ORBIT PERTURBATIONS

• INTRODUCTION

In the previous chapter focused on Keplerian motion, the spacecraft was assumed to be subject to the inverse square law:

$$\frac{d^2 \underline{r}}{dt^2} = -\frac{\mu}{r^2}\hat{r}$$

The same equation was proven to hold also for attracting bodies with a spherical mass distribution.

However, an Earth satellite actually experiences small but significant perturbations due to:

(a) EARTH ASPHERICITY

(b) THIRD BODY GRAVITATIONAL ATTRACTION (due mainly to Sun and Moon)

(c) AERODYNAMIC DRAG

(d) SOLAR RADIATION PRESSURE

While (a) and (b) have conservative nature, (c) and (d) yield surface perturbing effects, i.e. the related perturbing accelerations depend on the surface of the spacecraft that is subject to the interaction (either with atmosphere (c) or with solar radiation (d)).

Moreover, all perturbations (except (d)) depend on the instantaneous spacecraft altitude.

The main asphericity feature of the Earth is its OBLATENESS (related to the J_2 term of the gravitational potential, as shown in the following).

- Perturbations on different orbits

Perturbing effects have different magnitudes depending on the orbit altitude.

A. LOW EARTH ORBITS (altitude up to 700 km)

Earth oblateness and drag dominate

B. MEDIUM-ALTITUDE EARTH ORBITS (altitude from 700 to 10000 km)

Earth oblateness dominates

Solar radiation pressure is nonnegligible

Third body attraction is nonnegligible

Drag is negligible (although it is to be considered up to 1000 km)

C. HIGH-ALTITUDE EARTH ORBITS (altitudes greater than 10000 km)

Solar radiation pressure, third body attraction, and Earth oblateness are the main perturbations (although Earth oblateness effect decreases rapidly as the altitude increases)

D. GEOSTATIONARY ORBITS

The asphericity term J_{22}, related to ellipticity of Earth equator, dominates. This is due to the resonance. In fact, a geostationary satellite rotates together with the Earth (same period), therefore any longitudinal asphericity has an "amplified" resonant effect

As an example, at 1000 Km of altitudes, letting $a_0 = \frac{\mu}{r^2}$

(a) $a_{J2} \simeq 10^{-2} a_0$ (Earth oblateness)

(b) $a_{3B} \simeq 10^{-7} a_0$ (Sun and Moon attraction)

(c) $a_{RP} \simeq 10^{-9} a_0$ (solar radiation pressure)

(d) $a_D \simeq 10^{-10} a_0$ (aerodynamic drag)

LAGRANGE PLANETARY EQUATIONS (GAUSS FORM)

In the presence of orbit perturbations, the two integrals of motion

$$\underline{h} = \underline{r} \times \underline{v}$$

$$\underline{e} = -\hat{r} + \frac{\underline{v} \times \underline{h}}{\mu}$$

do not preserve any longer. The governing equation of the perturbing motion is

$$\frac{d^2\underline{r}}{dt^2} = -\frac{\mu}{r^3}\underline{r} + \underline{f} \qquad \text{Moreover, } \underline{h} = \underline{r} \times \underline{v} = \underline{w}\, r^2 - \underline{r}(\underline{w} \cdot \underline{r}) \; (*)$$

where $\underline{f}$ can include perturbing accelerations due to environment or even propulsive thrust. Using this relation

$$\frac{d\underline{h}}{dt} = \frac{d\underline{r}}{dt} \times \underline{v} + \underline{r} \times \frac{d\underline{v}}{dt} = \underline{r} \times \left[-\frac{\mu}{r^3}\underline{r} + \underline{f}\right] = \underline{r} \times \underline{f}$$

$$\frac{d\underline{e}}{dt} = -\frac{d\hat{r}}{dt} + \frac{1}{\mu}\left[\frac{d\underline{v}}{dt} \times \underline{h} + \underline{v} \times \frac{d\underline{h}}{dt}\right] =$$

$$= -\underline{w} \times \hat{r} + \frac{1}{\mu}\left[\left(-\frac{\mu}{r^3}\underline{r} + \underline{f}\right) \times \underline{h} + \underline{v} \times (\underline{r} \times \underline{f})\right] =$$

$$\overset{(*)}{=} -\frac{\underline{h}}{r^2} \times \hat{r} - \frac{\hat{r}}{r^2} \times \underline{h} + \frac{1}{\mu}\left[\underline{f} \times \underline{h} + \underline{v} \times (\underline{r} \times \underline{f})\right] =$$

$$= \frac{1}{\mu}\left[\underline{f} \times \underline{h} + \underline{v} \times (\underline{r} \times \underline{f})\right]$$

These two vector equations are independent of any reference frame, and may be useful for finding the time derivatives of the osculating orbit elements

$$\{a, e, i, \Omega, \omega, \vartheta_*\}$$

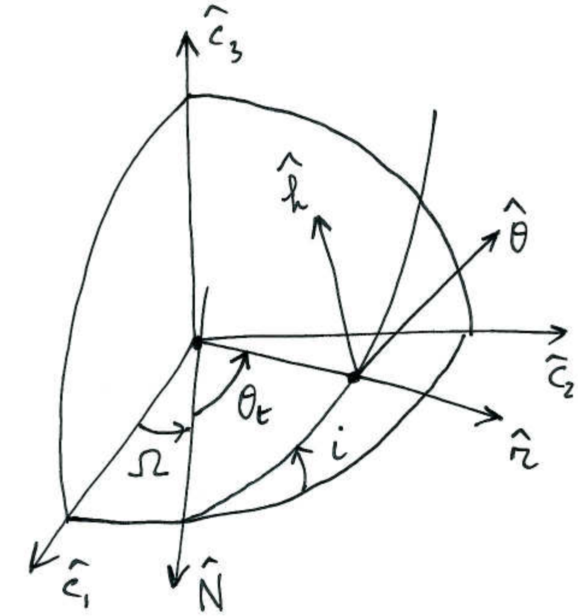
- <u>Equation for $\underline{h}$</u>

It is convenient to project the two previous vector equations into the LVLH-frame $(\hat{r}, \hat{\theta}, \hat{h})$.

