# RELATIVE ORBIT MOTION

## INTRODUCTION

The orbital motion of spacecraft placed in near circular orbits can be investigated with reference to a nearby circular orbit.

Linearized equations of motion can be obtained, which describe the spacecraft dynamics making reference to this neighboring circular orbit.

Linear orbit theory has many applications, i.e. midcourse guidance, close-range interception and rendezvous, and satellite evasive maneuvers.

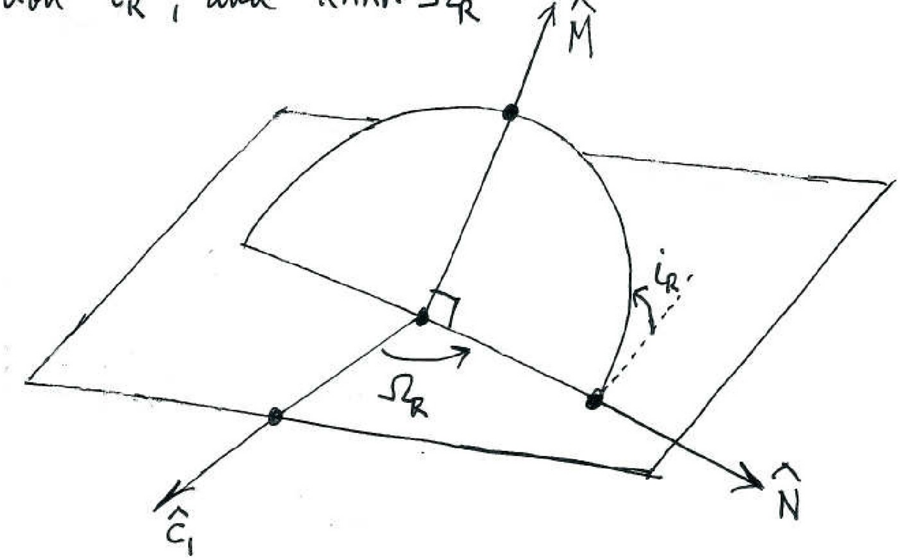
The great advantage in dealing with linear equations of motion is that superposition applies, i.e. the overall motion due to thrust and gravity can be obtained by superposition of the separated effects.

In this chapter, the Euler-Hill-Clohessy-Wiltshire equations (HCW henceforth) are formally obtained using spherical (curvilinear) coordinates, unlike what is found in many textbooks.

Although in this chapter the reference orbit is circular, some extensions and theoretical developments for reference elliptic orbits can be found in the scientific literature.

## EQUATIONS OF RELATIVE ORBIT MOTION

The equations of relative orbit motion can be derived with respect to a reference circular orbit, which has specified radius $R_R$, inclination $i_R$, and RAAN $\Omega_R$




This reference orbit is considered Keplerian at all times.

As a result, its orbit elements do not vary, as well as its orbital plane

$\Rightarrow$ axes $\hat{N}$ and $\hat{M}$ are inertial; together with $\hat{h}$ (not shown) they form a reference frame where the equations of motion can be written. One obtains the dynamics equations for $\{r, \tilde{\xi}, \tilde{\phi}, v_r, v_t, v_K\}$, where

$r$ = radius

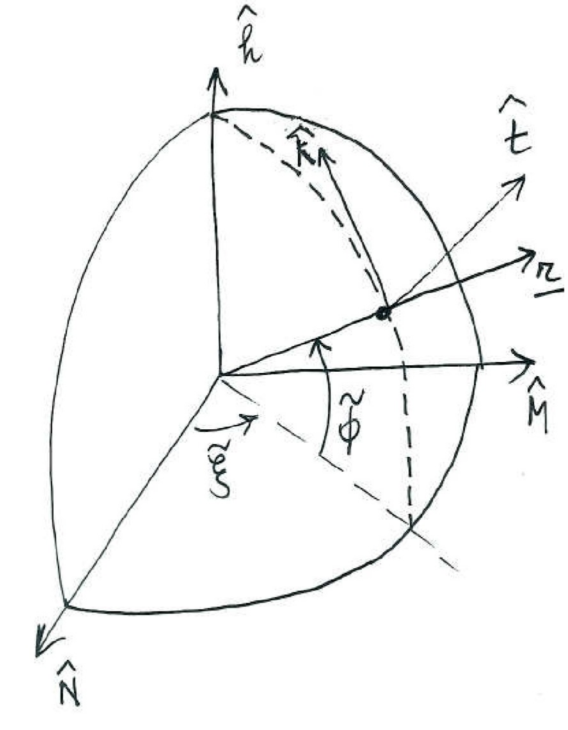
$\tilde{\xi}$ = right ascension

$\tilde{\phi}$ = declination

$v_r$ = radial (component of) velocity

$v_t$ = transverse (component of) velocity

$v_K$ = normal (component of) velocity



$$\begin{cases} \dot{r} = v_r \\ \dot{\tilde{\xi}} = \dfrac{v_t}{r c_{\tilde{\phi}}} \\ \dot{\tilde{\phi}} = \dfrac{v_k}{r} \end{cases} \qquad \begin{cases} \dot{v}_r = -\dfrac{\mu}{r^2} + \dfrac{v_t^2 + v_k^2}{r} + a_r \\ \dot{v}_t = \dfrac{v_t}{r}\left(v_k \tan\tilde{\phi} - v_r\right) + a_t \\ \dot{v}_k = -\dfrac{v_t^2}{r}\tan\tilde{\phi} - \dfrac{v_r v_k}{r} + a_k \end{cases}$$

where $\{a_r, a_t, a_k\}$ are the thrust or perturbing accelerations along the unit vectors $\{\hat{r}, \hat{t}, \hat{k}\}$.

