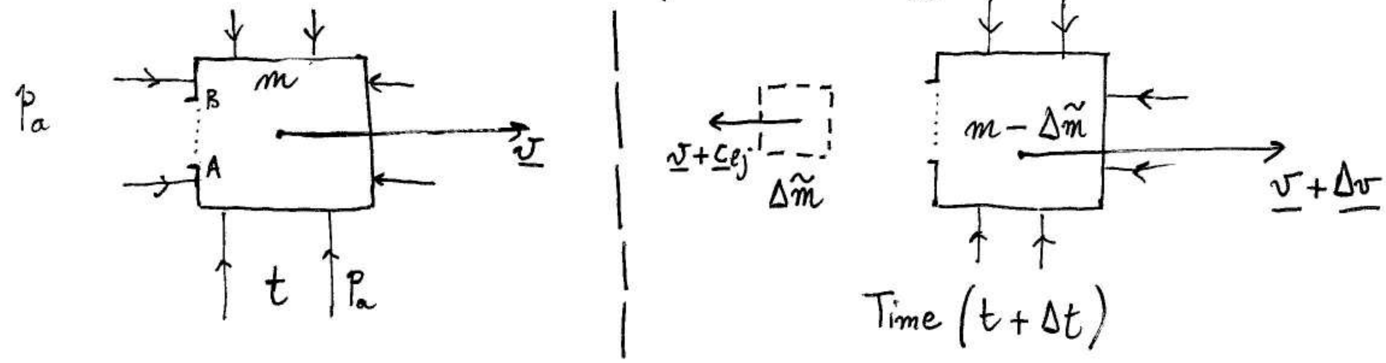
FUNDAMENTALS OF ROCKET DYNAMICS

NEWTON LAW WITH PROPULSION

A spacecraft travels with inertial velocity $\underline{v}$ at t and has mass m. The external pressure is p_a.



At the nozzle the pressure is p_e (section AB), area A_e

After Δt the overall system is composed of the ejected mass, which travels with velocity $(\underline{v}+\underline{c}_{ej})$, and the spacecraft, now with mass $(m-\Delta\tilde{m})$. $\underline{c}_{ej}$ is the exhausted mass velocity relative to the spacecraft.

The variation of momentum $\underline{M}$ is

$$\underline{M}(t+\Delta t)-\underline{M}(t) = \underbrace{(m-\Delta\tilde{m})(\underline{v}+\Delta\underline{v})+\Delta\tilde{m}(\underline{v}+\underline{c}_{ej})}_{\underline{M}(t+\Delta t)} - \underbrace{m\underline{v}}_{\underline{M}(t)}$$

The second Newton law is

$$\frac{d\underline{M}}{dt} = \underline{F_{ext}}$$ where the external forces are, in general,

$$\underline{F_{ext}} = \underline{G}+\underline{A}+\underline{P_{ext}}$$

i.e. $\underline{G}$ = gravitational force

$\underline{A}$ = aerodynamics force

$\underline{P_{ext}}$ = external pressure force

The latter term, $\underline{P_{ext}}$ has the following expression, with reference to the figure

$$\underline{P_{ext}} = \Big[\underbrace{p_a(A_1 - A_e) + p_e A_e}_{\text{pressure force on the left vertical surface}} \; \underbrace{- p_a A_1}_{\text{pressure force on the right vertical surface}}\Big]\hat{v} \qquad (A_1 \text{ is the vertical surface})$$

The forces on the horizontal surfaces balance, then their effect vanishes. Therefore

$$\underline{P_{ext}} = (p_e - p_a) A_e \hat{v}$$

The Newton law becomes

$$\lim_{\Delta t \to 0} \frac{-\cancel{\Delta\tilde{m}\,\underline{v}} + m\,\Delta\underline{v} - \Delta\tilde{m}\,\Delta\underline{v} + \cancel{\Delta\tilde{m}\,\underline{v}} + \Delta\tilde{m}\,\underline{c_{ej}} + \cancel{m\underline{v}} - \cancel{m\underline{v}}}{\Delta t} =$$

$$= m\frac{d\underline{v}}{dt} + \dot{\tilde{m}}\,\underline{c_{ej}} = \underline{G} + \underline{A} + A_e(p_e - p_a)\,\hat{v}$$

i.e.

$$\frac{d\underline{v}}{dt} = \frac{\underline{G} + \underline{A}}{m} + \frac{A_e(p_e - p_a)\hat{v} - \dot{\tilde{m}}\,\underline{c_{ej}}}{m}$$

But $\dot{\tilde{m}} = -\dot{m}$, hence

$(\Delta\tilde{m} = -\Delta m)$

$$\frac{d\underline{v}}{dt} = \frac{\underline{G} + \underline{A}}{m} + \frac{\dot{m}\,\underline{c_{ej}} + A_e\,(p_e - p_a)\,\hat{v}}{m}$$

Letting $\quad \underline{T} = \dot{m}\,\underline{c_{ej}} + A_e\left(p_e - p_a\right)\hat{v} = \dot{m}\,\underline{c}$

where $\quad \underline{c} = \underline{c_{ej}} + \dfrac{A_e\,(p_e - p_a)\,\hat{v}}{\dot{m}}$ is the effective exhaust velocity, which takes into account the pressure term

one obtains

$$\underline{T} = \dot{m}\,\underline{c} \quad // \left(-\underline{c}\right) \quad \text{because} \quad \dot{m} < 0$$

i.e.

$$\begin{cases} \dot{m} = -\dfrac{T}{c} \\ \\ \text{and} \quad \dfrac{d\underline{v}}{dt} = \dfrac{\underline{G} + \underline{A} + \underline{T}}{m} \end{cases}$$

$\underline{T}$ is the propulsive thrust, directed along $-\underline{c}$ (opposite to the direction where mass is ejected)

Usually, the effective exhaust velocity $\underline{c}$ is expressed in terms of SPECIFIC IMPULSE I_{sp} as

$c = g_0 I_{sp}$ where I_{sp} has the unit of sec and g_0 is the gravitational acceleration at sea level

- Tsiolkovsky's law

This fundamental law relates the propellant mass consumption with the required velocity change.