As a first step, $\underline{f}$ has components (f_r, f_θ, f_h) in this frame

$$\underline{f} = \begin{bmatrix} f_r & f_\theta & f_h \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{h} \end{bmatrix}$$

As a second step the vector rotation rate $\underline{\omega}$ can be written in terms of $(\hat{r}, \hat{\theta}, \hat{h})$, as follows.



$$\underline{\omega} = \dot{\Omega}\,\hat{c}_3 + \dot{i}\,\hat{N} + \dot{\theta}_t\,\hat{h}$$

Using the expression of $\hat{h}$ in terms of $\hat{c}_1, \hat{c}_2, \hat{c}_3$,

$$\hat{h} = \begin{bmatrix} s_i s_\Omega & -s_i c_\Omega & c_i \end{bmatrix} \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{bmatrix}$$

and the fact that

$$\hat{N} = c_\Omega \hat{c}_1 + s_\Omega \hat{c}_2$$

and also the definition of $R_A = R_3(\theta_t) R_1(i) R_3(\Omega)$

$$\begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{h} \end{bmatrix} = R_A \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{bmatrix}$$

one finally can obtain $\underline{\omega}$ written in $(\hat{r}, \hat{\theta}, \hat{h})$,

$$\underline{\omega} = \begin{bmatrix} \dot{\Omega} s_i s_{\theta_t} + \dot{i} c_{\theta_t} \\ \dot{\Omega} s_i c_{\theta_t} - \dot{i} s_{\theta_t} \\ \dot{\Omega} c_i + \dot{\theta}_t \end{bmatrix}^T \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{h} \end{bmatrix}$$

Let w_i denote the components of $\underline{\omega}$ along $(\hat{r}, \hat{\theta}, \hat{h})$

$$\underline{\omega} = w_1 \hat{r} + w_2 \hat{\theta} + w_3 \hat{h}$$

Now, the vector equation for $\underline{h}$ is projected onto $(\hat{r}, \hat{\theta}, \hat{h})$

left-hand-side $\quad \frac{d\underline{h}}{dt} = \dot{h} \hat{h} + \underline{\omega} \times \underline{h} = \dot{h} \hat{h} - r f_h \hat{\theta} + r f_\theta \hat{h}$

right-hand-side $\quad \underline{r} \times \underline{f} = \hat{h} \, r f_\theta - \hat{\theta} \, r f_h$

Therefore, one obtains (using the expressions of w_i):

$$\hat{r}) \quad h \left(\dot{\Omega} s_i c_{\theta_t} - \dot{i} s_{\theta_t} \right) = 0 \qquad (1)$$

$$\hat{\theta}) \quad h \left(\dot{\Omega} s_i s_{\theta_t} + \dot{i} c_{\theta_t} \right) = r f_h \qquad (2)$$

$$\hat{h}) \quad \dot{h} = r f_\theta \qquad (3)$$

Multiplying (2) by c_{θ_t} and (1) by s_{θ_t}, then subtracting

$$\dot{i} = r f_h \frac{c_{\theta_t}}{h}$$

Inserting the latter relation into (1)

$$\dot{\Omega} = r f_h \frac{s_{\theta_t}}{h s_i}$$

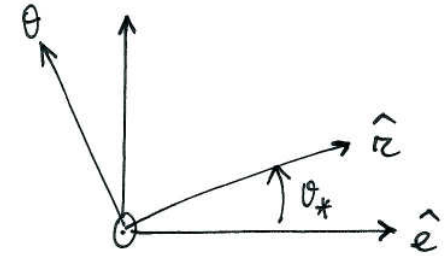
Moreover, as $h = \sqrt{\mu p}$

$$\dot{h} = \sqrt{\mu} \frac{\dot{p}}{2\sqrt{p}} \longrightarrow \quad \dot{p} = 2 \sqrt{\frac{p}{\mu}} \, r f_\theta$$

The latter equation will be useful for finding $\dot{a}$ at a later time.

- <u>Equation for $\underline{e}$</u>

Using again the LVLH frame, the eccentricity vector can be written as



$$\underline{e} = e \begin{bmatrix} c_{\theta_*} & -s_{\theta_*} & 0 \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{h} \end{bmatrix}$$

The related vector equation can be projected onto this frame

left-hand-side
$$\frac{d\underline{e}}{dt} = \dot{e}\left[c_{\theta_*}\hat{r} - s_{\theta_*}\hat{\theta}\right] + e\,\frac{d}{dt}\left[c_{\theta_*}\hat{r} - s_{\theta_*}\hat{\theta}\right] =$$

$$= \dot{e}\left[c_{\theta_*}\hat{r} - s_{\theta_*}\hat{\theta}\right] + e\left[-\dot{\theta}_* s_{\theta_*}\hat{r} + c_{\theta_*}\underline{\omega}\times\hat{r} - \dot{\theta}_* c_{\theta_*}\hat{\theta} - s_{\theta_*}\underline{\omega}\times\hat{\theta}\right] =$$

$$= \dot{e}\, c_{\theta_*}\hat{r} - \dot{e}\, s_{\theta_*}\hat{\theta} - e\dot{\theta}_* s_{\theta_*}\hat{r} - e\,\dot{\theta}_* c_{\theta_*}\hat{\theta} +$$

$$+ e\, c_{\theta_*}\left[\hat{\theta}\omega_3 - \hat{h}\omega_2\right] - e\, s_{\theta_*}\left[\hat{h}\omega_1 - \hat{r}\omega_3\right]$$

right-hand-side

$$\underline{f}\times\underline{h} = \hat{r}\, h f_\theta - \hat{\theta}\, h f_r$$

$$\underline{v}\times\left(\underline{r}\times\underline{f}\right) = \hat{r}\, r v_\theta f_\theta - \hat{\theta}\, r f_\theta v_r - \hat{h}\, r f_h v_r$$

$$\frac{1}{\mu}\left[\underline{f}\times\underline{h} + \underline{v}\times\left(\underline{r}\times\underline{f}\right)\right] = \left\{\hat{r}\left[h f_\theta + r v_\theta f_\theta\right] - \hat{\theta}\left[h f_r + r f_\theta v_r\right] - \hat{h}\, r f_h v_r\right\}\frac{1}{\mu}$$

Therefore, one obtains

$$\hat{r})\quad \dot{e}\, c_{\theta_*} - e\,\dot{\theta}_* s_{\theta_*} + e\, s_{\theta_*}\omega_3 = \frac{1}{\mu}\left[h f_\theta + r v_\theta f_\theta\right]$$

$$\hat{\theta})\quad -\dot{e}\, s_{\theta_*} - e\,\dot{\theta}_* c_{\theta_*} + e\, c_{\theta_*}\omega_3 = \frac{1}{\mu}\left[-h f_r - r f_\theta v_r\right]$$

$$\hat{h})\quad -e\, c_{\theta_*}\omega_2 - e\, s_{\theta_*}\omega_1 = -\frac{1}{\mu} r f_h v_r$$

The last relation is useless, because it is satisfied identically if the expressions for $\dot{\Omega}$ and $\dot{i}$ are inserted.

Instead, the first two relations allow finding $\dot{e}$ and $\dot{\theta}_*$; to this end, however, the expression of ω_3 must be used.