The reference orbit is circular, as previously stated, therefore

$$r_R = R_R \qquad \tilde{\xi}_R = \tilde{\xi}_0 + \sqrt{\frac{\mu}{R_R^3}}\,(t - t_0) \qquad \tilde{\phi}_R = 0$$

$$v_{r,R} = 0 \qquad v_{t,R} = \sqrt{\frac{\mu}{R_R}} \qquad v_{k,R} = 0$$

In the previous expressions the fact that the reference orbit lies on the $(\hat{N}, \hat{M})$ was used; $t_0$ is a generic initial time, and $\tilde{\xi}_0$ the corresponding initial value of $\tilde{\xi}$.

If a spacecraft is placed in a neighboring orbit, then the following displacements $\delta[\ ]$ can be simply added to the state variables of the reference orbit:

$$\begin{cases} r = r_R + \delta r = R_R + \delta r \\ \tilde{\xi} = \tilde{\xi}_R + \delta\xi = \tilde{\xi}_0 + \sqrt{\dfrac{\mu}{R_R^3}}\,(t - t_0) + \delta\xi \\ \tilde{\phi} = \tilde{\phi} + \delta\phi = \delta\phi \end{cases} \qquad \begin{cases} v_r = v_{r,R} + \delta v_r = \delta v_r \\ v_t = v_{t,R} + \delta v_t = \sqrt{\dfrac{\mu}{R_R}} + \delta v_t \\ v_k = v_{k,R} + \delta v_k = \delta v_k \end{cases}$$

These expressions are inserted in the kinematics and dynamics equations, to yield

$$\begin{cases} \delta\dot{r} = \delta v_r \\ \sqrt{\dfrac{\mu}{R_R^3}} + \delta\dot{\xi} = \dfrac{\sqrt{\dfrac{\mu}{R_R}} + \delta v_t}{(R_R + \delta r)\cos\delta\phi} \\ \delta\dot{\phi} = \dfrac{\delta v_K}{R_R + \delta r} \end{cases} \qquad \begin{cases} \delta\dot{v}_r = -\dfrac{\mu}{(R_R+\delta r)^2} + \dfrac{\left(\sqrt{\dfrac{\mu}{R_R}} + \delta v_t\right)^2}{R_R + \delta r} + a_r \\ \delta\dot{v}_t = \left[\sqrt{\dfrac{\mu}{R_R}} + \delta v_t\right] \dfrac{\delta v_K \tan\delta\phi - \delta v_r}{R_R + \delta r} + a_t \\ \delta\dot{v}_K = -\dfrac{\delta v_r\, \delta v_K}{R_R + \delta r} - \left[\sqrt{\dfrac{\mu}{R_R}} + \delta v_t\right]^2 \dfrac{\tan\delta\phi}{R_R + \delta r} + a_K \end{cases}$$

After expanding the trigonometric functions to first order, and after negleting all terms of order equal or greater than 2, one obtains

$$\begin{cases} \delta\dot{r} = \delta v_r \\ R_R\,\delta\dot{\xi} = \delta v_t - w_R\,\delta r \\ R_R\,\delta\dot{\phi} = \delta v_K \end{cases} \qquad \begin{cases} \delta\dot{v}_r = w_R^2\,\delta r + 2w_R\,\delta v_t + a_r \\ \delta\dot{v}_t = -w_R\,\delta v_r + a_t \\ \delta\dot{v}_K = -w_R^2 R_R\,\delta\phi + a_K \end{cases}$$

where $w_R := \sqrt{\dfrac{\mu}{R_R^3}}$ is the constant angular rate along the reference circular orbit. Letting

$$x = \delta r \qquad y = R_R\,\delta\xi \qquad z = R_R\,\delta\phi$$

the kinematics equations become

$$\dot{x} = \delta v_r \qquad \dot{y} = \delta v_t - w_R x \qquad \dot{z} = \delta v_K$$

After obtaining $\{\delta v_r, \delta v_t, \delta v_K\}$ in terms of $(x, y, z)$, from the dynamics equations, the following three second-order linear differential equations are obtained:

$$\begin{cases} \ddot{x} - 3w_R^2 x - 2w_R \dot{y} = a_r & (1) \\ \ddot{y} + 2w_R \dot{x} = a_t & (2) \\ \ddot{z} + w_R^2 z = a_K & (3) \end{cases}$$

HILL-CLOHESSY-WILTSHIRE equations of relative motion (in curvilinear coordinates)

## GENERAL SOLUTION (NATURAL MOTION)

The HCW equation represent a linear differential system, and can be solved using the general approach tailored to similar differential systems. Free motion is considered, i.e. $a_r = a_t = a_K = 0$

In this section, a more direct approach is used.

As a first step, the equation for z (3) is uncoupled from Eqs. (1) and (2), thus it can be solved separately.