Assumptions : - no aerodynamic force affects the motion
- no gravitational force affects the motion
- thrust aligned with the spacecraft velocity

$\Rightarrow$ Tsiolkovsky's law can be applied to

(a) impulsive maneuvers, where the thrust interval is negligible, which implies negligible aerodynamics and gravitational effects on the spacecraft

(b) motion in the absence of $\underline{G}$ and $\underline{A}$

Due to the assumptions, the Tsiolkovsky's law provides

(i) the maximum Δv attainable from a mass variation Δm

OR

(ii) the minimum propellant consumption for a given Δv

In other words, the Tsiolkovsky's law indicates the limiting performance attainable from a space vehicle

Due to the assumptions,

$$\left.\begin{aligned} &(a)\ \frac{dv}{dt} = \frac{T}{m} \\ &(b)\ \dot{m} = -\frac{T}{c} \end{aligned}\right\} \longrightarrow \dot{v} = -\frac{c\,\dot{m}}{m}$$

$$v_f - v_0 = -c \ln \frac{m_f}{m_0}$$

$$\rightarrow \quad v_f - v_0 = c \ln \frac{m_0}{m_f} \qquad \Delta v \text{ if } \frac{m_0}{m_f} \text{ is given}$$

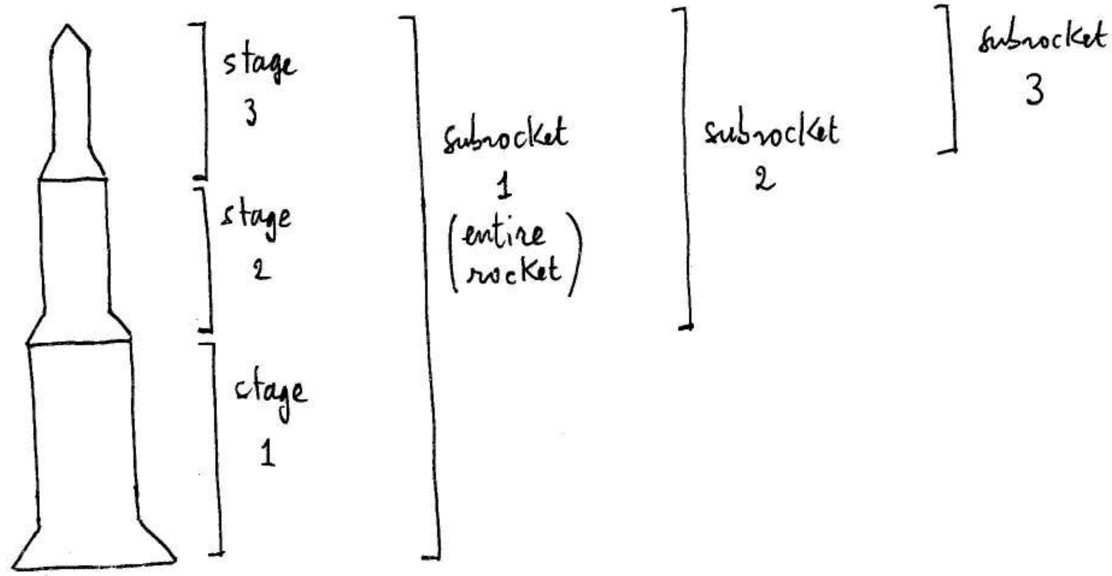
$$\rightarrow \quad \underbrace{m_f = m_0\, e^{-\frac{v_f - v_0}{c}}}_{\text{TSIOLKOVSKY'S LAW}} \qquad \frac{m_f}{m_0} \text{ if } \Delta v \text{ is given}$$

It is apparent that high values of c are desirable.

For a space vehicle the ratio $\mu_0 = \frac{m_f}{m_0}$ determines the maximum velocity increase that is attainable with a propellant mass $m_p = (m_0 - m_f)$

ROCKET STAGING

Rocket staging is aimed at increasing the overall Δv attainable from a launch vehicle, by throwing away structural masses that have become useless.



Each subrocket has initial mass $m_0^{(i)}$, composed of three terms:

$$m_0^{(i)} = m_s^{(i)} + m_p^{(i)} + m_U^{(i)}$$

$m_s^{(i)}$ = structural mass

$m_p^{(i)}$ = propellant mass

$m_U^{(i)}$ = initial mass of the succeeding subrocket or payload mass (for the last subrocket only)

i.e. $m_0^{(i+1)} = m_U^{(i)}$

The mass distribution is identified by means of two parameters

(a) $e_s^{(i)} = \dfrac{m_s^{(i)}}{m_s^{(i)} + m_p^{(i)}}$ (b) $u_0^{(i)} = \dfrac{m_f^{(i)}}{m_0^{(i)}} = \dfrac{m_s^{(i)} + m_U^{(i)}}{m_0^{(i)}}$

The structural coefficient ϵ_s is directly related to the technology available for building the stage i. Therefore, a lower bound exists for $\epsilon_s^{(i)}$: $\epsilon_s^{(i)} \geq \epsilon_{s\,min}^{(i)}$

Overall, given the following three relations:

$$(1) \quad m_0^{(i)} = m_s^{(i)} + m_p^{(i)} + m_v^{(i)}$$

$$(2) \quad \epsilon_s^{(i)} = \frac{m_s^{(i)}}{m_s^{(i)} + m_p^{(i)}}$$

$$(3) \quad \mu_0^{(i)} = \frac{m_s^{(i)} + m_v^{(i)}}{m_s^{(i)} + m_v^{(i)} + m_p^{(i)}} = \frac{m_0^{(i)} - m_p^{(i)}}{m_0^{(i)}}$$

one obtains

$$m_p^{(i)} = m_0^{(i)} \left[1 - \mu_0^{(i)}\right] \qquad \text{from } (3)$$

$$m_s^{(i)} = m_0^{(i)} \, \frac{\epsilon_s^{(i)}\left[1 - \mu_0^{(i)}\right]}{1 - \epsilon_s^{(i)}} \qquad \text{from } (2)$$

$$m_v^{(i)} = m_0^{(i)} \, \frac{\mu_0^{(i)} - \epsilon_s^{(i)}}{1 - \epsilon_s^{(i)}} \qquad \text{from } (1)$$