However, one can prove that

$$\omega_3 = \underline{\omega} \cdot \hat{h} = \sqrt{\frac{\mu}{p^3}} \left(1+e c_{\theta_*}\right)^2$$

In fact

- $\underline{v} = v_r \hat{r} + v_\theta \hat{\theta} \equiv \dot{r} \hat{r} + \underline{\omega} \times \underline{r} \Rightarrow v_\theta \hat{\theta} \equiv \underline{\omega} \times \underline{r}$

- $\underline{\omega} \cdot \hat{h} = \dfrac{\underline{\omega} \cdot \underline{h}}{h} = \dfrac{1}{h} \underline{\omega} \cdot \left[\underline{r} \times \left(v_r \hat{r} + v_\theta \hat{\theta}\right)\right] =$

$$= \frac{1}{h} \underline{\omega} \cdot \left[\underline{r} \times v_\theta \hat{\theta}\right] = \frac{1}{h} v_\theta \hat{\theta} \cdot \left(\underline{\omega} \times \underline{r}\right) = \frac{v_\theta^2}{h} = \frac{v_\theta}{r}$$

$$\left(\text{because} \quad h = |\underline{r} \times \underline{v}| = r v_\theta\right)$$

$$\Rightarrow \underline{\omega} \cdot \hat{h} = \frac{v_\theta}{r} = \sqrt{\frac{\mu}{p}} \left(1+e c_{\theta_*}\right) \frac{1+e c_{\theta_*}}{p} = \sqrt{\frac{\mu}{p^3}} \left(1+e c_{\theta_*}\right)^2$$

Incidentally, in the preceding pages the components of $\underline{\omega}$ where found, along $(\hat{r}, \hat{\theta}, \hat{h})$, and ω_3 was given by

$$\omega_3 = \dot{\Omega} c_i + \dot{\theta}_t$$

After equating this relation to the previous one,

$$\dot{\theta}_t = \omega_3 - \dot{\Omega} c_i = \sqrt{\frac{\mu}{p^3}} \left(1+e c_{\theta_*}\right)^2 - r f_h \frac{s_{\theta_t} c_i}{h s_i}$$

This last equation will be useful for finding $\dot{\omega}$ at a later time

Using the expression of $\omega_3 = \sqrt{\frac{\mu}{p^3}}\left(1+e\,c_{\theta_*}\right)^2$ into the equations for $\dot{e}$ and $\dot{\theta}_*$, one obtains, after several steps

$$\dot{e} = \sqrt{\frac{p}{\mu}}\, f_r\, s_{\theta_*} + \sqrt{\frac{p}{\mu}}\, f_\theta\, \frac{e + e\,c_{\theta_*}^2 + 2c_{\theta_*}}{1+e\,c_{\theta_*}}$$

$$\dot{\theta}_* = \sqrt{\frac{\mu}{p^3}}\left(1+e\,c_{\theta_*}\right)^2 + \frac{f_r}{e}\, c_{\theta_*}\sqrt{\frac{p}{\mu}} + f_\theta\sqrt{\frac{p}{\mu}}\, s_{\theta_*}\, \frac{-e\,c_{\theta_*}-2}{e\left(1+e\,c_{\theta_*}\right)}$$

Because $\dot{\theta}_t = \dot{\theta}_* + \dot{\omega}$ one obtains $\dot{\omega}$ from $\dot{\theta}_*$ and $\dot{\theta}_t$,

$$\dot{\omega} = -r f_h \frac{s_{\theta_t} c_i}{h\, s_i} - \frac{f_r}{e}\, c_{\theta_*}\sqrt{\frac{p}{\mu}} + f_\theta\sqrt{\frac{p}{\mu}}\, s_{\theta_*}\, \frac{e\,c_{\theta_*}+2}{e\left(1+e\,c_{\theta_*}\right)}$$

In the last expression one can replace r and h with

$$r = \frac{p}{1+e\,c_{\theta_*}} \quad \text{and} \quad h = \sqrt{\mu p}$$

- Gauss equations

In short, the following planetary equations (termed also Gauss eqs) hold:

$$\dot{p} = 2\sqrt{\frac{p}{\mu}}\, r\, f_\theta$$

$$\dot{e} = \sqrt{\frac{p}{\mu}}\, f_r\, s_{\theta_*} + \sqrt{\frac{p}{\mu}}\, f_\theta\, \frac{e + e\,c_{\theta_*}^2 + 2c_{\theta_*}}{1+e\,c_{\theta_*}}$$

$$\dot{i} = r f_h \frac{c_{\theta_t}}{h}$$

$$\dot{\Omega} = r f_h \frac{s_{\theta_t}}{h\, s_i}$$

$$\dot{\omega} = -r f_h \frac{s_{\theta_t} c_i}{h\, s_i} - \frac{f_r}{e}\, c_{\theta_*}\sqrt{\frac{p}{\mu}} + f_\theta\sqrt{\frac{p}{\mu}}\, s_{\theta_*}\, \frac{e\,c_{\theta_*}+2}{e\left(1+e\,c_{\theta_*}\right)}$$

$$\dot{\theta}_* = \sqrt{\frac{\mu}{p^3}}\left(1+e\,c_{\theta_*}\right)^2 + \frac{f_r}{e}\, c_{\theta_*}\sqrt{\frac{p}{\mu}} - f_\theta\sqrt{\frac{p}{\mu}}\, s_{\theta_*}\, \frac{e\,c_{\theta_*}+2}{e\left(1+e\,c_{\theta_*}\right)}$$

The first equation for $\dot{p}$ can be replaced with that for a

$$a = \frac{p}{1-e^2} \quad \rightarrow \quad \dot{a} = \frac{\dot{p}}{1-e^2} + \frac{p\, 2e\dot{e}}{(1-e^2)^2}$$

Using the equations for $\dot{e}$ and $\dot{p}$, finally one obtains

$$\dot{a} = \frac{2a^2}{h}\left(e\, s_{\theta_*} f_r + \frac{p}{r} f_\theta\right)$$

In the same way, using

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta_*}{2} \quad \text{and} \quad M = E - e\, s_E$$

one can obtain the equations for E and M

$$\dot{E} = \sqrt{\frac{\mu}{a}}\frac{1}{r} + \frac{p}{a\sqrt{1-e^2}\, h e}\left[a\left(c_{\theta_*} - e\right) f_r + \left(r+a\right) s_{\theta_*} f_\theta\right]$$

$$\dot{M} = \sqrt{\frac{\mu}{a^3}} + \frac{\sqrt{1-e^2}}{he}\left[\left(p\, c_{\theta_*} - 2re\right) f_r - \left(p+r\right) s_{\theta_*} f_\theta\right]$$

It is worth remarking that the previous equations are singular if

$e = 0$ (circular orbits)

$i = 0$ (equatorial orbits)

for which either ω or Ω are not defined.

In this case, an alternative set may be defined for numerical propagations: the NONSINGULAR EQUINOCTIAL elements, not affected by this problem.

