_i = \underline{a} & (1) \\ \sum m_i \underline{R}_i = \underline{a}\, t + \underline{b} & (2) \end{cases}$$

$\underline{a}, \underline{b}$ = constant vectors (depending only on the initial conditions)

$\underline{a}$ and $\underline{b}$ are equivalent to 6 scalar integrals.

The center of mass is defined as

$$\underline{R}_c = \frac{\sum_{i}^{N} m_i \underline{R}_i}{\sum_{i=1}^{N} m_i}$$

and, due to (1) and (2), moves in rectilinear uniform motion because $\dot{\underline{R}}_c = \frac{1}{M} \sum_{i=1}^{N} m_i \dot{\underline{R}}_i = \underline{a}$

Moreover, using the equation on energy of a system of masses

$$\frac{d\mathcal{E}}{dt} = \underline{F} \cdot \underline{\dot{R}_c} = 0 \qquad \text{because} \quad \underline{F} = \underline{0} \quad \text{(sum of external forces)}$$

$\Rightarrow \quad \mathcal{E} = \text{constant}$

For the problem of N bodies $\quad U = -\sum_{\substack{i=1 \\ j=i+1}}^{N,N} \frac{G\, m_i\, m_j}{r_{ij}}$

therefore $\quad \mathcal{E} = \frac{1}{2} \sum_{i=1}^{N} m_i\, \underline{\dot{R}_i} \cdot \underline{\dot{R}_i} - \sum_{\substack{i=1 \\ j=i+1}}^{N,N} \frac{G\, m_i\, m_j}{r_{ij}}$

The constant energy depends again only on the initial conditions, and represents another scalar quantity that preserves

Lastly, the angular momentum with respect to C (center of mass) can be evaluated. It is defined as

$$\underline{H_c} = \sum_{i=1}^{N} \underline{r_i} \times m_i \underline{\dot{R}_i} \qquad \text{where} \quad \underline{r_i} = \underline{R_i} - \underline{R_c}$$

and its time derivative is

$$\underline{\dot{H}_c} = \sum_{i=1}^{N} \left(\underline{\dot{R}_i} - \underline{\dot{R}_c}\right) \times m_i\, \underline{\dot{R}_i} + \sum_{i=1}^{N} \left(\underline{R_i} - \underline{R_c}\right) \times m_i\, \underline{\ddot{R}_i} =$$

$$= -\underline{\dot{R}_c} \times \sum_{i=1}^{N} m_i\, \underline{\dot{R}_i} + \sum_{\substack{i=1 \\ j=1, j\neq i}}^{N,N} \underline{R_i} \times \frac{G\, m_i\, m_j}{r_{ij}^3} \left(\underline{R_j} - \underline{R_i}\right) - \underline{R_c} \times \sum_{i=1}^{N} m_i \underline{\ddot{R}_i}$$

$$= -\underline{\dot{R}_c} \times M \underline{\dot{R}_c} = 0$$

In the last steps the double sum is zero because the terms simplify in pairs. Moreover, the fact $\sum_{i=1}^{N} m_i\, \underline{\ddot{R}_i} = \underline{0}$ was proven previously

The center of mass can be assumed also as the origin O' of an alternative inertial frame. Letting $O' \equiv C$

$$\underline{H_c} = \sum_{i=1}^{N} \underline{R_i} \times m_i \underline{\dot{R}_i}$$

$$\underline{\dot{H}_c} = \sum_{i=1}^{N} \underline{\dot{R}_i} \times m_i \underline{\dot{R}_i} + \sum_{\substack{i,j=1 \\ i \neq j}}^{N,N} \underline{R_i} \times \frac{m_i m_j G}{r_{ij}^3} \left(\underline{R_j} - \underline{R_i}\right) = \underline{0}$$

i.e. the same result is found.

In the end, $\underline{H_c}$ = constant, and this is equivalent to conservation of 3 scalar quantities.

The plane orthogonal to $\underline{H_c}$ is termed LAPLACE PLANE

In short, the total number of scalar integrals is 10, i.e.

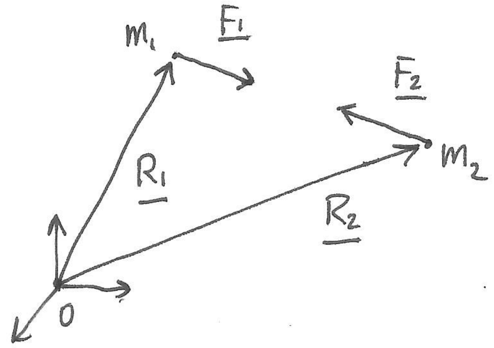
$$\underline{a}, \ \underline{b}, \ \underline{H_c}, \ \mathcal{E} \qquad \left(3 \text{ components for each vector} + \mathcal{E}\right)$$

$$\underline{R_c} = \frac{\underline{a}\, t + \underline{b}}{M}, \quad \underline{H_c} = \text{const}, \quad \mathcal{E} = \text{const}$$

These 10 integrals are specified once the initial conditions are known, and preserve in time regardless of the time evolution of each single point mass that composes the system.

- PROBLEM OF TWO BODIES

If two body are subject to their mutual attraction, then they obey :



$$\begin{cases} m_1 \dfrac{d^2 \underline{R_1}}{dt^2} = G \dfrac{m_1 m_2}{r_{12}^3} \left(\underline{R_2} - \underline{R_1}\right) =: + \underline{F_1} \\[2ex] m_2 \dfrac{d^2 \underline{R_2}}{dt^2} = G \dfrac{m_1 m_2}{r_{12}^3} \left(\underline{R_1} - \underline{R_2}\right) =: + \underline{F_2} \end{cases}$$

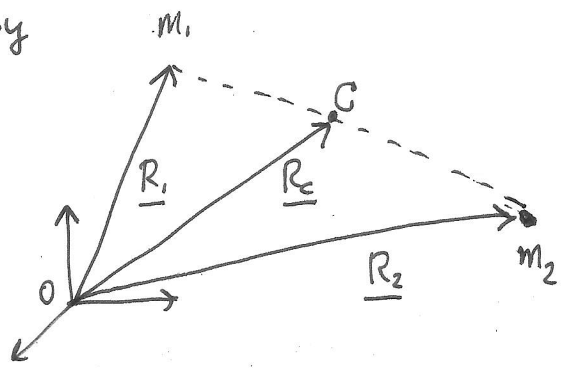
where $\underline{F_1} + \underline{F_2} = \underline{0}$

The motion of mass 2 relative to 1 is governed by the equation for $\left(\underline{R_2} - \underline{R_1}\right)$, i.e.

$$\left[\frac{d^2\left(\underline{R_2} - \underline{R_1}\right)}{dt^2} = -\frac{G\left(m_1 + m_2\right)}{r_{12}^3}\left(\underline{R_2} - \underline{R_1}\right)\right] \quad (1)$$

i.e. the motion of 2 relative to 1 occurs in way like if the total mass $M = m_1 + m_2$ is placed in the position of 1

The center of mass C is identified by



$$\underline{R_c} = \frac{m_1 \underline{R_1} + m_2 \underline{R_2}}{m_1 + m_2}$$

and can be proven to lie in the line that connects m_1 and m_2

In fact, (a) $\underline{R_1} - \underline{R_c} = \frac{m_2}{M}\left(\underline{R_1} - \underline{R_2}\right)$