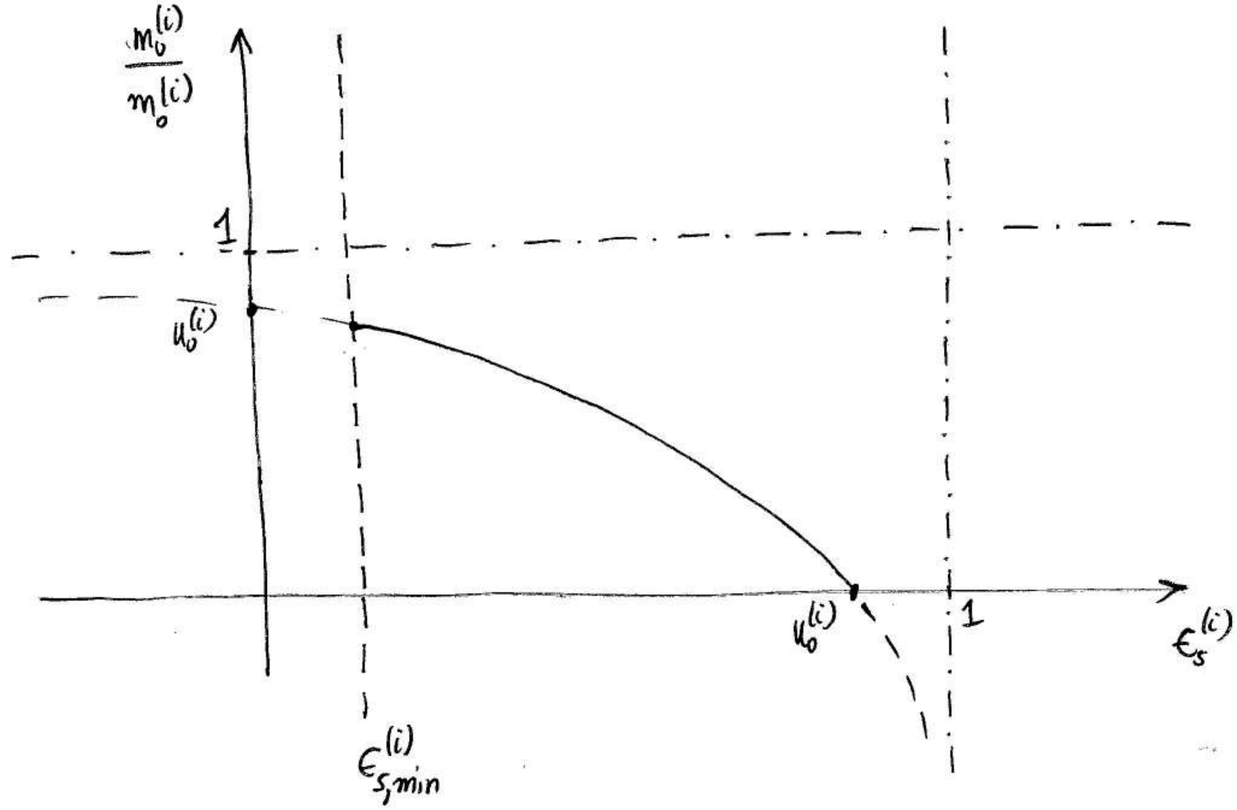
As a result of propulsion of the stage i (subrocket i) one gets a $\Delta v^{(i)}$,

$$\Delta v^{(i)} = -c_i \ln \mu_0^{(i)}$$

where c_i is the effective exhaust velocity for subrocket i

The expression for $m_U^{(i)}$ is $\quad m_U^{(i)} = m_0 \frac{u_0^{(i)} - \epsilon_S^{(i)}}{1 - \epsilon_S^{(i)}}$

Hence $m_U^{(i)} > 0$ only if $u_0^{(i)} > \epsilon_S^{(i)}$



From this plot, two facts are apparent

(a) $\quad \epsilon_S^{(i)} = u_0^{(i)} \implies \frac{m_U^{(i)}}{m_0^{(i)}} = 0$

(b) $\quad \frac{m_U^{(i)}}{m_0^{(i)}}$ is maximized at $\epsilon_S^{(i)} = \epsilon_{S,min}^{(i)}$

Point (b) means that small values of $\epsilon_S^{(i)}$ correspond to greater values of $\frac{m_U^{(i)}}{m_0^{(i)}}$

In other words, low structural coefficients are desirable in order to improve the subrocket performance, in terms of $\frac{m_U^{(i)}}{m_0^{(i)}}$

OPTIMAL ROCKET STAGING

This process is aimed at maximizing the overall Δv attainable from a multistage launch vehicle, by choosing the most appropriate mass distribution

In mathematical terms, the problem is in maximizing

$$\Delta v = \sum_{i=1}^{N} \Delta v^{(i)} = - \sum_{i=N}^{N} c_i \ln u_0^{(i)}$$

for given value of $\frac{m_U^{(N)}}{m_0^{(1)}}$ = ratio $\frac{\text{payload}}{\text{initial mass}}$

This value $\frac{m_U^{(N)}}{m_0^{(1)}}$ is given, therefore it can be regarded as a constraint for the optimization process,

$$\frac{m_U^{(N)}}{m_0^{(1)}} = \frac{m_U^{(N)}}{m_0^{(N)}} \frac{m_U^{(N-1)}}{m_0^{(N-1)}} \cdots \frac{m_U^{(1)}}{m_0^{(1)}} = \prod_{i=1}^{N} \frac{m_U^{(i)}}{m_0^{(i)}}$$

$$m_0^{(i)} = m_U^{(i-1)}$$

$$\rightarrow \frac{m_U^{(N)}}{m_0^{(1)}} = \prod_{i=1}^{N} \frac{u_0^{(i)} - \epsilon_s^{(i)}}{1 - \epsilon_s^{(i)}}$$

Unknowns are $\left\{ u_0^{(i)} \right\}_{i=1,..,N}$, whereas $\left\{ \epsilon_s^{(i)} \right\}$ are given

$\longrightarrow$ the problem is a constrained optimization problem

- Solution for a special case

Assume $\begin{cases} \epsilon_s^{(i)} = \epsilon_s & \text{same structural coefficient for all stages} \\ c_i = c & \text{same eff. exhaust velocity for all stages} \end{cases}$

the problem becomes

Maximize $\Delta v = -c \sum_{i=1}^{N} \ln u_0^{(i)}$

subject to $$\frac{m_U^{(N)}}{m_0^{(1)}} = \prod \frac{u_0^{(i)} - \epsilon_s}{1-\epsilon_s}$$

The problem can be formulated as follows:

min H where $$H = c \sum \ln u_0^{(i)} + \lambda \left[\frac{m_U^{(N)}}{m_0^{(1)}} - \frac{\prod_{i=1}^{N} \left[u_0^{(i)} - \epsilon_s \right]}{(1-\epsilon_s)^N} \right]$$

The minimizing solution must satisfy two conditions:

(a) $$\frac{\partial H}{\partial \lambda} = 0 \longrightarrow \frac{m_U^{(N)}}{m_0^{(1)}} = \frac{1}{(1-\epsilon_s)^N} \prod_{i=1}^{N} \left[u_0^{(i)} - \epsilon_s \right]$$

(b) $$\frac{\partial H}{\partial u_0^{(i)}} = 0 \longrightarrow \frac{c}{u_0^{(i)}} - \frac{\lambda}{(1-\epsilon_s)^N} \prod_{\substack{j=1 \\ j \neq i}}^{N} \left[u_0^{(j)} - \epsilon_s \right] \quad (i=1,..,N)$$

These are $(N+1)$ equations in $(N+1)$ unknowns, i.e.