In this section, the initial conditions

$$\{x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0\}$$

are assumed as known

### Out-of-plane motion

The out-of-plane equation (with $a_K = 0$) is

$$\ddot{z} + \omega_R^2 z = 0$$

and it is easily recognized as the harmonic motion equation, with solution ($\tau := t - t_0$)

$$z = K_1 \cos(\omega_R \tau) + K_2 \sin(\omega_R \tau)$$

As the initial conditions are known, the previous solution is evaluated at $t = t_0$ (i.e. $\tau = 0$), to yield

$$z(0) = K_1 = z_0$$

$$\dot{z}(0) = \omega_R K_2 = \dot{z}_0$$

$$\longrightarrow \quad z(\tau) = z_0 \cos(\omega_R \tau) + \frac{\dot{z}_0}{\omega_R} \sin(\omega_R \tau)$$

- In-plane motion

Equation (2), with $a_t = 0$, becomes

$$\ddot{y} + 2\omega_R \dot{x} = 0 \quad \Rightarrow \quad \dot{y} + 2\omega_R x = K_3 \quad \text{(constant)}$$

The constant $K_3$ can be easily expressed in terms of known quantities, because $x_0$ and $\dot{y}_0$ are known; therefore

$$K_3 = \dot{y}_0 + 2\omega_R x_0$$

Then the expression found for $\dot{y}$, $\quad \dot{y} = \dot{y}_0 + 2\omega_R x_0 - 2\omega_R x$

is inserted into (1),

$$\ddot{x} - 3\omega_R^2 x - 2\omega_R \left[ \dot{y}_0 + 2\omega_R x_0 - 2\omega_R x \right] = 0$$

$$\ddot{x} + \omega_R^2 x = 2\omega_R \dot{y}_0 + 4\omega_R^2 x_0$$

The latter equation is a linear differential equation of $2^{nd}$ order, whose solution is the sum of

(i) General solution of the homogeneous equation

$$\ddot{x} + \omega_R^2 x = 0$$

(ii) Particular solution of the complete, non-homogeneous equation

$$\ddot{x} + \omega_R^2 x = 2\omega_R \dot{y}_0 + 4\omega_R^2 x_0$$

For (i), the solution is

$$x_h(\tau) = K_4 \cos(\omega_R \tau) + K_5 \sin(\omega_R \tau) \qquad \tau = t - t_0$$

For (ii), a particular solution is

$$x_c(\tau) = \frac{2\dot{y}_0}{\omega_R} + 4x_0$$

Therefore, the general solution is

$$x(\tau) = K_4 \cos(\omega_R \tau) + K_5 \sin(\omega_R \tau) + \frac{2\dot{y}_0}{\omega_R} + 4 x_0$$

As the initial conditions are known, the previous solution is evaluated at $t=t_0$ (i.e. $\tau=0$), to yield

$$x(0) = K_4 + \frac{2\dot{y}_0}{\omega_R} + 4x_0 = x_0 \longrightarrow K_4 = -3x_0 - \frac{2\dot{y}_0}{\omega_R}$$

$$\dot{x}(0) = K_5 \omega_R = \dot{x}_0 \longrightarrow K_5 = \frac{\dot{x}_0}{\omega_R}$$

$$\longrightarrow x(\tau) = \left[-3x_0 - \frac{2\dot{y}_0}{\omega_R}\right] \cos(\omega_R \tau) + \frac{\dot{x}_0}{\omega_R} \sin(\omega_R \tau) + \frac{2\dot{y}_0}{\omega_R} + 4x_0$$

Moreover, the remaining equation for $y$ can now be integrated,

$$\dot{y} = \dot{y}_0 + 2\omega_R x_0 - 2\omega_R \left\{ \left[-3x_0 - \frac{2\dot{y}_0}{\omega_R}\right] \cos(\omega_R \tau) + \frac{\dot{x}_0}{\omega_R} \sin(\omega_R \tau) + \frac{2\dot{y}_0}{\omega_R} + 4x_0 \right\}$$

$$y = K_6 + \left[-3\dot{y}_0 - 6\omega_R x_0\right]\tau + \frac{2}{\omega_R}\left[3\omega_R x_0 + 2\dot{y}_0\right] \sin(\omega_R \tau) + \frac{2\dot{x}_0}{\omega_R} \cos(\omega_R \tau)$$

The latter equation is evaluated at $t=t_0$ $(\tau=0)$, in order to get $K_6$:

$$y(0) = K_6 + \frac{2\dot{x}_0}{\omega_R} = y_0 \longrightarrow K_6 = y_0 - \frac{2\dot{x}_0}{\omega_R}$$

and, finally the solution for $y$ is

$$y = y_0 - \frac{2\dot{x}_0}{\omega_R} - 3\left[\dot{y}_0 + 2\omega_R x_0\right]\tau + \frac{2}{\omega_R}\left[3\omega_R x_0 + 2\dot{y}_0\right] \sin(\omega_R \tau) + \frac{2\dot{x}_0}{\omega_R} \cos(\omega_R \tau)$$

In this way $\{x, y, z\}$ have an explicit expression as functions of $\{x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0\}$