(b) $\underline{R_2} - \underline{R_c} = \frac{m_1}{M}\left(\underline{R_2} - \underline{R_1}\right)$

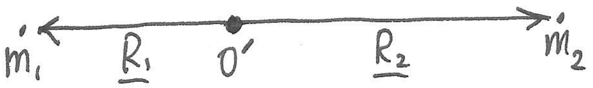
$\Rightarrow \quad \left(\underline{R_1} - \underline{R_c}\right) \uparrow\downarrow \left(\underline{R_2} - \underline{R_c}\right)$

The two relative positions are aligned and in opposite directions

Let $O' \equiv C$ be the origin of a new inertial frame

$$m_1 \underline{R_1} + m_2 \underline{R_2} = \underline{0}$$

$$\Downarrow$$



$m_1 R_1 = m_2 R_2$ at all times

Moreover, one obtains $\quad r_{12} = R_1 + R_2 = R_1\left(1+\frac{m_1}{m_2}\right) = R_2\left(1+\frac{m_2}{m_1}\right)$ (a)

and
$$\begin{cases} \underline{R_1} = -\dfrac{m_2}{m_1}\underline{R_2} & (b) \\[2ex] \underline{R_2} = -\dfrac{m_1}{m_2}\underline{R_1} & (c) \end{cases}$$

Using the previous relations (a), (b), (c), the motion of m_1 and m_2 with respect to O' is governed by

$$\begin{cases} m_1 \dfrac{d^2 \underline{R_1}}{dt^2} = -G \dfrac{m_1 m_2}{R_1^3 \dfrac{M^3}{m_2^3}} \left(\dfrac{m_1}{m_2}+1\right)\underline{R_1} = -G\dfrac{m_1 m_2^3}{M^2}\dfrac{\underline{R_1}}{R_1^3} \\[4ex] m_2 \dfrac{d^2 \underline{R_2}}{dt^2} = -G \dfrac{m_1 m_2}{R_2^3 \dfrac{M^3}{m_1^3}} \left(\dfrac{m_2}{m_1}+1\right)\underline{R_2} = -G\dfrac{m_2 m_1^3}{M^2}\dfrac{\underline{R_2}}{R_2^3} \end{cases}$$

i.e.

$$\left[(d) \quad \frac{d^2\underline{R_1}}{dt^2} = -G\frac{m_2^3}{M^2}\frac{\underline{R_1}}{R_1^3} \qquad \text{and} \qquad \frac{d^2\underline{R_2}}{dt^2} = -G\frac{m_1^3}{M^2}\frac{\underline{R_2}}{R_2^3} \quad (e) \right] \; 2$$

Equations (d) and (e) prove that the ABSOLUTE MOTION with respect to $G \equiv O'$ is Keplerian, with equivalent masses

$\dfrac{m_2^3}{M^2}$ acting on m_1 $\qquad$ and $\qquad$ $\dfrac{m_1^3}{M^2}$ acting on m_2

Previously, it was found that also the RELATIVE MOTION of 2 with respect to 1 and 1 with respect to 2 is Keplerian, with equivalent mass M (in both cases).

Moreover, $\dfrac{R_1}{R_2} = \dfrac{m_2}{m_1} = \text{const.}$, therefore each body reaches its apoapse (or periapse) at the same time as the other one

This means that

$$\frac{a_1(1-e_1)}{a_2(1-e_2)} = \frac{a_1(1+e_1)}{a_2(1+e_2)}$$ i.e. ratios at periapse and apoapse coincide

$$\rightarrow \quad (1-e_1)(1+e_2) = (1+e_1)(1-e_2) \longrightarrow \boxed{e_1 = e_2 \equiv e} \quad (3)$$

The two conics (in their absolute motion) have the same eccentricity. Moreover $\dfrac{a_1}{a_2} = \left\{\dfrac{m_2}{m_1}(r_{P2}+r_{A2})\right\} / (r_{P2}+r_{A2}) = \dfrac{m_2}{m_1}$.

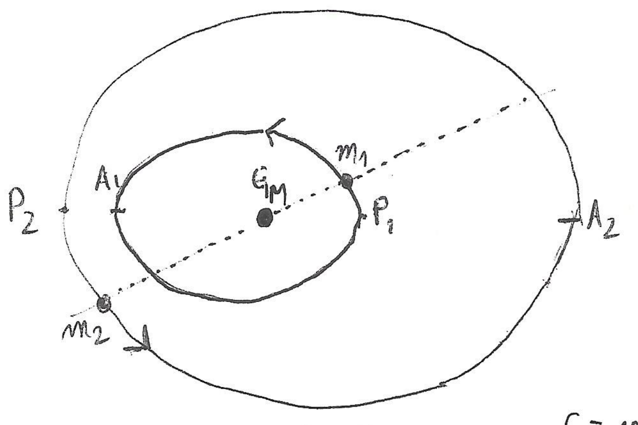
The greater mass 1 has motion completely inside the ellipse of body 2 (of smaller mass) if

$$a_1(1+e) < a_2(1-e)$$

As $\dfrac{a_1}{a_2} = \dfrac{m_2}{m_1}$ one gets $\dfrac{m_2}{m_1}(1+e) < 1-e \rightarrow$

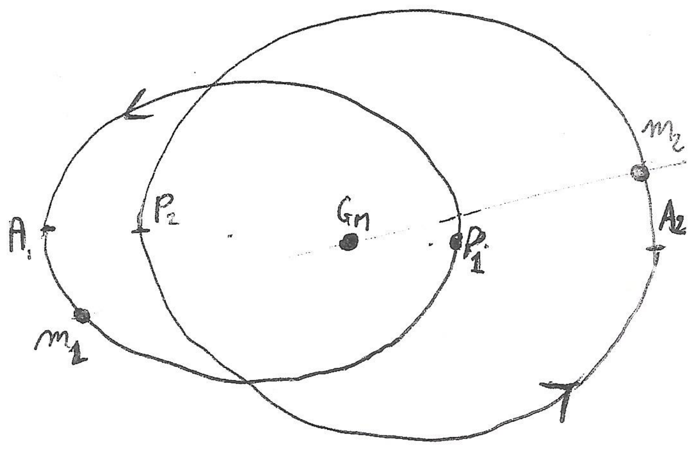
$$\rightarrow \quad e\left(1+\frac{m_2}{m_1}\right) < 1-\frac{m_2}{m_1} \longrightarrow e < \frac{m_1-m_2}{M}$$

(written under the assumption that $m_1 > m_2$)



$e < \dfrac{m_1-m_2}{M}$

G_M = center of mass



$e > \dfrac{m_1-m_2}{M}$

The mean motion is given by

$$w^2 = \frac{G\frac{m_2^3}{M^2}}{a_1^3} = \frac{G\frac{m_1^3}{M^2}}{a_2^3}$$

but

$$\frac{m_2}{M} = \frac{m_2}{m_1+m_2} = \frac{\frac{m_2}{m_1}}{1+\frac{m_2}{m_1}} = \frac{\frac{a_1}{a_2}}{1+\frac{a_1}{a_2}} = \frac{a_1}{a_1+a_2}$$

$$\frac{m_1}{M} = \frac{m_1}{m_1+m_2} = \frac{\frac{m_1}{m_2}}{1+\frac{m_1}{m_2}} = \frac{\frac{a_2}{a_1}}{1+\frac{a_2}{a_1}} = \frac{a_2}{a_1+a_2}$$

hence, one obtains

$$\left[w^2 = \frac{GM}{(a_1+a_2)^3} \right] \quad (4) \quad \text{Mean motion}$$