$$\left\{ u_0^{(i)} \right\}_{i=1,..,N} \text{ and } \lambda$$

As a first step, λ is derived from (b)

$$\lambda = \frac{c\left(1-\epsilon_s\right)^N}{u_0^{(1)} \prod_{\substack{j=1 \\ j\neq 1}}^{N} \left[u_0^{(j)} - \epsilon_s\right]} = \dots = \frac{c\left(1-\epsilon_s\right)^N}{u_0^{(N)} \prod_{\substack{j=1 \\ j\neq N}}^{N} \left[u_0^{(j)} - \epsilon_s\right]}$$

which is equivalent to

$$\frac{c\left(1-\epsilon_s\right)^N \left[u_0^{(1)} - \epsilon_s\right]}{u_0^{(1)}} = \dots = \frac{c\left(1-\epsilon_s\right)^N \left[u_0^{(N)} - \epsilon_s\right]}{u_0^{(N)}}$$

i.e.

$$\frac{u_0^{(k)} - \epsilon_s}{u_0^{(k)}} = \frac{u_0^{(1)} - \epsilon_s}{u_0^{(1)}} \quad \text{i.e.} \quad u_0^{(k)} u_0^{(1)} - \epsilon_s u_0^{(1)} = u_0^{(1)} u_0^{(k)} - \epsilon_s u_0^{(k)}$$

$\Rightarrow \quad u_0^{(k)} = u_0^{(1)} \quad$ i.e. $\quad u_0^{(k)} = u_0 \Rightarrow$ all subrockets

must have the same parameter u_0

If now this result is inserted into the constraint equation, one obtains

$$\frac{m_U^{(N)}}{m_0^{(1)}} = \left(\frac{u_0 - \epsilon_s}{1-\epsilon}\right)^N \rightarrow \quad u_0 = \epsilon + \left(1-\epsilon\right)\left[\frac{m_U^{(N)}}{m_0^{(1)}}\right]^{\frac{1}{N}}$$

and the maximum Δv is

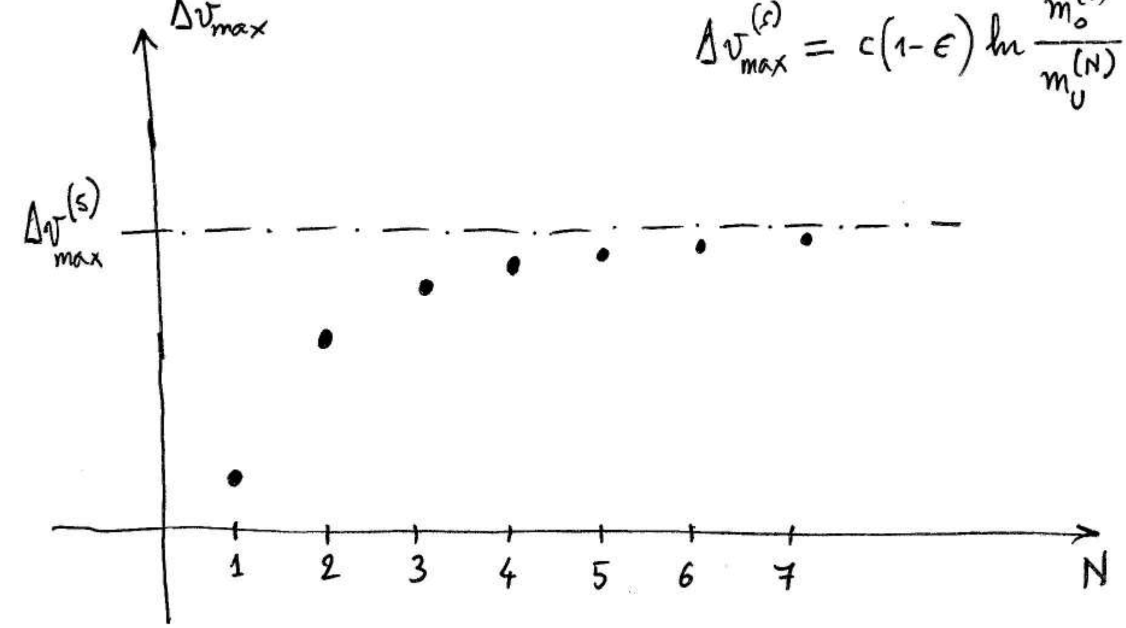
$$\Delta v_{max} = -cN \ln\left\{\epsilon + \left(1-\epsilon\right)\left[\frac{m_U^{(N)}}{m_0^{(1)}}\right]\right\}^{\frac{1}{N}}$$

This function as a finite limit has $N \rightarrow \infty$

$\Rightarrow$ there $\exists$ a limiting value for Δv; this "supremum" value is the upper bound, which can be never outperformed

It is interesting to plot Δv_{max} as a function of the number of stages N

$$\Delta v_{max}^{(s)} = c(1-\epsilon)\ln\frac{m_o^{(1)}}{m_U^{(N)}}$$



Increasing the number of stages implies increasing Δv_{max}

However, the advantage of increasing N is smaller and smaller as N increases.

Conversely, structural and operational complexity increases as N increases.

These two considerations explain why multistage launch vehicles usually are composed of no more than 5 stages (often, 4 or 3 stages).

ASCENT TRAJECTORY OF A LAUNCH VEHICLE

This section outlines some basic characteristics of the ascent path of a multistage launch vehicle.

This path includes two main phases:

(a) ATMOSPHERIC arc, when the atmospheric density is nonnegligible (altitudes lower than 40 km)

(b) EXOATMOSPHERIC (SPACE) arc, where the atmospheric density has negligible effects (typically, over 40 km)

During phase (a), suitable countermeasures must be taken in order to preserve the structural integrity of the launch vehicle.

Phases of the ascent trajectory

Typical phases of an ascent path are

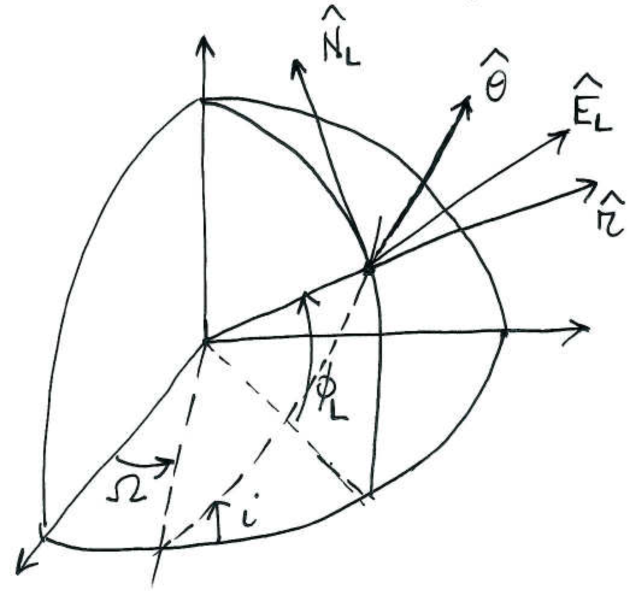
(a) VERTICAL ARC at launch: right after ignition of the engines of the first stage, the launch vehicle ascends vertically, for two main reasons:

(i) Launch site security: exhausted gases must be directed vertically at ignition; each launch site has a mandatory duration of the vertical arc

(ii) Avoid premature rotation of the velocity vector toward a horizontal flight condition

(b) ROLL MANEUVER : several launch vehicles roll about their longitudinal axis, in order to assume the correct attitude for the subsequent pitch maneuver

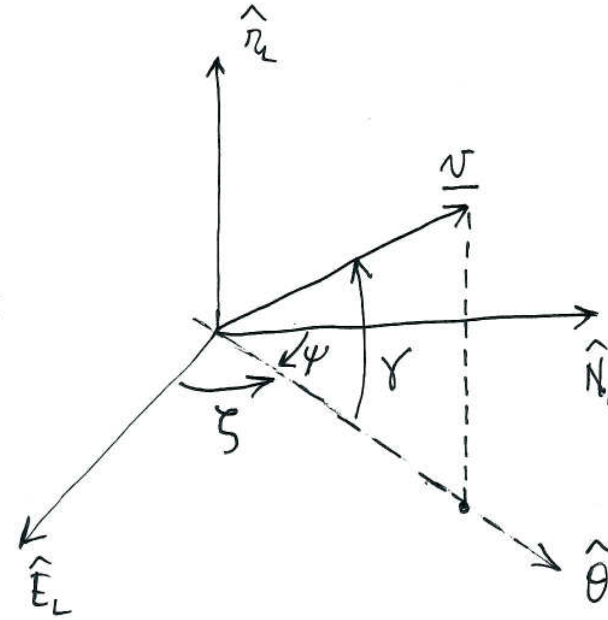
(c) PITCH MANEUVER : this second attitude maneuver is needed in order to complete the selection of the orbit plane. In fact, we can assume that after (a) and (b) the velocity of the launch vehicle is directed radially, i.e. along $\hat{r}$. The pitch maneuver has the effect of rotating the velocity in order that proper values of γ and ζ be selected (cf. lower figure, which refers to the end of the pitch maneuver). In this figure $(\hat{E}_L, \hat{N}_L)$ identify the local horizontal plane, where $\hat{\theta}$ lies; the angle ψ, taken clockwise from $\hat{N}_L$ is the AZIMUTH angle. In Section 2 (Keplerian trajectories) it was found that



ϕ_L = latitude of the launch site

$$\cos i = \cos \zeta \cos \phi$$

$$\Rightarrow \cos i = \cos \phi \sin \psi$$



γ = flight path angle ζ = heading angle

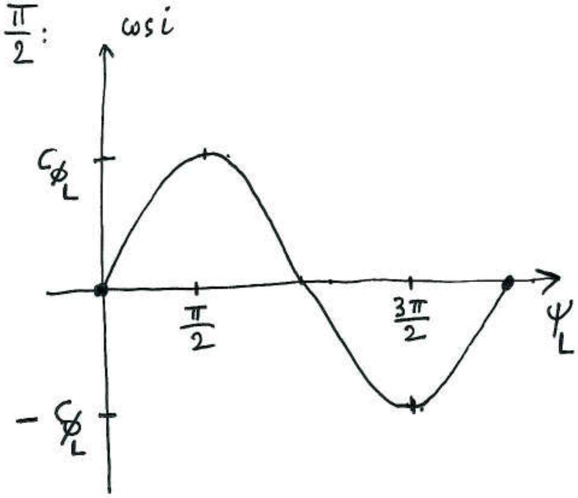
The latter relation, written with reference to the latitude of the launch site is

$$\cos i = \cos\phi_L \sin\psi_L \qquad \text{or} \qquad \cos i = \cos\phi_L \cos\zeta_L$$

and is termed EQUATION of the launch site.

Its meaning is that the orbit plane inclination can be selected by choosing ψ_L (azimuth after pitch maneuver), or equivalently ζ_L (heading after pitch maneuver)

However, it is apparent that the minimum inclination for a given latitude ϕ_L occurs at $\psi_L = \frac{\pi}{2}$:



ⓐ $$i_{min} = \mathrm{acos}\left[c_{\phi_L}\right] = \begin{cases} \phi_L & \text{if } \phi_L > 0 \\ -\phi_L & \text{if } \phi_L < 0 \end{cases}$$

LAUNCH TOWARD EAST ($\psi_L = \frac{\pi}{2}$)

whereas the maximum inclination for a given latitude ϕ_L occurs at $\psi_L = \frac{3\pi}{2}$:

ⓑ $$i_{max} = \mathrm{acos}\left[-c_{\phi_L}\right] = \begin{cases} \pi - \phi_L & \text{if } \phi_L > 0 \\ \pi + \phi_L & \text{if } \phi_L < 0 \end{cases}$$

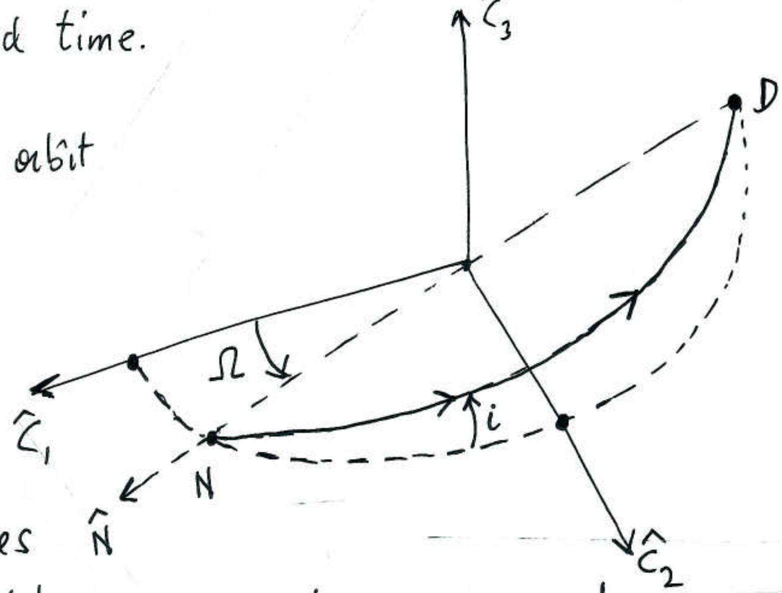
LAUNCH TOWARD WEST ($\psi_L = \frac{3\pi}{2}$)

It is worth remarking that launches toward West are really not frequent, because the Earth rotation is toward East, and therefore launching toward East is more advantageous.