- State transition matrix

The general solution found for $\{x, y, z\}$ can be formally rewritten using the state transition matrix. As a preliminary step, however, $\{\dot{x}, \dot{y}, \dot{z}\}$ must be obtained:

$$\dot{x} = +\left[+3x_0\omega_R + 2\dot{y}_0\right]\sin(\omega_R\tau) + \dot{x}_0\cos(\omega_R\tau)$$

$$\dot{y} = -3\left[\dot{y}_0 + 2\omega_R x_0\right] + 2\left[3\omega_R x_0 + 2\dot{y}_0\right]\cos(\omega_R\tau) - 2\dot{x}_0\sin(\omega_R\tau)$$

$$\dot{z} = -z_0\omega_R\sin(\omega_R\tau) + \dot{z}_0\cos(\omega_R\tau)$$

These equations are considered together with those for $\{x, y, z\}$, in order to obtain a form in which dependancy on initial condition emerges, i.e. in order to write them in terms of the state transition matrix:

$$\begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \underbrace{\begin{bmatrix} 4-3\cos(\omega_R\tau) & 0 & 0 & \frac{\sin(\omega_R\tau)}{\omega_R} & \frac{2}{\omega_R}\left[1-\cos(\omega_R\tau)\right] & 0 \\ 6\left[\sin(\omega_R\tau)-\omega_R\tau\right] & 1 & 0 & \frac{2}{\omega_R}\left[\cos(\omega_R\tau)-1\right] & \frac{4}{\omega_R}\sin(\omega_R\tau)-3\tau & 0 \\ 0 & 0 & \cos(\omega_R\tau) & 0 & 0 & \frac{\sin(\omega_R\tau)}{\omega_R} \\ 3\omega_R\sin(\omega_R\tau) & 0 & 0 & \cos(\omega_R\tau) & 2\sin(\omega_R\tau) & 0 \\ 6\omega_R\left[\cos(\omega_R\tau)-1\right] & 0 & 0 & -2\sin(\omega_R\tau) & 4\cos(\omega_R\tau)-3 & 0 \\ 0 & 0 & -\omega_R\sin(\omega_R\tau) & 0 & 0 & \cos(\omega_R\tau) \end{bmatrix}}_{\text{State transition matrix } \Phi(\tau) \quad (\tau = t - t_0)} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ \dot{x}_0 \\ \dot{y}_0 \\ \dot{z}_0 \end{bmatrix}$$

- Compact form

The in-plane and out-of-plane motion (with no acceleration) can be expressed in a different, more compact form:

$$(1) \quad x(\tau) = c_1 \cos\left(-\omega_R \tau + \varphi\right) + 4x_0 + \frac{2\dot{y}_0}{\omega_R}$$

$$(2) \quad y(\tau) = 2c_1 \sin\left(-\omega_R \tau + \varphi\right) + y_0 - \frac{2\dot{x}_0}{\omega_R} - 3\left(\dot{y}_0 + 2\omega_R x_0\right)\tau$$

$$(3) \quad z(t) = c_3 \cos\left(\omega_R \tau + \chi\right)$$

where

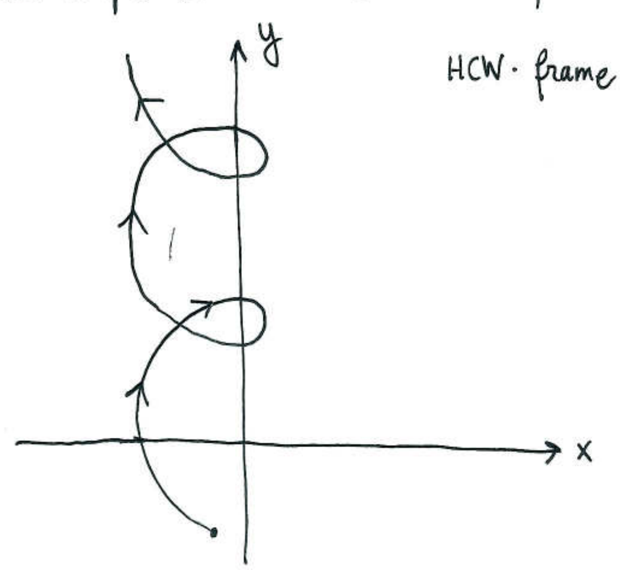
$$c_1 = \sqrt{\left(3x_0 + \frac{2\dot{y}_0}{\omega_R}\right)^2 + \left(\frac{\dot{x}_0}{\omega_R}\right)^2} \qquad c_3 = \sqrt{z_0^2 + \left(\frac{\dot{z}_0}{\omega_R}\right)^2}$$