- Planet - satellite problem

If m_1 = planet mass and m_2 = satellite mass, then $m_1 \gg m_2 \simeq 0$ and the center of mass coincides with the planet center. Equation (2) becomes

$$\frac{d^2\underline{R_2}}{dt^2} = -\frac{Gm_1}{R_2^3}\underline{R_2}$$, i.e. the classical equation with a single attracting body

while

$$w^2 = \frac{Gm_1}{a^3}$$

The planet-satellite problem is also termed RESTRICTED PROBLEM OF TWO BODIES, because $m_2 \ll m_1$ (thus m_2 does not affect m_1)

• CIRCULAR RESTRICTED PROBLEM OF THREE BODIES (CR3BP)

Motion of a 3^{rd} body that has negligible mass with respect to two massive bodies, termed the PRIMARIES, i.e.

$$m := m_3 \ll m_2 < m_1$$

In other words, the third body does not affect the remaining two bodies 1 and 2, which are assumed to describe circular orbits around their mass center

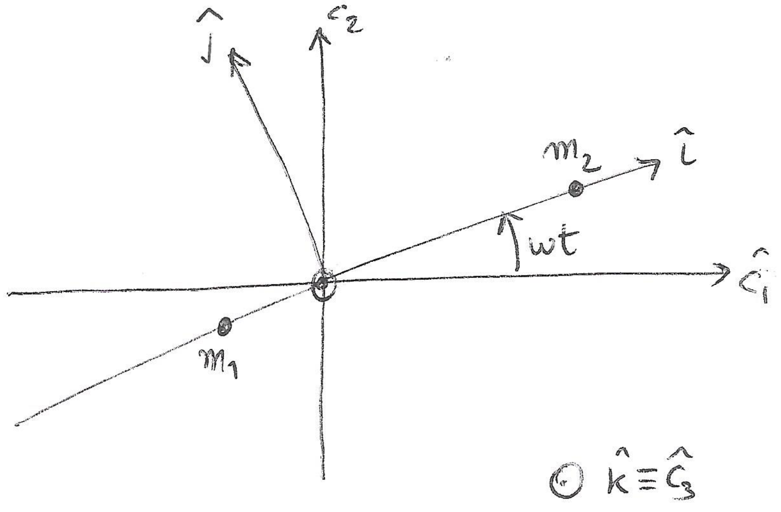
The angular velocity of the two primaries is

$\omega = \sqrt{\frac{GM}{R^3}}$ where R = their constant distance

• Frames

SYNODIC REFERENCE FRAME

$\{\hat{i}, \hat{j}, \hat{k}\}$ rotates together with the two primaries with $\hat{k} \uparrow\uparrow \underline{H}$ (angular momentum $\underline{H}$)

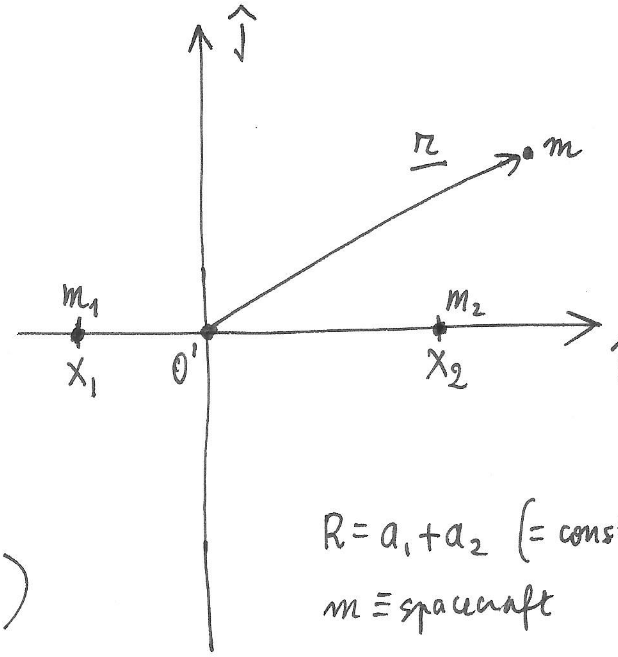


$\{\hat{c}_1, \hat{c}_2, \hat{c}_3\}$ inertial axes
$\{\hat{i}, \hat{j}, \hat{k}\}$ synodic axes

At $t=0$ they coincide

$$\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{bmatrix}$$

The position of m_1 and m_2 with respect to O' (center of mass) is given by



$$\begin{cases} x_2 - x_1 = R \\ m x_1 + m_2 x_2 = 0 \end{cases} \longrightarrow$$

$$\rightarrow \begin{cases} x_1 = -\dfrac{m_2}{M} R \\[2ex] x_2 = \dfrac{m_1}{M} R \end{cases}$$

$R = a_1 + a_2$ (= const)

$m \equiv$ spacecraft

Letting $\mu = \dfrac{m_2}{m_1 + m_2}$ (mass parameter)

and $\begin{cases} DU = R & \text{distance unit} \\ TU = \omega^{-1} & \text{time unit} \end{cases} \Rightarrow G(m_1 + m_2) = 1 \dfrac{DU^3}{TU^2}$

one obtains

$$\begin{cases} x_1 = -\mu R = -\mu\, DU \\ x_2 = (1-\mu) R = (1-\mu)\, DU \end{cases} \qquad \begin{cases} G m_2 = \mu \dfrac{DU^3}{TU^2} \\[2ex] G m_1 = (1-\mu) \dfrac{DU^3}{TU^2} \end{cases}$$

One can choose 1 and 2 such that $m_1 > m_2 \Rightarrow \mu < \frac{1}{2}$

- <u>Equations of motion</u>

$$\frac{d^2 \underline{r}}{dt^2} = -\frac{(1-\mu)(\underline{r} - \underline{R}_1)}{|\underline{r} - \underline{R}_1|^3} - \frac{\mu(\underline{r} - \underline{R}_2)}{|\underline{r} - \underline{R}_2|^3}$$

omitting DU and TU henceforth

The position vector can be written in terms of its components in the rotating frame $(\hat{\imath}, \hat{\jmath}, \hat{k})$,

$$\underline{r} = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{bmatrix} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

Because $\underline{\omega} \times \hat{\imath} = \hat{\jmath}\omega$, $\underline{\omega} \times \hat{\jmath} = -\hat{\imath}\omega$, $\underline{\omega} \times \hat{k} = \underline{0}$

$(\underline{\omega} = \omega\hat{k})$ the left hand side of the previous vector equation becomes

$$\frac{d^2\underline{r}}{dt^2} = \frac{d}{dt}\left[\dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k} + x(\underline{\omega}\times\hat{\imath}) + y(\underline{\omega}\times\hat{\jmath}) + z(\underline{\omega}\times\hat{k})\right] =$$

$$= \frac{d}{dt}\left[(\dot{x} - \omega y)\hat{\imath} + (\dot{y} + \omega x)\hat{\jmath} + \dot{z}\right] =$$

$$= (\ddot{x} - \omega\dot{y})\hat{\imath} + (\ddot{y} + \omega\dot{x})\hat{\jmath} + \ddot{z} + \omega(\dot{x} - \omega y)\hat{\jmath} - \omega(\dot{y} + \omega x)\hat{\imath} =$$

$$= (\ddot{x} - 2\omega\dot{y} - \omega^2 x)\hat{\imath} + (\ddot{y} + 2\omega\dot{x} - \omega^2 y)\hat{\jmath} + \ddot{z}$$