In conclusion, for a given ϕ_L, inclinations i in the range

$$i_{min} \le i \le i_{max} \qquad \text{are allowed}$$

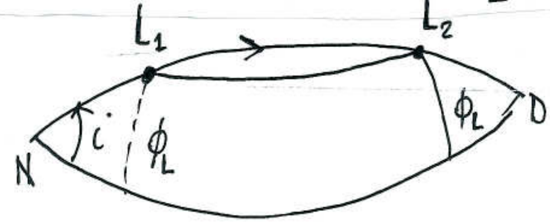
It is worth remarking that the RAAN Ω, which is the other angle needed to identify the orbit plane, can be selected by choosing the launch date and time.



In conclusion, for a desired orbit with specified Ω and i;

$$|\phi_L| \leq i \leq \pi - |\phi_L|$$

and two launch opportunities exist (at most), from L_1 and L_2

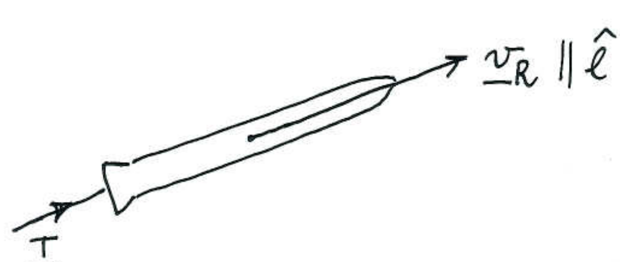


These two points are the two intersections of the desired orbit plane with the circle described by the launch site during the Earth rotation.

Desired orbit is traveled in the direction $\rightarrow$

N = ascending node D = descending n.

(d) GRAVITY TURN: after completing the pitch maneuver the velocity is usually aligned with the longitudinal axis of the vehicle, for the purpose of avoiding flexure stresses that could be destructive (these appear if $\underline{v}$ is not aligned with the longitudinal axis).



Gravity turn flight condition

As the thrust direction is usually near-coincident with the longitudinal axis, the gravity turn condition implies aligning $\underline{T}$ (thrust direction) and $\underline{v}_R$ (velocity relative to the atmosphere)

In this phase, which usually involves the first and second subrocket, gravity turns the trajectory, and the name of this phase is due to this.

Some military-derived rocket can bear also a limited incidence angle α between $\underline{v}$ and $\hat{\ell}$ (longitudinal axis). In this case the condition for safe flight is usually expressed as

$$q\alpha \leq (q\alpha)_{max} \qquad q = \frac{1}{2}\rho v_R^2 \quad \left(\rho = \begin{array}{c}\text{atmosph.}\\ \text{density}\end{array}\right)$$

where q is dynamic pressure and $(q\alpha)_{max}$ is a characteristic limiting value that cannot be exceeded.

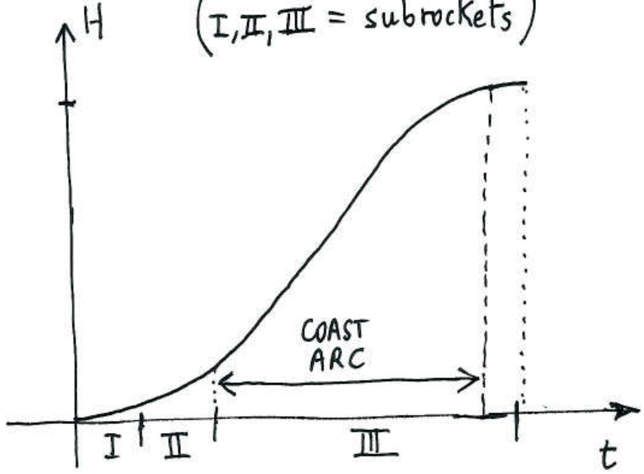
(e) RAREFIED ATMOSPHERIC ARC: when the atmosphere becomes rarefied, an incidence angle can exist, while satisfying the constraint on $(q\alpha)$ described previously; this arc usually ends at burnout of the second or third stage, when atmospheric density is negligible

(f) COAST ARC: in this ballistic arc, no aerodynamic force affects the space vehicle, and the motion is (nearly) Keplerian. The launch vehicle increases its altitude while velocity drops.

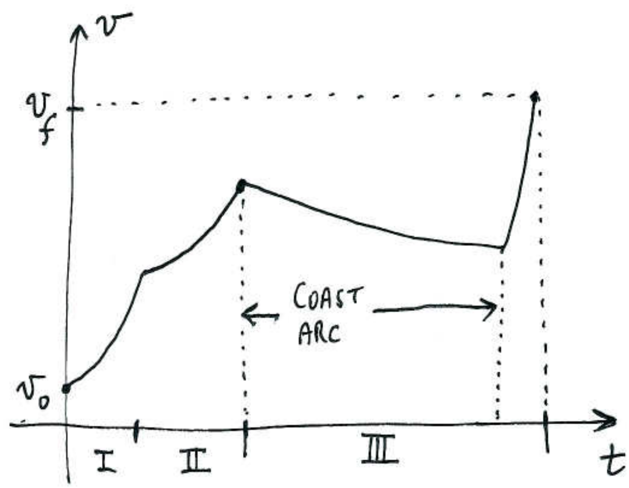
(g) INJECTION ARC: the upper stage is finally ignited, for a short time interval (usually tens of seconds or a few minutes, at most). This ignition occurs when the launch vehicle has reached approximately the altitude of the final desired circular orbit. During this powered arc,

> the velocity increases up to reaching the desired orbital value
> the altitude remains substantially unaltered

(I, II, III = subrockets)