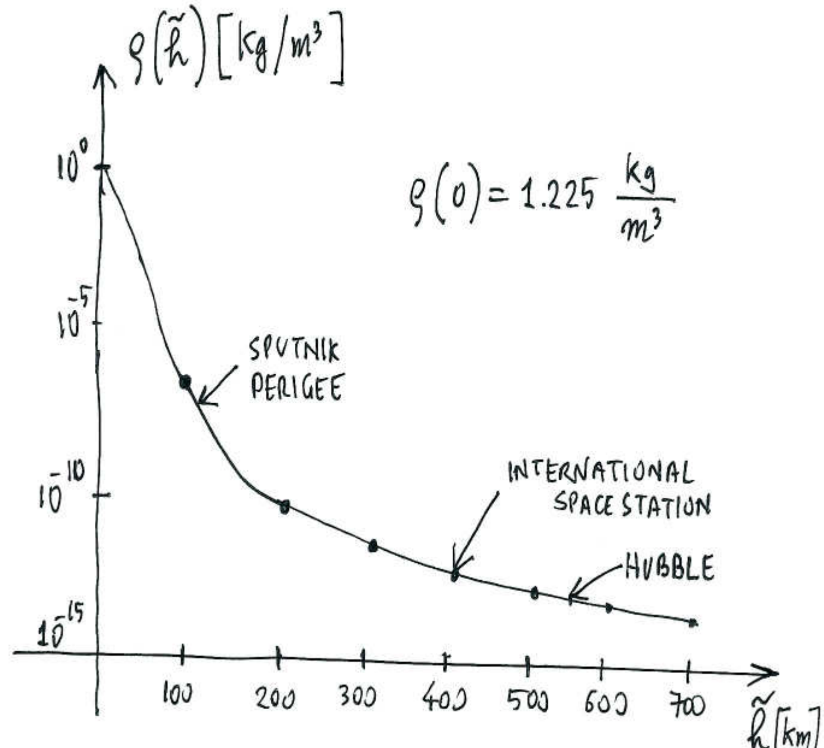
AERODYNAMIC DRAG

By convention, the altitude at which space "begins" is 100 km. In fact over 99.9999 % of the Earth atmosphere lies below 100 km. Nevertheless, over 100 km and up to 1000 km, aerodynamic drag affects orbital motion, and this perturbing effect increases considerably as the spacecraft altitude reduces (and more dense atmosphere is encountered).

A suitable atmospheric model is to be defined for the purpose of evaluating the aerodynamic drag perturbation. The common, sufficiently accurate approach for atmospheric density interpolation is based on using a piecewise exponential function of the altitude:

$$\rho(\tilde{h}) = \rho_{R,i} \exp\left[-\frac{\tilde{h}-\tilde{h}_{R,i}}{H_{R,i}}\right]$$

where $\tilde{h}_{R,i}$ is a reference altitude at which the density ρ equals $\rho_{R,i}$ whereas $H_{R,i}$ is the scale altitude, related to the decrease rate of ρ



Using tabular data and setting $\{\tilde{h}_{R,i}\}$ with $\rho(\tilde{h}_{R,i})$ known, $\{H_{R,i}\}$ can be found. These data can be taken from the US Standard Atmosphere 1976 (USSA76).

- Exact derivation of drag perturbing acceleration

The aerodynamic drag acceleration on an orbiting spacecraft is given by:

$$\underline{a_D} = \frac{D}{m} = -\frac{1}{2} C_D \frac{S}{m} \rho v_R^2 \widehat{v_R}$$

where S = aerodynamic surface (i.e. spacecraft cross section)

C_D = drag coefficient

ρ = atmospheric density

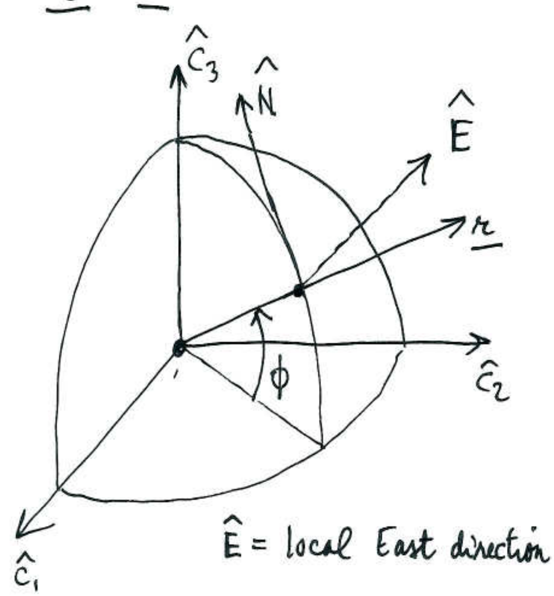
m = spacecraft mass

$\underline{v_R}$ = spacecraft velocity relative to the atmosphere

In general, S can vary depending on the spacecraft geometry and attitude; m varies if propulsion is employed (for instance). In rarefied flow regime $C_D \simeq 2.2$ (with a minimum value $C_D = 2$ associated with spheric spacecraft).

The velocity $\underline{v_R}$ equals

$\underline{v_R} = \underline{v} - \underline{v_a}$ where $\underline{v_a} = \underline{\omega_E} \times \underline{r}$ is the local velocity of atmosphere at position $\underline{r}$, whereas $\underline{v}$ is the spacecraft inertial velocity.



$\underline{\omega_E} = \omega_E \hat{c}_3$

$\rightarrow \underline{\omega_E} \times \underline{r} = \omega_E r \cos\phi \, \hat{E}$

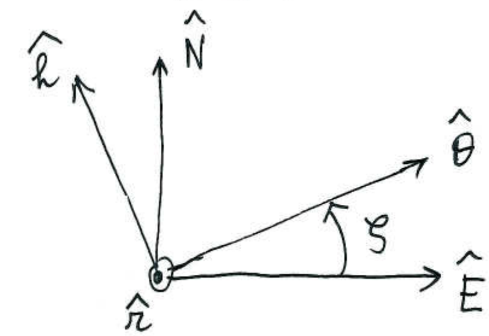
$\hat{E}$ = local East direction

The previous equation for $\underline{v_a} = \omega_E r c_\phi \hat{E}$ is found by assuming that the atmosphere rotates together with the Earth.

As a result, $\underline{v_R}$ has the following components along the local $(\hat{r}, \hat{E}, \hat{N})$-frame

$$\underline{v_R} = \begin{bmatrix} v_r & v_E - \omega_E r c_\phi & v_N \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{E} \\ \hat{N} \end{bmatrix}$$

Moreover, the spacecraft velocity components v_r, v_E, v_N are



$$v_r = \sqrt{\frac{\mu}{p}}\, e\, s_{\theta_*}$$

$$v_E = \sqrt{\frac{\mu}{p}} \left(1 + e\, c_{\theta_*}\right) c_\varsigma$$

$$v_N = \sqrt{\frac{\mu}{p}} \left(1 + e\, c_{\theta_*}\right) s_\varsigma$$

Then, the two angles ϕ and ς can be written in terms of orbit elements (see the Chapter "Keplerian Trajectories").