Therefore, along the three rotating axes

$$\hat{\imath})\quad \ddot{x} - 2\omega\dot{y} - \omega^2 x = -\frac{(1-\mu)(x+\mu)}{\left[(x+\mu)^2 + y^2 + z^2\right]^{3/2}} - \frac{\mu(x+\mu-1)}{\left[(x+\mu-1)^2 + y^2 + z^2\right]^{3/2}}$$

$$\hat{\jmath})\quad \ddot{y} + 2\omega\dot{x} - \omega^2 y = -\frac{(1-\mu)\,y}{\left[(x+\mu)^2 + y^2 + z^2\right]^{3/2}} - \frac{\mu y}{\left[(x+\mu-1)^2 + y^2 + z^2\right]^{3/2}}$$

$$\hat{k})\quad \ddot{z} = -\frac{(1-\mu)\,z}{\left[(x+\mu)^2 + y^2 + z^2\right]^{3/2}} - \frac{\mu z}{\left[(x+\mu-1)^2 + y^2 + z^2\right]^{3/2}}$$

In the previous expressions the denominators contain the instantaneous distance from mass 1 and mass 2.

The physical unit of $(1-\mu)$ and μ in numerators is $\frac{DU^3}{TU^2}$

The physical unit of $(x+\mu)$ and $(x+\mu-1)$ in denominators is DU as well as in numerators

• <u>Jacobi integral</u>

Letting $$\Omega = \frac{\omega^2}{2}(x^2+y^2) + \frac{1-\mu}{[(x+\mu)^2+y^2+z^2]^{1/2}} + \frac{\mu}{[(x+\mu-1)^2+y^2+z^2]^{1/2}}$$

(Ω is also termed "potential function")

the equations of motion can be rewritten as

$$\begin{cases} \ddot{x} - 2\omega\dot{y} = \dfrac{\partial\Omega}{\partial x} & (1) \\ \ddot{y} + 2\omega\dot{x} = \dfrac{\partial\Omega}{\partial y} & (2) \\ \ddot{z} = \dfrac{\partial\Omega}{\partial z} & (3) \end{cases}$$

(1) is multiplied by $\dot{x}$, (2) by $\dot{y}$, (3) by $\dot{z}$, then one adds and obtains

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z} = \dot{x}\frac{\partial\Omega}{\partial x} + \dot{y}\frac{\partial\Omega}{\partial y} + \dot{z}\frac{\partial\Omega}{\partial z}$$

$$\rightarrow \frac{1}{2}\frac{d}{dt}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) = \frac{d\Omega}{dt} \longrightarrow \frac{d}{dt}\left[\Omega - \frac{1}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)\right] = 0$$

This means that the quantity

$$C := 2\Omega - \left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) \quad \text{is CONSTANT}$$

This is referred to as the JACOBI INTEGRAL.

As $C \propto -\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)$ it is intuitive that C is related to energy. In fact, C decreases as the energy increases; of course, for specified initial conditions, the value of C does not change in time, and, due to this, C is an INTEGRAL of motion in the CR3BP

- **Zero velocity surfaces and curves**

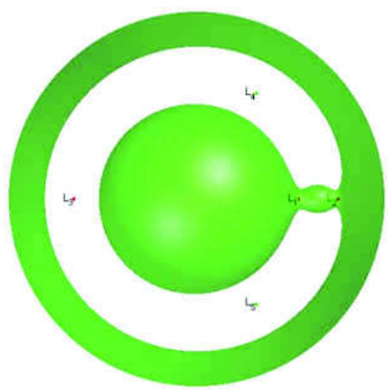
Zero velocity surfaces (in 3-d) and curves (in 2-d) are the loci where $\dot{x} = \dot{y} = \dot{z} = 0$.

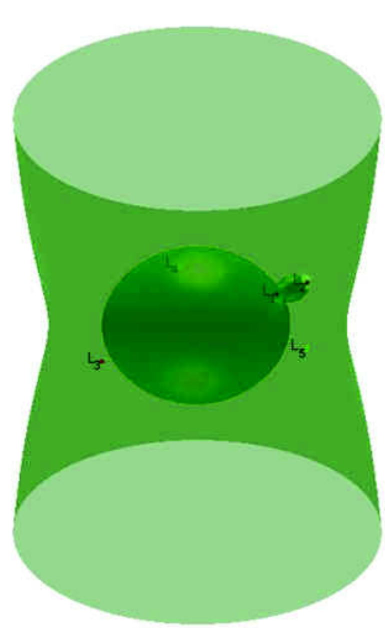
These surfaces (and curves) constrain the region where the spacecraft motion can take place. In fact

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 2\Omega(x,y,z) - C \geq 0$$

Because Ω is a function of the space coordinates only (x,y,z), the inequality at the right-hand-side defines the region of allowed motion, which is termed also HILL'S REGION.

Looking at $\quad 2\Omega = w^2(x^2+y^2) + \dfrac{2(1-\mu)}{[(x+\mu)^2+y^2+z^2]^{1/2}} + \dfrac{2\mu}{[(x+\mu-1)^2+y^2+z^2]^{1/2}}$





(i) if x, y are large
→ first term prevails, and is associated with a cylinder with axis z

(ii) if $(x+\mu)^2+y^2+z^2$ is small
OR
if $(x+\mu-1)^2+y^2+z^2$ is small
→ either 2^nd OR 3^rd term prevails, with two associated surfaces:

⇒ 2^nd term is a near-sphere about primary 1
3^rd term is a near-sphere about primary 2

In the previous figure, the zero velocity surfaces are illustrated for a particular value of C. Motion is allowed

(i) In the proximity of primary 1, i.e. inside the near-sphere that surrounds primary 1;

(ii) In the proximity of primary 2, i.e. inside the near-sphere that surrounds primary 2;

(iii) Outside the near-cylindrical surface with axis z.

Zero velocity curves are the sections of zero velocity surfaces with the (x,y)-plane, and will be described in greater detail in the following.

- <u>Libration points</u>

Libration (or Lagrange) points are equilibrium points in the synodic frame, where the 3rd body (i.e. the spacecraft) remains indefinitely, provided that it is located at these points with $\dot{x} = \dot{y} = \dot{z} = 0$ (zero velocity in (x,y,z))

These points are sought in the (x,y)-plane, i.e.

$z = 0$ and $\dot{z} = 0$ hold in the following.