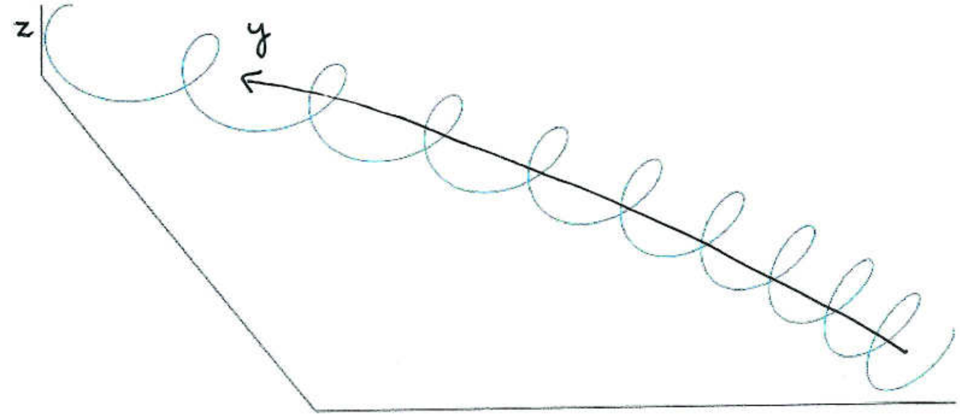
$$\begin{cases} \sin\varphi = \dfrac{\dot{x}_0}{\omega_R c_1} \\ \cos\varphi = -\left(3x_0 + \dfrac{2\dot{y}_0}{\omega_R}\right)\dfrac{1}{c_1} \end{cases} \qquad \begin{cases} \sin\chi = -\dfrac{\dot{z}_0}{\omega_R c_3} \\ \cos\chi = \dfrac{z_0}{c_3} \end{cases}$$

It is apparent that (3) represent an out-of-plane harmonic motion with amplitude $c_3$ and period $T = \frac{2\pi}{\omega_R}$, i.e. the orbital period of the reference circular orbit.

Instead, (1) and (2) are coupled and the respective curve is a CYCLOID, in general. Its form in the $(x, y)$-plane depends on the values of $(c_1, \varphi, \dot{y}_0, x_0)$

However, several special cases can be identified.






The above figure represents the general solution along the $(x, y, z)$-axes. Its projection into the $(x, y)$-plane is the cycloid portrayed at the previous page.

It is worth remarking that

(i) The in-plane motion includes two terms:

(a) an OSCILLATORY TERM (both in $x$ and $y$), related to trigonometric functions

(b) a SECULAR TERM (in $y$ only), related to the term $-3(\dot{y}_0 + 2\omega_R x_0)\tau$ : this represents a steady drift along $y$

The oscillation in $x$ represents a varying differential radius of a neighboring orbit, whereas the secular term describes an orbit of period slightly different from $T = \frac{2\pi}{\omega_R}$

(ii) The out-of-plane motion includes an OSCILLATORY TERM only.

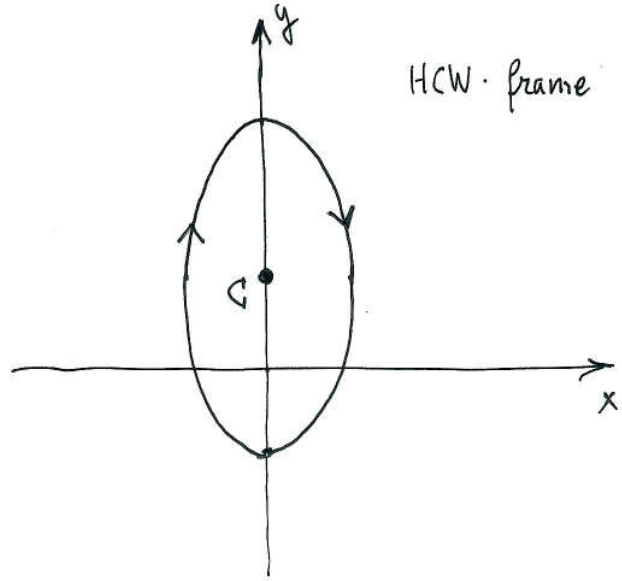
The combined effects of relative motion in all components represents the general case of a neighboring orbit, which is ELLIPTIC, with different period, and with a slightly displaced plane with respect to the reference circular orbit.

- SPECIAL SOLUTIONS

If the initial conditions satisfy certain relations, then several special solutions can be identified

(1) RELATIVE ELLIPTIC MOTION if $\dot{y}_0 + 2\omega_R x_0 = 0$

No drift along $y$ takes place and the motion is periodic both in the $(x,y)$-plane and along $z$. The period is $T = \frac{2\pi}{\omega_R}$, i.e. the orbit period of the reference circular orbit. This neighboring path represents an ELLIPTIC, PLANE-DISPLACED orbit with same period as that of the reference orbit. The relative elliptic orbit in the $(x,y)$-plane, i.e. projection on this plane, is traveled in clockwise sense




In fact, the general solution becomes

$$\begin{cases} x(t) = c_1 \cos(-\omega_R t + \varphi) \\ y(t) = 2c_1 \sin(-\omega_R t + \varphi) + y_0 - \frac{2\dot{x}_0}{\omega_R} \end{cases}$$

and the angle in the trigonometric functions decreases as $t$ increases

The ellipse has semimajor and semiminor axes with ratio 2:1, and center at $C\left(0, y_0 - \frac{2\dot{x}_0}{\omega_R}\right)$

As special case, if $c_3 = 0$, then relative motion takes plane entirely on the $(x,y)$-plane, with all the previous considerations still valid.