Altitude time history



Inertial velocity time history

The inertial velocity has initial value

$$v_0 = R_E \, \omega_E \, c_{\phi_L}$$

ω_E = Earth rotation rate
R_E = Earth radius

due to the Earth rotation velocity at the launch site, and the final velocity v_f is

$$v_f = \sqrt{\frac{\mu_E}{R_f}}$$

μ_E = Earth gravitat parameter

R_f = radius of the final (circular) orbit

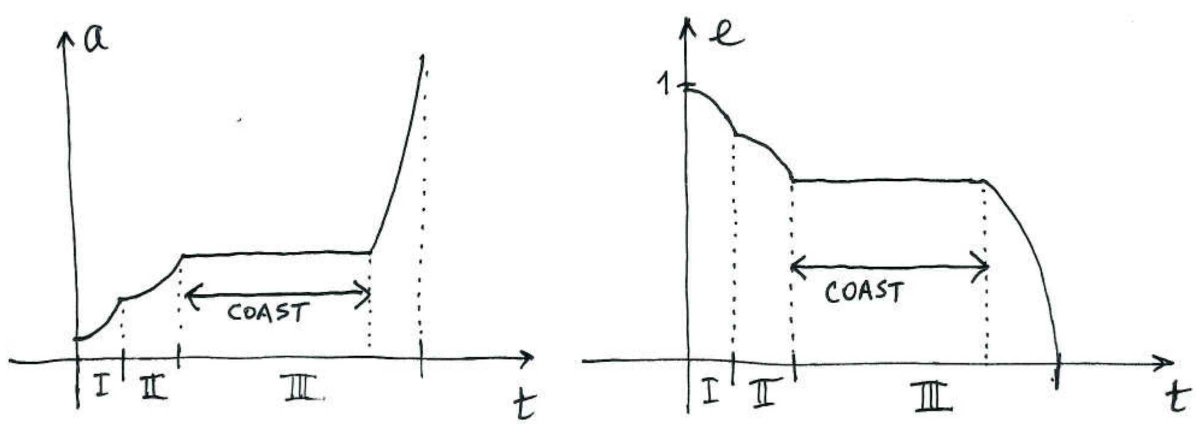
- <u>Osculating orbit elements</u>

The orbit elements vary during the ascent path. Their instantaneous (osculating) value can be regarded as the value of the orbit element if any thrust or perturbing force would cease at time t and hence forward.

During the ascent path

(a) Ω and i vary in the early phases, and remain substantially unaltered in the subsequent arcs
→ orbit plane selection occurs in the very first phases (this is consistent with what was discussed with regard to the roll and pitch maneuvers)

(b) a and e vary considerably in the last phases; in particular, during the injection phase (g)
→ orbit shape is changed during the injection arc (g), which circularizes the orbit



Right after launch, $e \simeq 1$, i.e. the trajectory is nearly-rectilinear.

- Losses equation for rockets

The Tsiolkovsky formula provides the limiting (maximum) Δv attainable for a specified mass distribution.

It is found under several assumptions, i.e.

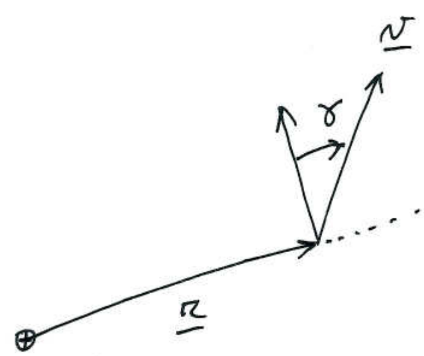
(a) Thrust as the only force that affects the motion, and

(b) Thrust aligned with $\underline{v}$ at all times

However, these hypotheses are not met for launch vehicles, and the general vector equation is to be considered:

$$\frac{d\underline{v}}{dt} = \frac{\underline{G} + \underline{A} + \underline{T}}{m}$$

where
$$\begin{cases} \underline{G} = -\frac{\mu_E}{r^2}\hat{r} & \text{gravitational force} \\ \underline{A} = \text{aerodynamic force} \\ \underline{T} = \text{Thrust} \end{cases}$$



This vector equation is projected along $\hat{v}$, unit vector aligned with $\underline{v}$, to yield

$$\frac{dv}{dt} = -\frac{\mu}{r^2}S_\gamma + \frac{T}{m}C_\alpha - \frac{D}{m}$$

where α = incidence angle (or misalignment angle) between $\underline{T}$ and $\underline{v}$

D = aerodynamic drag force magnitude

(other components, such as lift and side force, are assumed as negligible)

The previous equation is rewritten as

$$\frac{dv}{dt} = -\frac{\mu}{r^2} s_\gamma - \frac{T}{m}\left(1 - c_\alpha\right) - \frac{D}{m} + \frac{T}{m}$$

$$\rightarrow \quad \Delta v = \int_{v_0}^{v_f} dv = \underbrace{\int_{t_0}^{t_f} \frac{T}{m}\, dt}_{\Delta v_T} - \underbrace{\int_{t_0}^{t_f} \frac{\mu}{r^2} s_\gamma\, dt}_{\Delta v_G} - \underbrace{\int_{t_0}^{t_f} \frac{T}{m}\left(1 - c_\alpha\right) dt}_{\Delta v_M} - \underbrace{\int_{t_0}^{t_f} \frac{D}{m}\, dt}_{\Delta v_D}$$

$$\rightarrow \quad \Delta v = \Delta v_T - \Delta v_G - \Delta v_M - \Delta v_D$$

where
$$\begin{cases} \Delta v_T = \text{velocity change of Tsiolkovsky} \\ \Delta v_G = \text{gravity loss} \\ \Delta v_M = \text{misalignment loss} \\ \Delta v_D = \text{drag loss} \end{cases}$$

The last three terms are losses of Δv due to the

(i) action of gravity (ii) misalignment angle α (iii) drag

During the early stages of ascent, γ is large and the gravitation loss dominates, whereas the gravity turn flight condition $(\alpha = 0)$ implies $\Delta v_M = 0$. Then, as the velocity increases, D increases, while γ starts decreasing and so does Δv_G. When $\alpha \neq 0$, also the misalignment loss plays a role.

The losses equation holds for each subrocket, although the previous comments regard the first subrocket.

For the VEGA launch vehicle, the following typical values are obtained:

$$\Delta v_T = 3.054 \ \frac{km}{sec}$$

$$\Delta v_G = 0.683 \ \frac{km}{sec}$$

$$\Delta v_M = 0.019 \ \frac{km}{sec}$$

$$\Delta v_D = 0.107 \ \frac{km}{sec}$$

In conclusion, the following comments can be drawn:

(a) The thicker layers of atmosphere are crossed along a near-radial trajectory, in order to limit drag losses;

(b) As sooon as atmospheric density decreases, the velocity rotates so as to decrease γ, in order to avoid excessive gravitational losses.

EQUATIONS OF SPACEFLIGHT

The previous vector equation for $\underline{v}$ is accompanied by the Kinematics equation for $\underline{r}$,

(1) $$\frac{d\underline{r}}{dt} = \underline{v}$$ Kinematics equations

(2) $$\frac{d\underline{v}}{dt} = \frac{\underline{G}+\underline{A}+\underline{T}}{m}$$ Dynamics equations

These two vector equations can be projected along proper frames, in order to obtain the three-dimensional equations of spaceflight.