Equilibrium means that $\quad \dot{x} = 0 \quad$ and $\quad \dot{y} = 0$

and also $\quad \ddot{x} = 0 \quad$ and $\quad \ddot{y} = 0$

at libration points

These conditions yield

$$(A) \quad \ddot{x} = \omega^2 x - \frac{(1-\mu)(x+\mu)}{\left[(x+\mu)^2+y^2\right]^{3/2}} - \frac{\mu\,(x+\mu-1)}{\left[(x+\mu-1)^2+y^2\right]^{3/2}} = 0$$

$$(B) \quad \ddot{y} = \omega^2 y - \frac{(1-\mu)\,y}{\left[(x+\mu)^2+y^2\right]^{3/2}} - \frac{\mu\, y}{\left[(x+\mu-1)^2+y^2\right]^{3/2}} = 0$$

(1) COLLINEAR LIBRATION POINTS

Equilibrium points are sought along the x-axis (i.e. $y=0$).
Only (A) is needed, because (B) is satisfied if $y=0$;
(A) becomes

$$\omega^2 x - \frac{(1-\mu)(x+\mu)}{|x+\mu|^3} - \frac{\mu\,(x+\mu-1)}{|x+\mu-1|^3} = 0$$

Three cases occur

(a) $x+\mu<0 \longrightarrow x<-\mu$

(b) $x+\mu>0$ and $x+\mu-1<0 \longrightarrow -\mu<x<1-\mu$

(c) $x+\mu-1>0 \longrightarrow x>1-\mu$

In each case a quintic equation can be found (not reported for the sake of brevity): the only real admissible solution in the respective range (a, b, or c) provides the x-coordinate of the equilibrium point, in the previous 3 cases:

(a) Left exterior collinear libration point, denoted with L_3
(b) Interior collinear libration point, denoted with L_1
(c) Right exterior collinear libration point, denoted with L_2

(2) TRIANGULAR (OR EQUILATERAL) LIBRATION POINTS

If $y \neq 0$, then both (A) and (B) must vanish.

Letting
$$\begin{cases} r_1 := \sqrt{(x+\mu)^2 + y^2} & \text{distance from primary 1} \\ r_2 := \sqrt{(x+\mu-1)^2 + y^2} & \text{distance from primary 2} \end{cases}$$

$$\text{(A)} \quad \omega^2 x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x+\mu-1)}{r_2^3} = 0$$

where $\omega = 1\ TU^{-1}$

$$\text{(B)} \quad \omega^2 y - \frac{\mu y}{r_2^3} - \frac{(1-\mu) y}{r_1^3} = 0$$

(due to definition of TU)

Using $\omega = 1\ TU^{-1}$ (B) becomes

$$y\left[r_1^3 r_2^3 - \mu r_1^3 - r_2^3 + \mu r_2^3 \right] = 0$$

the term in parentheses vanishes if $r_1 = r_2 = 1$ (DU) regardless of μ. Using $r_1 = r_2 = 1$ (DU) in (A), one gets

$$x - (1-\mu)(x+\mu) - \mu(x+\mu-1) = 0$$

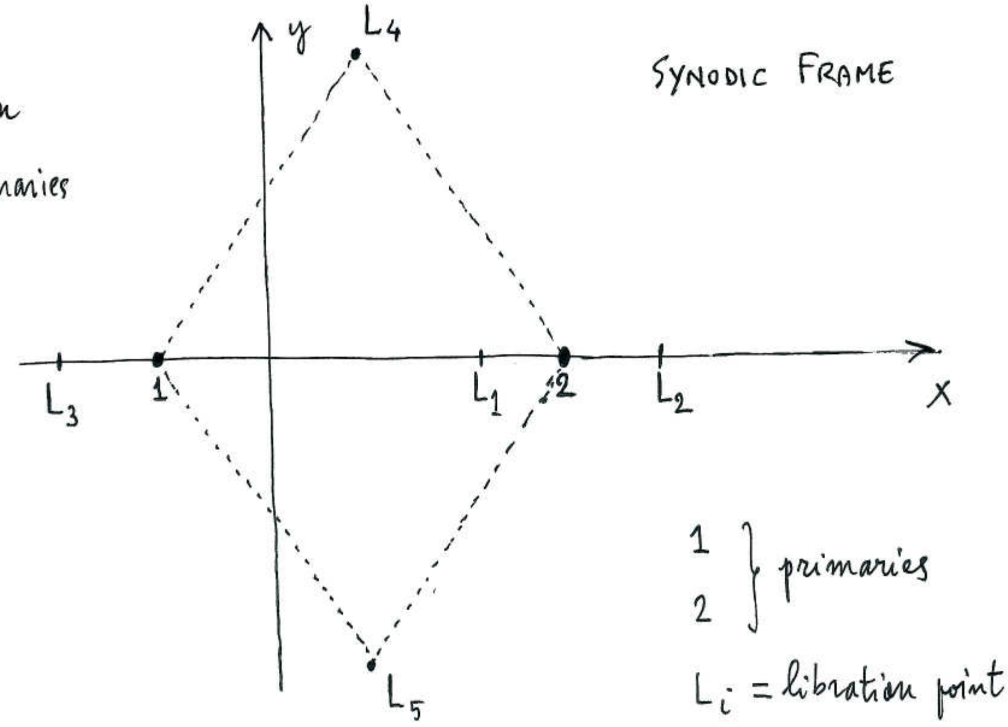
Therefore also (A) is fulfilled, and this means that in the (x, y) the points $r_1 = r_2 = 1$ DU are equilibrium points. Two such points exist, located above and below the x-axis, and termed equilateral or triangular points because each triangular libration point forms an equilateral triangle with the two primaries. It is common to denote with

L_4 the triangular point above the x-axis

L_5 the triangular point under the x-axis

Location of libration point for two primaries (when $\mu < 0.5$)

L_4 and L_5 form two equilateral triangles



• Function Ω at L_i

The function Ω is stationary at L_i ; in fact

$$\frac{\partial \Omega}{\partial x} = 0 \quad \text{and} \quad \frac{\partial \Omega}{\partial y} = 0 \quad \text{at } L_i$$

However, Ω can have a minimum or maximum at L_i (or can be simply stationary). In order to find out if Ω has min or max at L_i, these are the steps :

(a) Calculate symbolically $\Omega_{xx}, \Omega_{yy}, \Omega_{xy}$ and evaluate these at L_i

(b) Calculate $$\det \begin{bmatrix} \Omega_{xx} & \Omega_{xy} \\ \Omega_{yx} & \Omega_{yy} \end{bmatrix} =: H$$

(c) Four cases can occur :

(i) $\Omega_{xx}, \Omega_{yy} > 0$ and $H > 0$ $\rightarrow$ Ω has min value at L_i

(ii) $\Omega_{xx}, \Omega_{yy} < 0$ and $H > 0$ $\rightarrow$ Ω has max value at L_i

(iii) $H = 0$ $\rightarrow$ further derivatives needed

(iv) $H < 0$ $\rightarrow$ not max nor min

The results of the study of Ω at L_i are

(a) At L_1, L_2, L_3 (collinear points) Ω has not a max or min value, i.e. it is simply stationary

(b) At L_4, L_5 Ω has the minimum value Ω_{min}

$$\Omega_{min} = \Omega(L_4, L_5) = \frac{3}{2} - \frac{\mu}{2}(1-\mu)$$

As Ω has the minimum value at L_4 and L_5, the inequality $2\Omega - C \geq 0$ (HILL'S REGION of allowed motion) is satisfied in the entire space if

$$C < 2\Omega_{min} = 3 - \mu(1-\mu)$$

In other words, if the initial conditions for the spacecraft are such that $C < 2\Omega_{min}$, then it can travel in the entire space, because no zero velocity surface exists.

- <u>Special values of C</u>

If the spacecraft is placed at L_i with zero velocity, then

$$C_i = 2\Omega(L_i)$$

Because the velocity is zero at L_i, the libration point belongs to the zero velocity surface (and curve, in the (x, y)-plane)

From the previous discussion

$$C_{5,4} = 2\Omega(L_4, L_5) \leqslant 2\Omega \quad \text{at all points}$$

Hence, the motion can take place in the entire space if

$$C < C_{4,5}$$

This means also that the zero velocity surfaces (and curves) disappear at $C = C_{4,5}$.