Finally, $\underline{v_R}$ is written in terms of its components along $(\hat{r}, \hat{\theta}, \hat{h})$

$$\underline{v_R} = \begin{bmatrix} v_r & v_E - \omega_E r c_\phi & v_N \end{bmatrix} R_1^T(\varsigma) \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{h} \end{bmatrix}$$ and this leads to

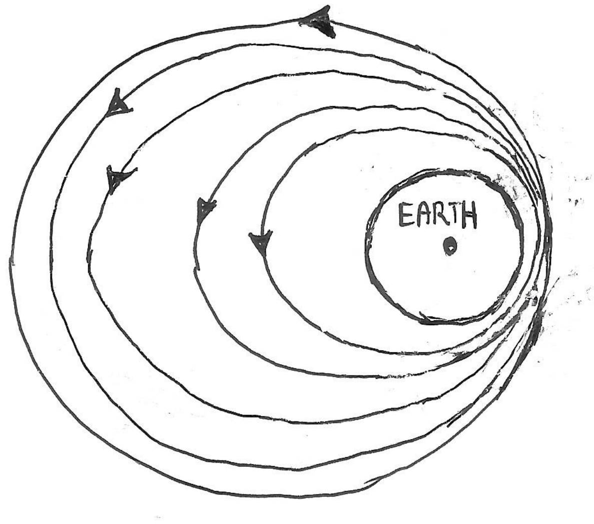
identifying the perturbing drag acceleration components

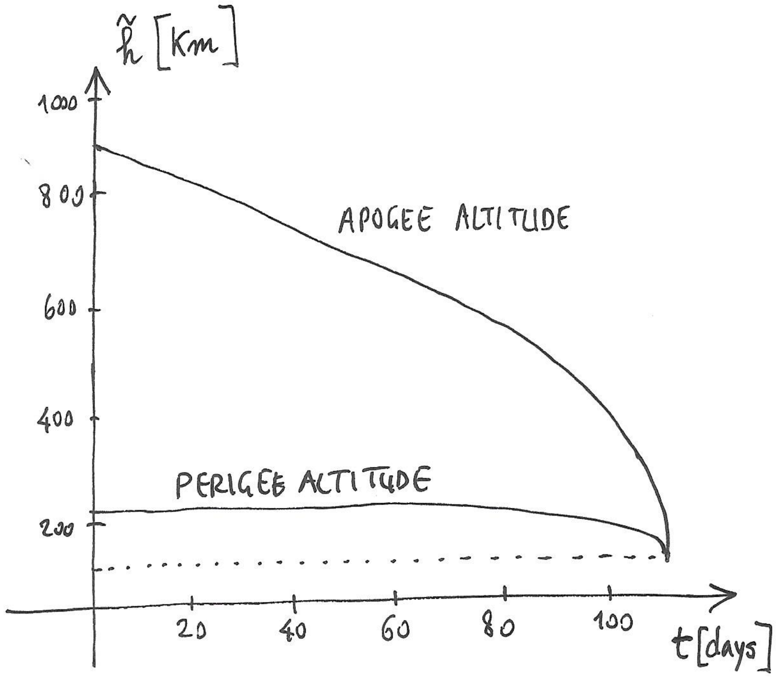
$$\left\{ f_r^{(D)},\ f_\theta^{(D)},\ f_h^{(D)} \right\}$$

which can be inserted into the Lagrange planetary equations for numerical integration.

- <u>Effect on orbit elements</u>

The term $c_D \frac{S}{m}$ is referred to as BALLISTIC COEFFICIENT, and is directly related to the action of drag, which is greater if this coefficient increases.





The aerodynamic drag action is concentrated at perigee for an elliptic orbit, and has the effect of decreasing the apogee altitude. As density is low at apogee, the perigee altitude is marginally reduced. By inspecting the previous typical behavior along elliptic orbits it is apparent that drag implies

$a \downarrow$ reduction of semimajor axis

$e \downarrow$ reduction of eccentricity

Along low-Earth circular orbits, the long term effect of drag yields a spiral trajectory that slowly decreases in altitude.

Usually, the term $f_h^{(D)}$ is very small, therefore $\dot{i}$ and $\dot{\Omega}$ nearly equal 0. This means that the aerodynamic drag yields only a modest (usually negligible) variation of the orbit plane.

- Approximate analysis for near-circular orbits

If the orbit is near-circular, then

$$a \simeq R \qquad \text{and} \qquad v \simeq \sqrt{\frac{\mu}{a}}$$

Moreover, if one assumes that $\underline{v}_R \simeq \underline{v}$ (by neglecting $\underline{v}_a$, which is small with respect to typical orbital velocities), one obtains

$$\underline{v} \simeq \sqrt{\frac{\mu}{a}}\,\hat{\theta} \quad \longrightarrow \quad f_\theta^{(D)} = -\frac{1}{2} C_D \frac{S}{m} \rho \frac{\mu}{a} \;;\; f_r^{(D)} = f_h^{(D)} = 0$$

Now, the Lagrange equation for a can be considered, to yield

$$\dot{a} = \frac{2a^2}{h} \frac{p}{r} f_\theta^{(D)} \underset{\uparrow}{=} \frac{2a^{3/2}}{\sqrt{\mu}} f_\theta^{(D)} = -C_D \frac{S}{m} \rho \sqrt{\mu a}$$

$p \simeq a$ (because $e \simeq 0$); $r \simeq a$

The previous differential equation can be integrated numerically. However, for a small variation of a, one can assume $\rho \simeq$ const, and the previous equation can be integrated analytically:

$$\frac{\dot{a}}{a^{1/2}} = -C_D \frac{S}{m} \rho \quad \longrightarrow \quad a_{fin}^{1/2} - a_{ini}^{1/2} = -\frac{C_D S}{2m} \rho \sqrt{\mu} \left(t_{fin} - t_{ini}\right)$$

The previous approximate solution holds for

> nearly-circular orbits

> limited variations of semimajor axis (otherwise ρ cannot be assumed as constant)

SOLAR RADIATION PRESSURE

The radiated power intensity at the Sun photosphere is

$$S_0 = 63.15 \cdot 10^6 \frac{W}{m^2}$$

Electromagnetic radiation follows the inverse square law, therefore at distance R from the Sun center, the radiation intensity S is

$$S = S_0 \left(\frac{R_0}{R}\right)^2$$ where $R_0 = 696000$ Km is the radius of the photosphere

As a result, along the Earth orbit (which is nearly circular with radius of $149.6 \cdot 10^6$ Km), the radiation intensity S_E is

$$S_E = 1367 \frac{W}{m^2}$$

S_E is also termed SOLAR CONSTANT.

If S_E is divided by c (the speed of light), one obtains the solar radiation pressure P_{SR}

$$P_{SR} = \frac{S_E}{c} = 4.56 \cdot 10^{-6} \frac{N}{m^2}$$

For the sake of simplicity, one can use the cannonball model, i.e. the satellite is assumed to be a sphere of radius σ. In this case the perturbing acceleration due to P_{SR} is

$$\underline{f}^{(SR)} = -\nu \frac{P_{SR} A}{m} C_R \hat{r}_{SUN}$$

where ν = shadow function = $\begin{cases} 1, & \text{spacecraft illuminated} \\ 0, & \text{spacecraft in shadow} \end{cases}$

A = reference surface, which is illuminated $= \pi \sigma^2$ (cannonball)
m = spacecraft mass
$\hat{r}_{SUN}$ = unit vector pointing from spacecraft to the Sun

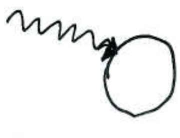
and, finally, C_R = radiation pressure coefficient, ranging from 1 to 2.