(2) MOTION ALONG A NEIGHBORING CIRCULAR ORBIT if $c_1 = c_3 = 0$

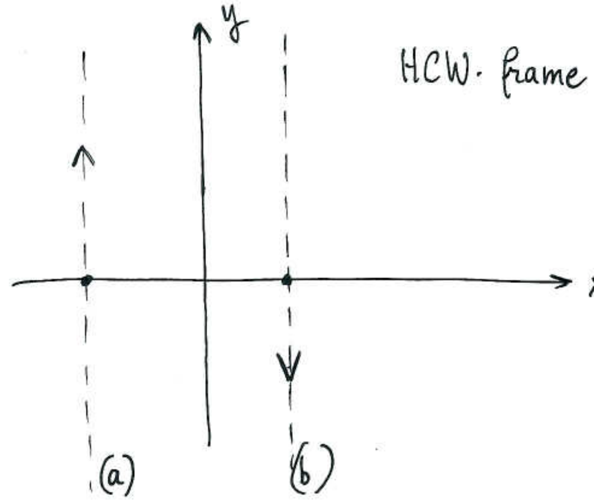
No oscillatory term exists, and the solution becomes

$$\begin{cases} x(\tau) = 4x_0 + \dfrac{2\dot{y}_0}{\omega_R} = x_0 & \text{because } c_1 = 0 \Rightarrow \begin{cases} 3x_0 + \dfrac{2\dot{y}_0}{\omega_R} = 0 \\ \dot{x}_0 = 0 \end{cases} \\ y(\tau) = y_0 - 3\left(\dot{y}_0 + 2\omega_R x_0\right)\tau = y_0 - \dfrac{3}{2}\omega_R x_0 \tau & \text{because } c_1 = 0, \text{ again} \end{cases}$$

This solution is associated with a neighboring circular orbit with radius $R = R_R + x_0$.

In the figure

(a) is a lower circular orbit and drift takes place along $+y$; in fact a lower circular orbit is traveled faster than the reference orbit




(b) is a higher circular orbit and drift takes place along $-y$; in fact, a higher circular orbit is traveled slower than the reference orbit.

It is worth emphasizing that this solution remains sufficiently accurate for arbitrary large values of $y$ (provided that $x_0$ is small). In fact, while deriving all the linear equations HCW, the term $y = R_R \delta\xi$ never appeared, thus it is not necessary that $y$ be small.

Instead, in general $\{x, z, \dot{x}, \dot{y}, \dot{z}\}$ must be small in order that the HCW be accurate enough.

As a special case, if $x_0 = 0$, then the spacecraft travels along a circular orbit that is coplanar, with same period $T = \frac{2\pi}{\omega_R}$, and a constant displacement $y_0$.
In fact, the previous solution reduces to

$$x(\tau) = 0 \qquad \text{and} \qquad y(\tau) = y_0$$

(3) OUT-OF-PLANE CIRCULAR MOTION if

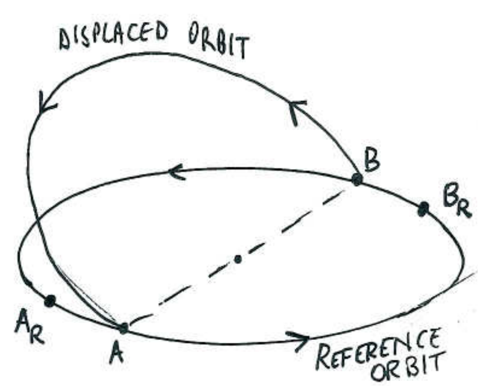
$$c_1 = 0 \qquad \text{and} \qquad \dot{y}_0 + 2\omega_R x_0 = 0 \qquad \text{and} \qquad c_3 \neq 0$$

The solution becomes

$$x(\tau) = 0 \qquad y(\tau) = y_0 \qquad z(\tau) = c_3 \cos\left(+\omega_R \tau + \chi\right)$$

and is associated with a neighboring circular orbit with same radius (and orbital period), but a slightly different orbital plane, which is the reason for the oscillatory relative dynamics along z. The two orbits, i.e. that of the spacecraft and the reference orbit intersect at those points where z vanishes. (A and B in fig)

When the spacecraft is at A, the "reference vehicle" along the reference orbit is at $A_R$ the length of arc $A_R A$ is given by $R_R y_0$. Same considerations hold for B and $B_R$.



Inertial frame

- LINEAR IMPULSIVE RENDEZVOUS

In a previous section, the state transition matrix was introduced, and is especially useful for investigating orbit interception and rendezvous involving two spacecraft that travel in the proximity of a circular orbit.

In fact, the state transition matrix $\Phi(\tau)$ yields the relative position and velocity as functions of their initial values:

$$\begin{bmatrix} \delta \underline{r}(\tau) \\ \delta \underline{v}(\tau) \end{bmatrix} = \begin{bmatrix} M(\tau) & N(\tau) \\ S(\tau) & T(\tau) \end{bmatrix} \begin{bmatrix} \delta \underline{r_0} \\ \delta \underline{v_0} \end{bmatrix}$$

where $\delta \underline{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\delta \underline{v} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$ and $M, N, S, T$ are the (3 by 3) partitions of $\Phi$, i.e., letting $s := \sin(w_R\tau)$ and $c := \cos(w_R\tau)$:

$$M(\tau) = \begin{bmatrix} 4-3\cos(w_R\tau) & 0 & 0 \\ 6\left[\sin(w_R\tau)-w_R\tau\right] & 1 & 0 \\ 0 & 0 & \cos(w_R\tau) \end{bmatrix} =: \begin{bmatrix} M_{ip}(\tau) & & 0 \\ & & 0 \\ 0 & 0 & c \end{bmatrix}$$