As exoatmospheric motion is being considered, $\underline{A} = \underline{0}$ (no aerodynamic force). Moreover, the mass equation holds, in addition to (1) and (2) :

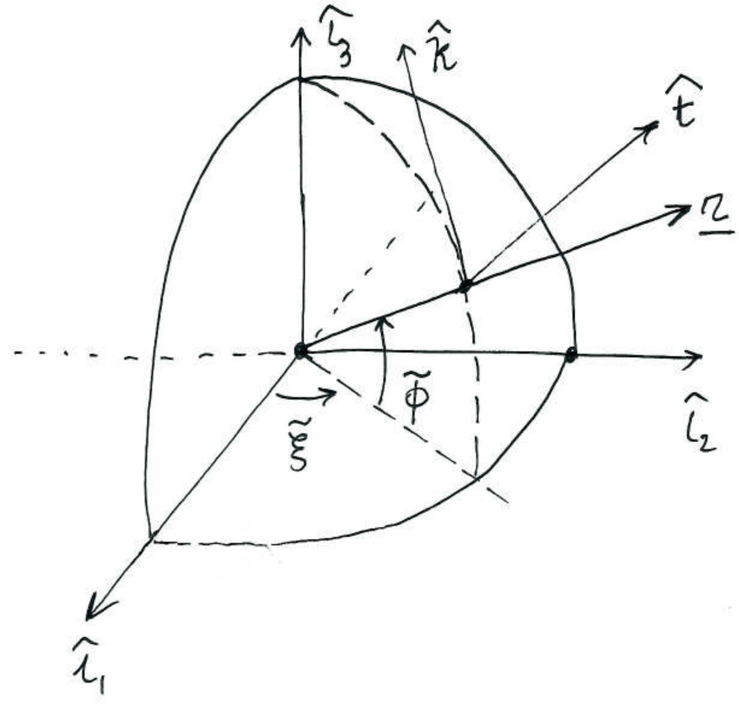
$$\dot{m} = -\frac{T}{c}$$

T = thrust magnitude

c = effective exhaust velocity

Let $(\hat{\iota}_1, \hat{\iota}_2, \hat{\iota}_3)$ denote three inertial directions (not necessarily those of the ECI-frame)

$\underline{r}$ = spacecraft position vector



The position vector $\underline{r}$ is identified by means of

$$r = |\underline{r}|$$

$\tilde{\xi}$ = right ascension, taken counterclockwise from $\hat{\iota}_1$

$\tilde{\phi}$ = declination

whereas the velocity vector $\underline{v}$ can be projected along $(\hat{r}, \hat{t}, \hat{k})$

$$\underline{v} = \begin{bmatrix} v_r & v_t & v_k \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{t} \\ \hat{k} \end{bmatrix}$$

These unit vectors are obtained after 2 elementary rotations, starting from $(\hat{\iota}_1, \hat{\iota}_2, \hat{\iota}_3)$:

(a) Counterclockwise rotation about axis 3 by angle $\tilde{\xi}$

(b) Clockwise rotation about axis 2 by angle $\tilde{\phi}$

$$\begin{bmatrix} \hat{r} \\ \hat{t} \\ \hat{k} \end{bmatrix} = R_2(-\tilde{\phi})\, R_3(\tilde{\xi}) \begin{bmatrix} \hat{\iota}_1 \\ \hat{\iota}_2 \\ \hat{\iota}_3 \end{bmatrix}$$

Using the Poisson formula and the transport theorem, the left-hand sides of (1) and (2) can be handled, to yield

$$\begin{cases} \dot{r} = v_r \\[2ex] \dot{\tilde{\xi}} = \dfrac{v_t}{r c_{\tilde{\phi}}} \\[2ex] \dot{\tilde{\phi}} = \dfrac{v_k}{r} \end{cases}$$

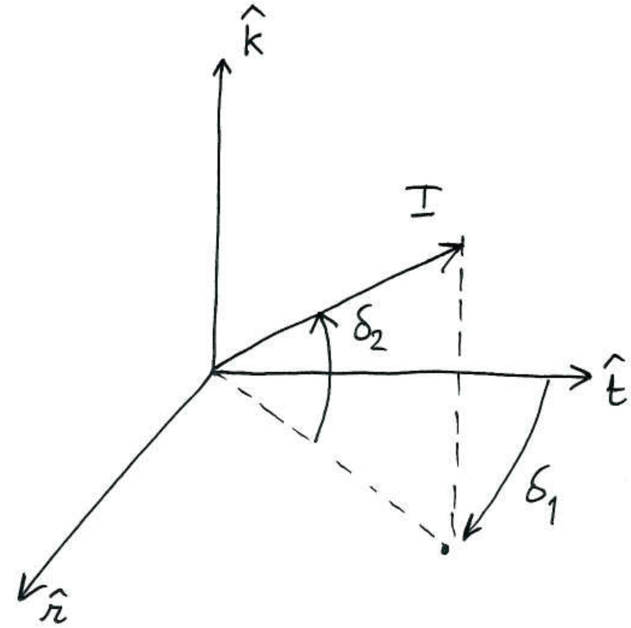
KINEMATICS EQUATIONS

$$\begin{cases} \dot{v}_r = -\dfrac{\mu}{r^2} + \dfrac{v_t^2 + v_k^2}{r} + \dfrac{T}{m} \sin\delta_1 \cos\delta_2 \\[2ex] \dot{v}_t = \dfrac{v_t}{r}\left(v_k \tan\tilde{\phi} - v_r\right) + \dfrac{T}{m} \cos\delta_1 \cos\delta_2 \\[2ex] \dot{v}_k = -\dfrac{v_t^2}{r} \tan\tilde{\phi} - \dfrac{v_r v_k}{r} + \dfrac{T}{m} \sin\delta_2 \end{cases}$$

DYNAMICS EQUATIONS

where the angles δ_1 and δ_2 define the thrust direction in the frame $(\hat{r}, \hat{t}, \hat{n})$, i.e.

$$\underline{T} = T \left[c_{\delta_2} s_{\delta_1} \quad c_{\delta_2} c_{\delta_1} \quad s_{\delta_2} \right] \begin{bmatrix} \hat{r} \\ \hat{t} \\ \hat{k} \end{bmatrix}$$



In conclusion, for the study and optimization of spaceflight trajectories,

$$\left.\begin{array}{l} T = \text{thrust magnitude} \\ \left.\begin{array}{l} \delta_1 \\ \delta_2 \end{array}\right\} \text{thrust angles} \end{array}\right] \text{represent CONTROL variables}$$

These control variables time histories must be selected in order to obtain the desired behavior of the dynamical system at hand, i.e. the launch vehicle. Its dynamical STATE is described through $\underline{r}$ and $\underline{v}$, i.e. $\{r, \tilde{\xi}, \tilde{\Phi}, v_r, v_t, v_k\}$, governed by the previously found Kinematics and dynamics equations.

In this study, attitude is not considered, but actually only through proper attitude maneuvers the thrust direction can assume the desired values in time.