In particular: (i) total absorbing surface $\Rightarrow$ $C_R = 1$

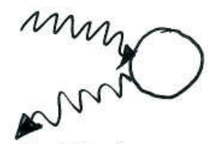
(ii) total reflecting surface $\Rightarrow$ $C_R = 2$

These two values derive from conservation of linear momentum, because radiation can be regarded as the impact of photons on the spacecraft. If photons are reflected, the action on the spacecraft is to be counted twice, i.e. the photon releases twice the linear momentum variation to the spacecraft of a totally absorbed photon.

(a) $C_R = 1$



Absorption

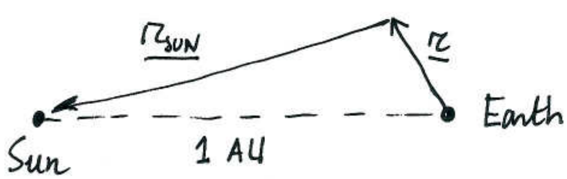


(b) $C_R = 2$

Reflection

Moreover, the unit vector $\hat{r}_{SUN}$ can be identified with the unit vector going from the Earth to the Sun.

In fact, an orbiting satellite has radius $r = |\underline{r}| \ll 1$ AU

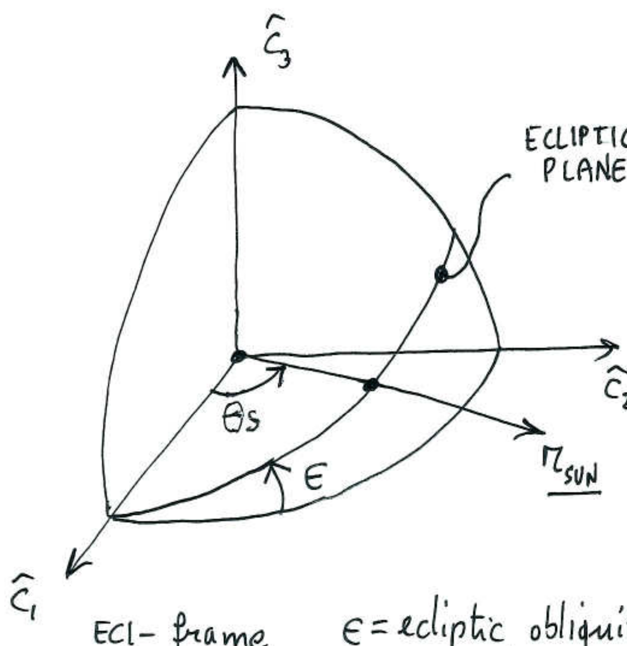


Therefore $\underline{r}_{SUN}$ can be regarded as the position vector of the Sun relative to the Earth.

In the right figure θ_S identifies the instantaneous position of the Sun in the ECI frame. If the Earth orbit is approximated as circular

$$\theta_S = \theta_{S0} + \frac{2\pi}{1\,sy}(t - t_0)$$

θ_{S0} = angle θ_S at t_0 ; 1 sy = 1 sidereal year



ECI-frame $\quad \epsilon$ = ecliptic obliquity = 23.45 deg

In the previous steps ϵ (ecliptic obliquity) was assumed as constant and θ_s (ecliptic longitude) was approximated as linearly time-varying. These are accurate approximations. However, algorithms exist for finding the actual values of ϵ and θ_s, as functions of the Julian date.

If ϵ and θ_s are specified, then $\underline{r}_{SUN}$ is given by

$$\underline{r}_{SUN} = 1\text{ AU} \begin{bmatrix} C_{\theta_s} & C_\epsilon S_{\theta_s} & S_\epsilon S_{\theta_s} \end{bmatrix} \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{bmatrix}$$

However, in the Chapter "Keplerian Trajectories" the following relation was found:

$$\begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{h} \end{bmatrix} = R_A \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{bmatrix} \Rightarrow \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{bmatrix} = R_A^T \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{h} \end{bmatrix}$$

Therefore $\underline{r}_{SUN}$ can be written in terms of $(\hat{r}, \hat{\theta}, \hat{h})$,

$$\underline{r}_{SUN} = 1\text{ AU} \underbrace{\begin{bmatrix} C_{\theta_s} & C_\epsilon S_{\theta_s} & S_\epsilon S_{\theta_s} \end{bmatrix} R_A^T}_{} \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{h} \end{bmatrix}$$

and the lower-bar corresponds to the three components of $\underline{r}_{SUN}$ along $(\hat{r}, \hat{\theta}, \hat{h})$. Once these are known, the solar radiation pressure perturbing acceleration, which has direction $-\hat{r}_{SUN}$ can be written in terms of its components $\left\{ f_r^{(SR)}, f_\theta^{(SR)}, f_h^{(SR)} \right\}$.

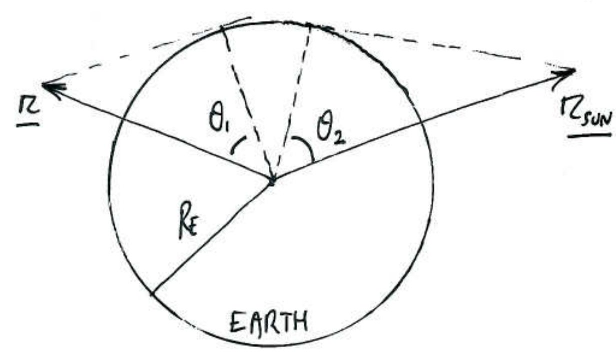
These can be inserted into the Lagrange planetary equations for numerical integration.

- Shadow computation

The angle between the satellite position $\underline{r}$ and the Sun position $\underline{r_{SUN}}$ is

$$\Theta = \operatorname{acos}\left(\frac{\underline{r} \cdot \underline{r_{SUN}}}{r\, r_{SUN}}\right)$$

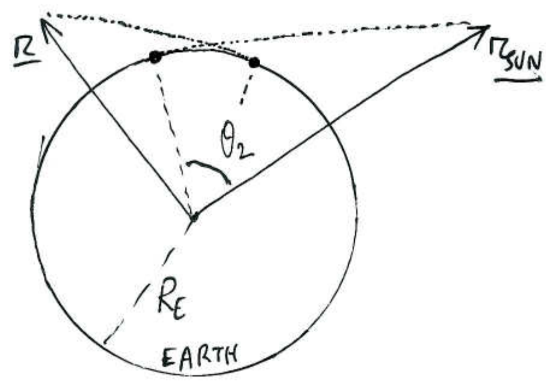
(a) SHADOW



Moreover, with reference to the figures

$$\begin{cases} \theta_1 = \operatorname{acos} \dfrac{R_E}{r} \\ \theta_2 = \operatorname{acos} \dfrac{R_E}{r_{SUN}} \end{cases}$$

(b) ILLUMINATION



θ_1 not shown

Two cases can occur:

(a) $\Theta > \theta_1 + \theta_2 \rightarrow$ SHADOW

(b) $\Theta \leq \theta_1 + \theta_2 \rightarrow$ ILLUMINATION

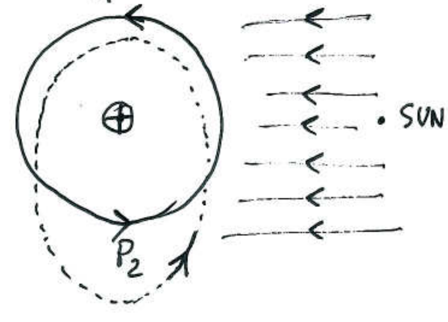
- Overview of the perturbing effect on orbit elements

In general, the solar radiation pressure yields variations for all orbit elements

For a circular orbit, an eccentricity is created, which is orthogonal to $\hat{\underline{r}}_{SUN}$:

at P_1, SRP "adds" velocity (positive acceleration)

at P_2, SRP "subtracts" velocity (negative acceleration)



$\rightarrow$ an elliptic orbit is obtained due to SRP, with $\underline{e}$ (not shown) orthogonal to $\underline{r_{SUN}}$ (elliptic orbit is denoted with ----).

Due to Earth rotation about the Sun, also the eccentricity vector $\underline{e}$ rotates

Earth gravitational harmonics

Several gravitational models of increasing fidelity have been employed in the past to describe the gravitational field of the Earth, e.g. WGS-84, EGM-96, and JGM-2, to name a few. All of them are based upon using the expression of the gravitational potential written in terms of harmonics, associated with Legendre polynomials.

In general, a celestial body with a specified geometry and mass distribution generates a gravitational potential that is the integral of the contribution of each infinitesimal mass *dm* that composes the body itself. With reference to the Earth,

$$U = G \int_{Earth} \frac{dm}{r}$$

where r is the distance between mass dm and the point at which the potential is evaluated, and G is the universal gravitation constant. After several analytical steps, one obtains the following expression for U

$$U = \frac{\mu_E}{r} - \frac{\mu_E}{r} \sum_{l=2}^{\infty} \left(\frac{R_e}{r} \right)^l J_l P_{l0} \left(\sin\phi \right) + \frac{\mu_E}{r} \sum_{l=2}^{\infty} \sum_{m=1}^{l} \left(\frac{R_e}{r} \right)^l J_{lm} P_{lm} \left(\sin\phi \right) \cos\left[m \left(\lambda_g - \lambda_{lm} \right) \right]$$

where μ_E $\left(= 398600.4 \text{ km}^3/\text{sec}^2\right)$ and R_E $\left(= 6378.136 \text{ km}\right)$ are the Earth gravitational parameter and equatorial radius, P_{lm} is the Legendre polynomial of degree l and order m, ϕ is the latitude, λ_g is the geographical longitude of the point at which the potential is evaluated, and r is its distance from the mass center of the attracting body; J_l is the coefficient associated with harmonic l, whereas J_{lm} and λ_{lm} are coefficients associated with harmonics of degree l and order m. All these coefficients depend on the actual geometry and mass distribution of the Earth. The polynomials P_{lm} can be calculated through the following relations:

$$P_{00} = 1$$

$$P_{lm} = \frac{1}{l!2^l} \left(1 - x^2\right)^{\frac{m}{2}} \frac{d^{l+m}}{dx^{l+m}} \left[\left(x^2 - 1\right)^l \right], \text{ with } x = \sin\phi$$

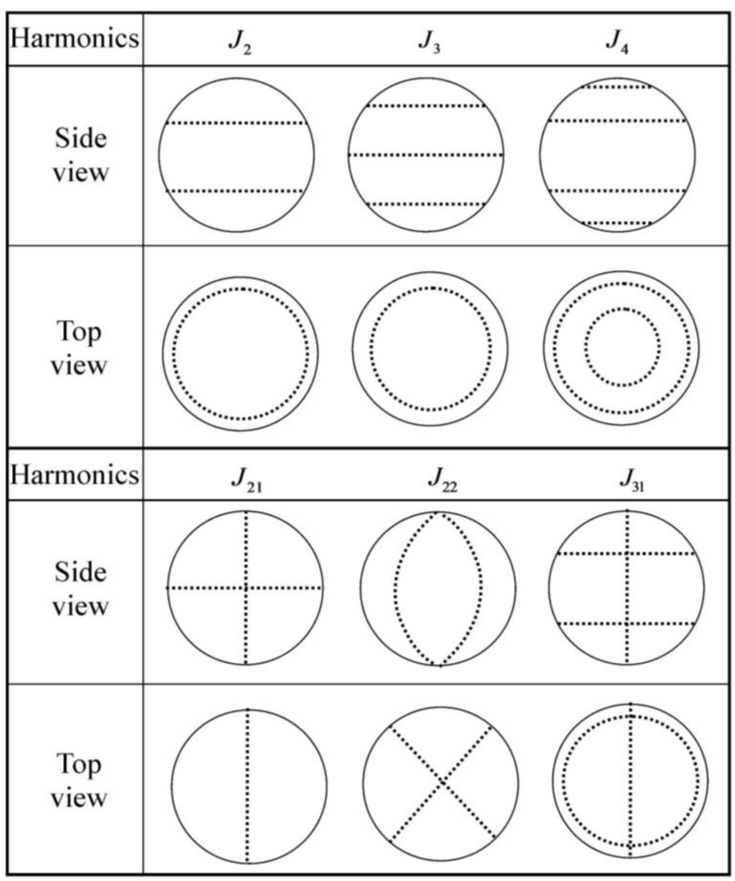
In the previous expression of U different contributions can be distinguished:

(a) *zonal* harmonics, depending only on latitude and associated with terms J_l,

(b) *sectoral* harmonics, depending only on longitude and associated with terms J_{ll} (i.e. terms J_{lm} when $l = m$), and

(c) *tesseral* harmonics, depending on both longitude and latitude, and associated with terms J_{lm}.

Zonal harmonics correspond to the terms of the geopotential that vanish at certain values of latitude. For instance, the term J_2 vanishes at latitudes $\pm 35.3\,\text{deg}$ and is representative of the Earth oblateness. Sectoral harmonics vanish at certain values of geographical longitude. As an example, harmonic J_{22} vanishes at the geographical longitudes of 30.1, 120.1, 210.1, and 300.1 deg, and is related to the (modest) eccentricity of the Earth equator. Tesseral harmonics vanish at given latitudes and geographical longitudes in a way such that the equipotential lines divide the Earth in tiles. As a general rule, harmonic J_{lm} has $(l-m)$ parallels and m meridians as equipotential lines; some of them are illustrated in the next figure.