The geometry of the zero velocity curves vary as C varies, i.e. when the spacecraft initial conditions change

Let C_i = value of C when the zero velocity curve contains L_i.

several cases can occur:

(1) $C < C_{4,5}$: motion allowed in the entire space

(2) $C_{4,5} < C < C_3$: motion forbidden only in the proximity of L_4 and L_5

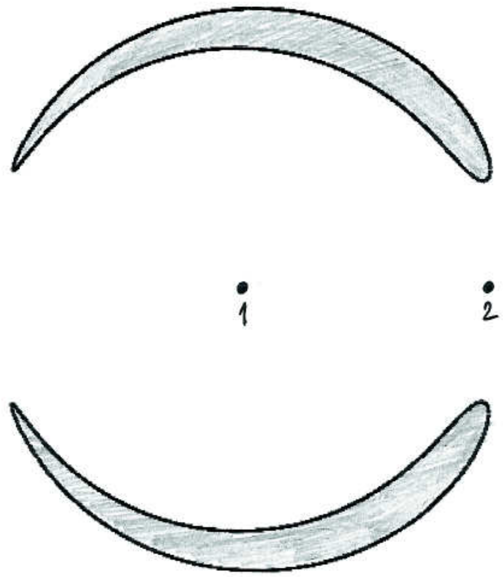
(3) $C_3 < C < C_2$: interior and exterior transfers from primary 1 to 2 are feasible (interior transfers pass close to L_1, exterior transfers pass close to L_2)

(4) $C_2 < C < C_1$: interior transfers from 1 to 2 are feasible; motion outside the greater curve is feasible no exterior transfer at L_2 is feasible

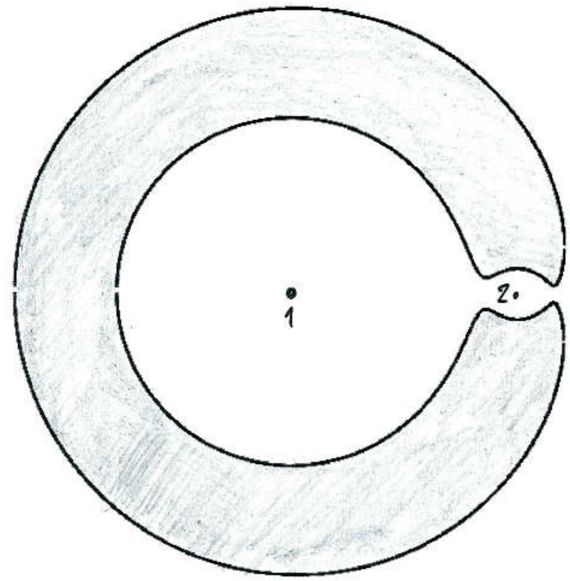
(5) $C_1 < C$: no interior or exterior transfer is feasible; the spacecraft remains confined either (i) in the proximity of primary 1, or (ii) in the proximity of primary 2, or (iii) outside the greater zero velocity curve

It is apparent that $C_1, C_2, C_3, C_{4,5}$ represent very meaningful values for understanding feasibility of a trajectory.

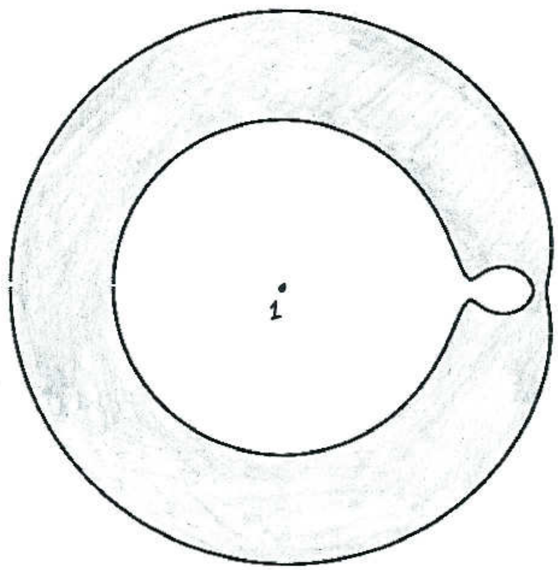
(2)



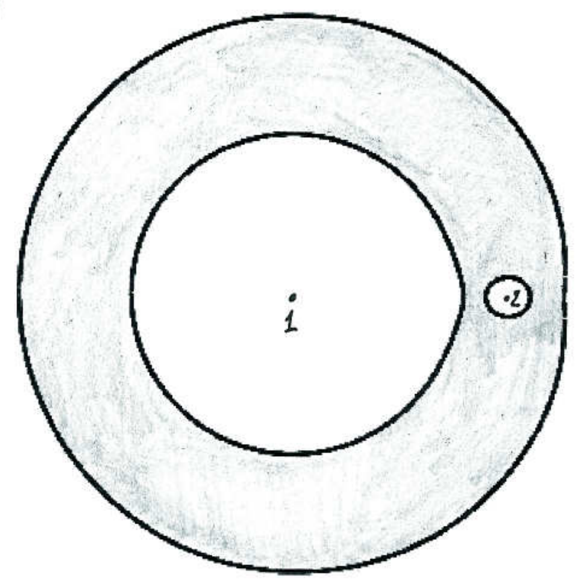
(3)



(4)



(5)



In these figures the forbidden region is shaded (grey).

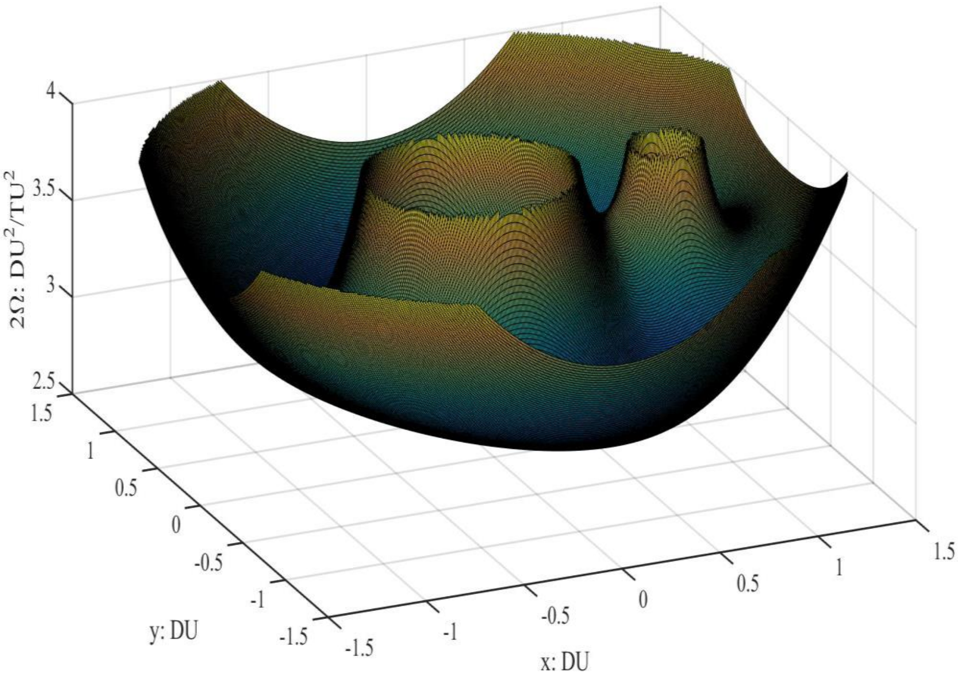
From (2) to (5) the energy decreases, and this means that C increases

The next figure portrays the coordinates of L_i as μ varies.