$$N(\tau) = \begin{bmatrix} \frac{\sin(w_R\tau)}{w_R} & \frac{2}{w_R}\left[1-\cos(w_R\tau)\right] & 0 \\ \frac{2}{w_R}\left[\cos(w_R\tau)-1\right] & \frac{4}{w_R}\sin(w_R\tau)-3\tau & 0 \\ 0 & 0 & \frac{\sin(w_R\tau)}{w_R} \end{bmatrix} =: \begin{bmatrix} N_{ip}(\tau) & & 0 \\ & & 0 \\ 0 & 0 & \frac{s}{w_R} \end{bmatrix}$$

$$S(\tau) = \begin{bmatrix} 3w_R \sin(w_R\tau) & 0 & 0 \\ 6w_R[\cos(w_R\tau)-1] & 0 & 0 \\ 0 & 0 & -w_R \sin(w_R\tau) \end{bmatrix} =: \begin{bmatrix} S_{ip}(\tau) & & 0 \\ & & 0 \\ 0 & 0 & -w_R S \end{bmatrix}$$

$$T(\tau) = \begin{bmatrix} \cos(w_R\tau) & 2\sin(w_R\tau) & 0 \\ -2\sin(w_R\tau) & 4\cos(w_R\tau)-3 & 0 \\ 0 & 0 & \cos(w_R\tau) \end{bmatrix} =: \begin{bmatrix} T_{ip}(\tau) & & 0 \\ & & 0 \\ 0 & 0 & C \end{bmatrix}$$

Displaced positions and velocities of two neighboring spacecraft can be written in terms of these matrices.

As an example, let (a) $\{\delta \underline{r_0}, \delta \underline{v_0}^-\}$ be the initial position and velocity of spacecraft 1; let (b) $\{\delta \underline{r_f}, \delta \underline{v_f}^+\}$ be the position and velocity of spacecraft 2 at specified time $t_f$. The time of flight is $(t_f - t_0)$ and is assumed as prescribed.

(i) ORBIT INTERCEPTION requires a single impulsive velocity change at $t_0$ such that $\delta \underline{r}(\tau_f) = \delta \underline{r_f}$ $\quad (\tau_f = t_f - t_0)$

(ii) ORBIT RENDEZVOUS requires two velocity variations, at $t_0$ and $t_f$ such that
$$\begin{cases} \delta \underline{r}(\tau_f) = \delta \underline{r_f} \\ \delta \underline{v}(\tau_f) = \delta \underline{v_f}^+ \end{cases}$$

the first velocity variation is the same as that needed for orbit interception, the second velocity variation is needed to acquire the final velocity of spacecraft 2, i.e. $\delta \underline{v_f}^+$

As a result, the first velocity change is such that

$$\underline{\delta r}(\tau_f) = \underline{\delta r_f} \longrightarrow M(\tau_f)\,\underline{\delta r_0} + N(\tau_f)\,\underline{\delta v_0^+} = \underline{\delta r_f}$$

$$\rightarrow \quad \underline{\delta v_0^+} = N^{-1}(\tau_f)\left[\underline{\delta r_f} - M(\tau_f)\,\underline{\delta r_0}\right]$$

$$\rightarrow \quad \underline{\Delta v_1} = \underline{\delta v_0^+} - \underline{\delta v_0^-} = -\underline{\delta v_0^-} + N^{-1}(\tau_f)\left[\underline{\delta r_f} - M(\tau_f)\,\underline{\delta r_0}\right]$$

After this impulse, interception (i) occurs at time $t_f = t_0 + \tau_f$.
Instead, if rendezvous is desired, i.e. one aims at arriving at spacecraft 2 with zero relative velocity, then a second velocity change is requested, right before arriving at $\underline{\delta r_f}$
The space vehicle 1 has velocity $\underline{\delta v_f^-}$ when it reaches the final position $\underline{\delta r_f}$:

$$\underline{\delta v_f^-} = S(\tau_f)\,\underline{\delta r_0} + T(\tau_f)\,\underline{\delta v_0^+} = S(\tau_f)\underline{\delta r_0} + T(\tau_f)N^{-1}(\tau_f)\left[\underline{\delta r_f} - M(\tau_f)\,\underline{\delta r_0}\right]$$

whereas the final desired velocity at rendezvous is $\underline{\delta v_f^+}$. Therefore

$$\underline{\Delta v_2} = \underline{\delta v_f^+} - \left\{S(\tau_f)\,\underline{\delta r_0} + T(\tau_f)\,N^{-1}(\tau_f)\left[\underline{\delta r_f} - M(\tau_f)\,\underline{\delta r_0}\right]\right\}$$

In summary:

(i) ORBIT INTERCEPTION is performed with a single velocity change, $\underline{\Delta v_1}$

(ii) ORBIT RENDEZVOUS is performed with two velocity changes, i.e. $\underline{\Delta v_1}$ and $\underline{\Delta v_2}$

It is worth noting that these two velocity changes can be obtained provided that $N(\tau_f)$ is nonsingular, otherwise $N^{-1}(\tau_f)$ does not exist.

By calculating $\det N(\tau_f)$ one obtains

$$\det N(\tau_f) = \left\{ 8\left[1-\cos(\omega_R\tau_f)\right] - 3\,\omega_R\tau_f \sin(\omega_R\tau_f)\right\} \sin(\omega_R\tau_f)$$

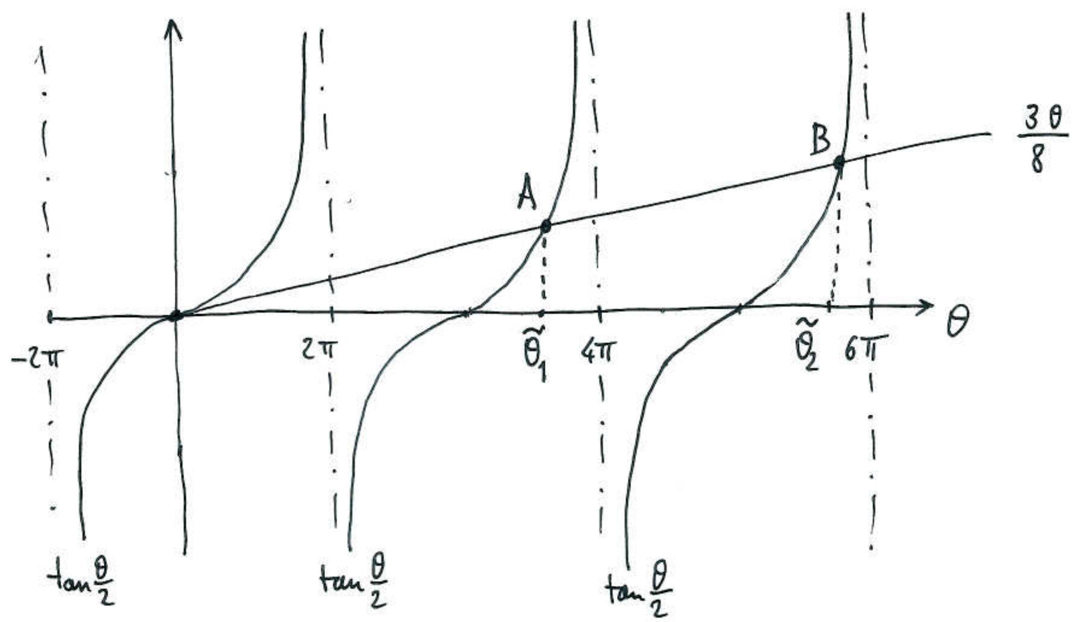
Letting $\theta = \omega_R\tau_f$ the previous relation identifies infinite (discrete) values of $\theta$ at which $N(\tau_f)$ is singular, i.e.

$$\theta = k\pi \quad (k \text{ integer}) \quad \text{and} \quad \theta \text{ such that } \quad 8(1-C_\theta) - 3\theta S_\theta = 0$$

The latter relation can be rewritten as

$$\frac{1-C_\theta}{S_\theta} = \frac{3}{8}\theta \quad \longrightarrow \quad \frac{S_\theta}{1+C_\theta} = \frac{3}{8}\theta \quad \text{because } \frac{S_\theta}{1+C_\theta} = \frac{1-C_\theta}{S_\theta}$$

$$\longrightarrow \quad \tan\frac{\theta}{2} = \frac{3\theta}{8} \quad \Rightarrow \quad \left\{\tilde{\theta}_k\right\} \text{ solutions (to find numerically)}$$




A and B are associated with the first solutions of the trascendental equation $\tan\frac{\theta}{2} = \frac{3\theta}{8}$

• <u>Planar impulsive rendezvous</u>

If $z=0$ and $\dot{z}=0$ at all times the relative motion can be investigated by taking into account $(x, y)$ only (in-plane motion).

All the previous considerations on velocity variations still hold, but only the partitions of $M, N, S, T$ related to $(x, y)$ are needed. This means that

$$\underline{\Delta v_1} = \delta \underline{v_0}^+ - N_{ip}^{-1}(\tau_f)\left[\delta \underline{r_f} - M_{ip}(\tau_f)\delta \underline{r_0}\right]$$

$$\underline{\Delta v_2} = \delta \underline{v_f}^+ - \left\{ S_{ip}(\tau_f)\delta \underline{r_0} + T_{ip}(\tau_f) N_{ip}^{-1}(\tau_f)\left[\delta \underline{r_f} - M_{ip}(\tau_f)\delta \underline{r_0}\right]\right\}$$

where $\begin{cases} \delta \underline{r_0} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\ \delta \underline{v_0}^- = \begin{bmatrix} \dot{x}_0^- \\ \dot{y}_0^- \end{bmatrix} \end{cases}$ are the specified relative position and velocity of spacecraft 1 at $t_0$ (before $\underline{\Delta v_1}$)

$\begin{cases} \delta \underline{r_f} = \begin{bmatrix} x_f \\ y_f \end{bmatrix} \\ \delta \underline{v_f}^+ = \begin{bmatrix} \dot{x}_f^+ \\ \dot{y}_f^+ \end{bmatrix} \end{cases}$ are the specified relative position and velocity of spacecraft 2 at $t_f$ (after $\underline{\Delta v_2}$)

It is easy to obtain that matrix $N_{ip}(\tau)$ becomes singular if $\omega_R \tau_f$ $(=: \theta)$ satisfies

$$\tan\frac{\theta}{2} = \frac{3\theta}{8}$$ (see previous page).