Moreover, for the Earth-Moon system $\left(\mu = \frac{1}{82.27}\right)$ the characteristic values of C_1 through C_4 are reported.

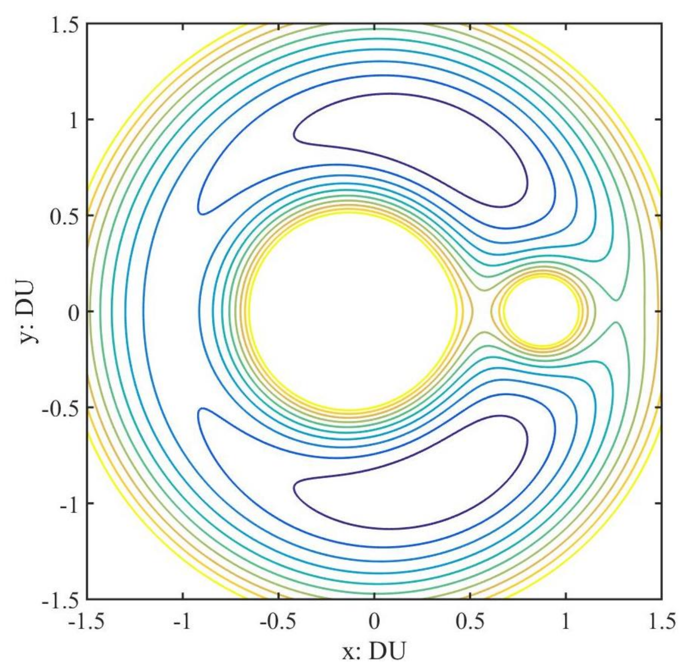
Function 2Ω(x,y)

If this surface is cut with a horizontal plane associated with a specific value of C, then the region of allowed motion corresponds to the portion of surface above this plane.

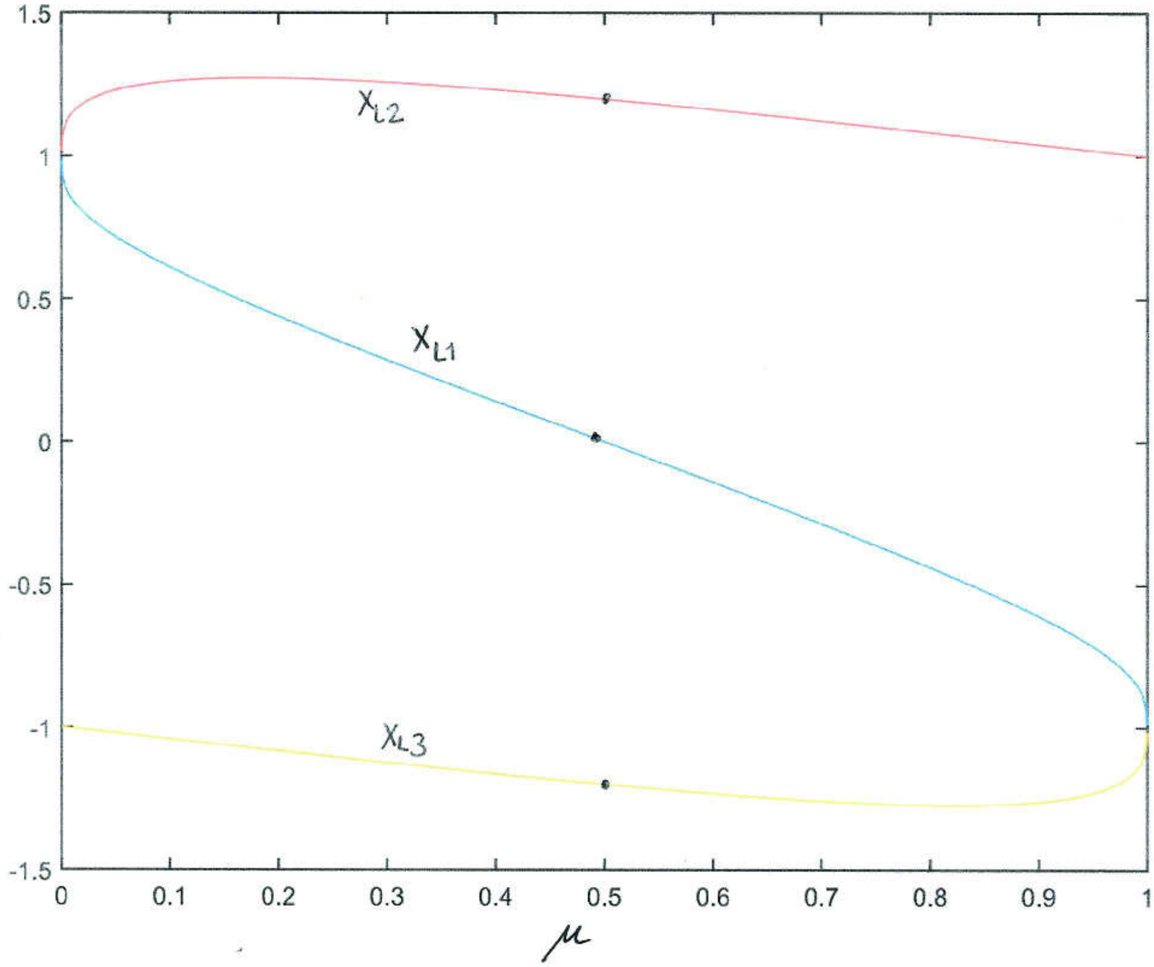


Contour plot of 2Ω(x,y)

This is obtained by cutting the preceding surface with different horizontal planes. Each plane is associated with a different value of C, therefore each curve corresponds to a different value C of the Jacobi integral. This means that in fact these curves are the zero velocity curves at different values of C



Position of the libration points (collinear) as a function of μ.



$\mu = \frac{1}{82.27}$ for the Moon

$C_1 = 3.18838273477815 \ \frac{DU^2}{TU^2}$ when $C_1 = 2\Omega(L_1)$

$C_2 = 3.17219608074121 \ \frac{DU^2}{TU^2}$ $C_2 = 2\Omega(L_2)$

$C_3 = 3.01215166144792 \ \frac{DU^2}{TU^2}$ $C_3 = 2\Omega(L_3)$

$C_4 = 2.9879926473692 \ \frac{DU^2}{TU^2}$ $C_4 = 2\Omega(L_4)$

- Stability of libration points

Let (x_0, y_0) be the coordinates of a libration point;
(ξ, η) are small displacements, i.e.

$$\begin{cases} x = x_0 + \xi \\ y = y_0 + \eta \end{cases} \qquad \underline{r_0} \leftrightarrow (x_0, y_0)$$

Using $\omega = 1\ \mathrm{TU}^{-1}$, the equations of motion are expanded to first order, to yield

$$\begin{cases} \ddot{x} - 2\dot{y} = \Omega_x \longrightarrow \ddot{x}_0 + \ddot{\xi} - 2(\dot{y}_0 + \dot{\eta}) = \Omega_x\big|_{\underline{r_0}} + \Omega_{xx}\big|_{\underline{r_0}}\xi + \Omega_{xy}\big|_{\underline{r_0}}\eta \\ \ddot{y} + 2\dot{x} = \Omega_y \longrightarrow \ddot{y}_0 + \ddot{\eta} + 2(\dot{x}_0 + \dot{\xi}) = \Omega_y\big|_{\underline{r_0}} + \Omega_{yx}\big|_{\underline{r_0}}\xi + \Omega_{yy}\big|_{\underline{r_0}}\eta \end{cases}$$

The partial derivatives of Ω are evaluated at $\underline{r_0}$, here and henceforth, but $|_{\underline{r_0}}$ is omitted, for the sake of brevity.
Because $\ddot{x}_0 = \ddot{y}_0 = 0$ and $\Omega_x = \Omega_y = 0$ (at $\underline{r_0}$), one obtains the linear, time-independent differential system

$$\ddot{\xi} - 2\dot{\eta} = \Omega_{xx}\xi + \Omega_{xy}\eta$$

$$\ddot{\eta} + 2\dot{\xi} = \Omega_{xy}\xi + \Omega_{yy}\eta$$

Letting $\underline{z} = [\xi \ \ \dot{\xi} \ \ \eta \ \ \dot{\eta}]^T$, the previous system can be rewritten as

$$\underline{\dot{z}} = A\underline{z}, \quad \text{where} \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \Omega_{xx} & 0 & \Omega_{xy} & 2 \\ 0 & 0 & 0 & 1 \\ \Omega_{yx} & -2 & \Omega_{yy} & 0 \end{bmatrix}$$

Stability of this linear differential system depends on the eigenvalues of A, given by solving

$$\det(A - \lambda I) = 0$$

$$\rightarrow \quad \lambda^4 - \lambda^2 \left(\Omega_{xx} + \Omega_{yy} - 4 \right) + \Omega_{xx} \Omega_{yy} - \Omega_{xy}^2 = 0$$

where $\{\Omega_{xx}, \Omega_{yy}, \Omega_{xy}\}$ are evaluated at L_i

(1) COLLINEAR LIBRATION POINTS

After several calculations one can obtain

$$\begin{cases} \Omega_{xx} + \Omega_{yy} - 4 = k_1 - 2 \\ \Omega_{xx} \Omega_{yy} - \Omega_{xy}^2 = (1 + 2k_1)(1 - k_1) \end{cases}$$

where k_1 is a constant, which can be proven to be $k_1 > 1$ for all collinear points.

Because $k_1 > 1$, $\Omega_{xx}\Omega_{yy} - \Omega_{xy}^2 < 0$ and therefore a solution for λ^2 exists such that $\lambda^2 > 0$

But $\lambda^2 > 0 \Rightarrow$ a positive and a negative real eigenvalue

$\Rightarrow$ λ real and positive implies INSTABILITY

i.e. collinear libration points are UNSTABLE

This means that if the spacecraft is placed in the proximity of L_i $(i = 1, 2, 3)$, with small ξ and η, then its dynamics is divergent (UNSTABLE equilibrium)

Linear periodic solutions are found only if the spacecraft is placed "along" the stable eigenvector.

(2) TRIANGULAR LIBRATION POINTS

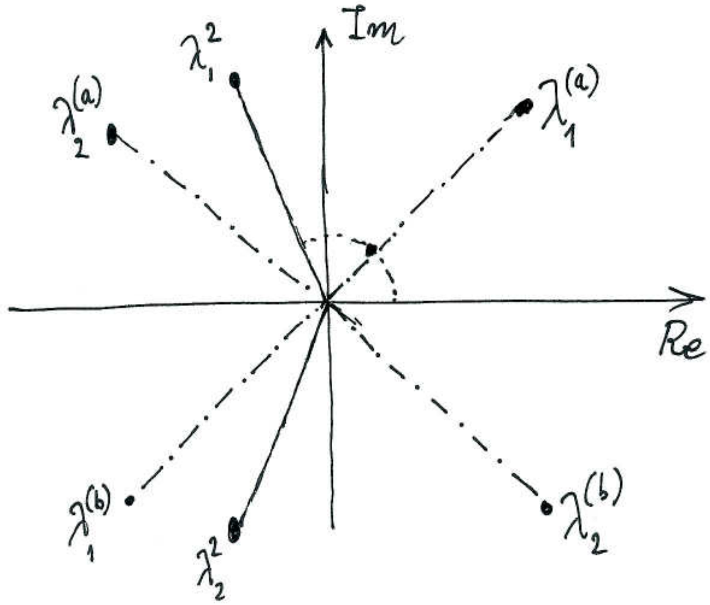
After several calculations one obtains

$$\Omega_{xx} = \frac{3}{4} \qquad \Omega_{yy} = \frac{9}{4} \qquad \Omega_{xy} = \pm \frac{3\sqrt{3}}{4}(1-2\mu)$$

$$\rightarrow \begin{cases} \Omega_{xx} + \Omega_{yy} - 4 = -1 < 0 & (a) \\ \Omega_{xx}\Omega_{yy} - \Omega_{xy}^2 = \frac{27}{4}\mu(1-\mu) > 0 & (b) \end{cases}$$

Condition (a) implies that λ^2 has negative real part

However any pair of complex conjugate $\lambda^2_{1,2}$ with negative real part admits $\{\lambda_1^{(a)}, \lambda_1^{(b)}, \lambda_2^{(a)}, \lambda_2^{(b)}\}$ with positive real part (2 of them, see figure)



The only way for $\{\lambda_1^{(a)}, \lambda_2^{(b)}, \lambda_2^{(a)}, \lambda_2^{(b)}\}$ with real part ≤ 0 is having $\lambda^2_{1,2}$ real and negative. In addition to (a) and (b) the following condition must hold

$$(\Omega_{xx} + \Omega_{yy} - 4)^2 - 4(\Omega_{xx}\Omega_{yy} - \Omega_{xy}) \geq 0$$

$$\rightarrow \quad 27\mu^2 - 27\mu + 1 \geq 0$$

This inequality has the following solution

$$0 \leq \mu \leq \underbrace{\frac{1}{2} - \frac{\sqrt{69}}{18}}_{\sim 0.0385} \qquad \text{OR} \qquad \underbrace{\frac{1}{2} + \frac{\sqrt{69}}{18}}_{\sim 0.9615} \leq \mu \leq 1$$

(a) If the two primaries have μ that satisfies one of these two inequalities, the equilibrium around L_4, L_5 is NEUTRALLY STABLE (according to the linear analysis)

(b) Instead, if $\frac{1}{2} - \frac{\sqrt{69}}{18} < \mu < \frac{1}{2} + \frac{\sqrt{69}}{18}$

then the equilibrium about L_4, L_5 is UNSTABLE

It is easy to check that for the Earth-Moon system condition (a) is satisfied

In summary

(a) Around L_1, L_2, L_3 : UNSTABLE equilibrium

(b) Around L_4, L_5: equilibrium is

> NEUTRALLY STABLE if $0 \leq \mu \leq \frac{1}{2} - \frac{\sqrt{69}}{18}$ or $\frac{1}{2} + \frac{\sqrt{69}}{18} \leq \mu \leq 1$

> UNSTABLE if $\frac{1}{2} - \frac{\sqrt{69}}{18} \leq \mu \leq \frac{1}{2} + \frac{\sqrt{69}}{18}$