Several harmonics of the gravitational field have been extensively studied and can be proven to be responsible of secular or periodic effects on the orbit elements of spacecraft orbiting the Earth. It is worth remarking that the first term of U, termed U_K $\left(=\mu_E/r\right)$ hence forward, corresponds to the potential generated by a body with spherical symmetry (both in geometry and in mass distribution). Due to the Newton's law, U_K yields the well known law of gravitation

$$\frac{d^2\boldsymbol{r}}{dt^2}=\nabla\left(\frac{\mu_E}{r}\right)=-\frac{\mu_E}{r^3}\boldsymbol{r}$$

where $\boldsymbol{r}$ identifies the position of a generic point with respect to the center of mass of the attracting body. The previous equation governs Keplerian motion, which is an excellent approximation of the actual motion of the planets around the Sun and represents an adequate approximation for analyzing exoatmospheric orbital motion around the Earth, at least for limited time intervals.

Accuracy of the Earth gravitational model depends on the accuracy of the coefficients J_{lm}. Recently, as a result of a consistent measurement campaign from orbiting satellites, the EGM2008 gravitational model has been introduced. The first coefficients of the EGM2008 model are reported in the following:

$J_2=1.083\cdot10^{-3}$ $\quad J_{21}=1.807\cdot10^{-9}$ $\quad J_{22}=1.816\cdot10^{-6}$

$J_3=-2.532\cdot10^{-6}$ $\quad J_{31}=2.209\cdot10^{-6}$ $\quad J_{32}=3.774\cdot10^{-7}$ $\quad J_{33}=2.214\cdot10^{-7}$

$J_4=-1.620\cdot10^{-6}$ $\quad J_{41}=6.786\cdot10^{-7}$ $\quad J_{42}=1.676\cdot10^{-7}$ $\quad J_{43}=6.042\cdot10^{-8}$ $\quad J_{44}=7.644\cdot10^{-9}$

$\lambda_{21}=1.719$ $\quad \lambda_{22}=-0.261$

$\lambda_{31}=0.122$ $\quad \lambda_{32}=-0.300$ $\quad \lambda_{33}=0.366$

$\lambda_{41}=-2.418$ $\quad \lambda_{42}=0.542$ $\quad \lambda_{43}=-0.067$ $\quad \lambda_{44}=0.530$

It is worth noting that the term J_2, related to Earth oblateness, dominates among all harmonics.

The expression of the Earth gravitational potential, expanded to a suitable order, yields the gravitational force (per mass unit) to the desired accuracy,

$$\boldsymbol{g}=\nabla U$$

where the operator ∇ can be expressed either in an inertial or in a rotating reference frame.

• J2 HARMONIC (EARTH OBLATENESS)

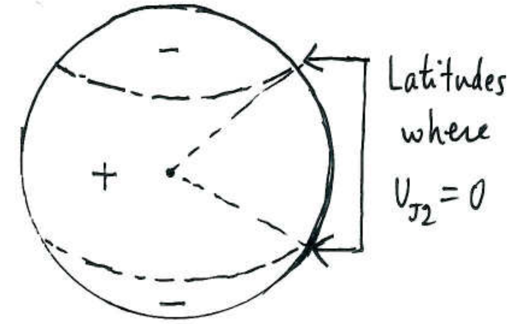
The J_2 harmonic of the Earth gravitational field has dominant effect as a perturbation, at low and medium orbit altitudes

It is related to Earth oblateness.

In fact, using the general expression for Legendre polynomials,

$$P_{20} = \frac{1}{2 \cdot 4} \frac{d^2}{dx^2}\left[\left(x^2-1\right)^2\right] = \frac{3x^2-1}{2} = \frac{3S_\phi^2-1}{2}$$

and the gravitational potential associated with J_2 is



$$U_{J2} = -\frac{\mu_E}{r}\left(\frac{R_E}{r}\right)^2 J_2 \frac{3S_\phi^2-1}{2}$$

and it turns out that

$$U_{J2} \begin{cases} <0 & \text{if } \operatorname{asin}\left(\frac{1}{\sqrt{3}}\right) < \phi < \frac{\pi}{2} \text{ or } -\frac{\pi}{2} < \phi < -\operatorname{asin}\left(\frac{1}{\sqrt{3}}\right) & (a) \\ >0 & \text{if } -\operatorname{asin}\left(\frac{1}{\sqrt{3}}\right) < \phi < \operatorname{asin}\left(\frac{1}{\sqrt{3}}\right) & (b) \end{cases}$$

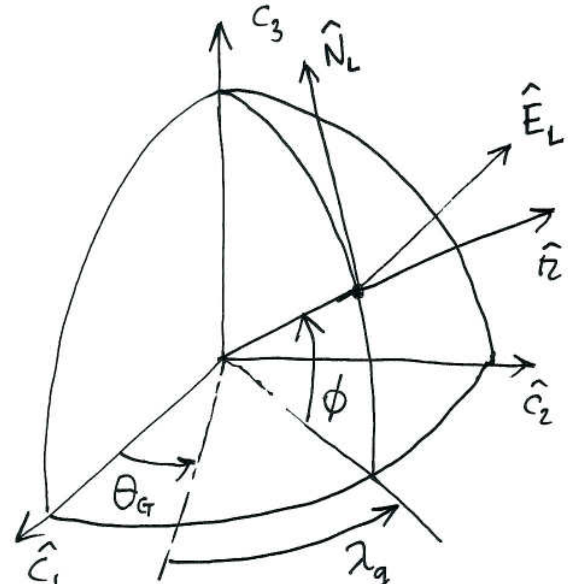
Therefore the term J_2 is associated with additional mass (b) around the equator, and less mass at the poles (a)

The term J_2 dominates because J_2 is three orders of magnitude greater than other harmonics of the geopotential.

- Exact derivation of the perturbing acceleration

For all harmonics of geopotential, it is convenient to use the ∇ operator written in spherical coordinates as

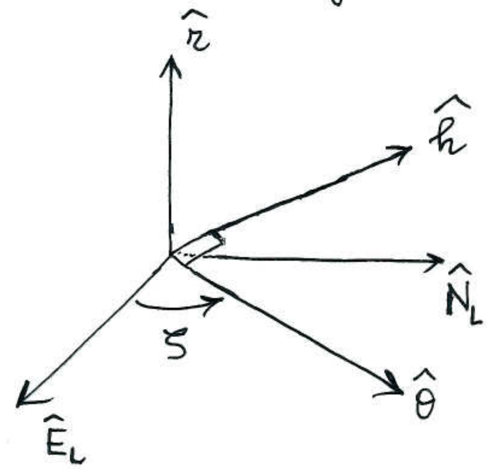
$$\nabla = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{E}_L}{r c_\phi}\frac{\partial}{\partial \lambda_g} + \frac{\hat{N}_L}{r}\frac{\partial}{\partial \phi}$$



where $\begin{cases} \lambda_g = \text{geographical longitude} \\ \phi = \text{latitude} \end{cases}$

and $\begin{cases} \hat{r} \leftrightarrow \text{radial direction} \\ \hat{E}_L \leftrightarrow \text{Eastward direction} \\ \hat{N}_L \leftrightarrow \text{Northward direction} \end{cases}$

In the right figure Θ_G is the Greenwich sidereal time (i.e., the instantaneous absolute longitude of the Greenwich meridian)

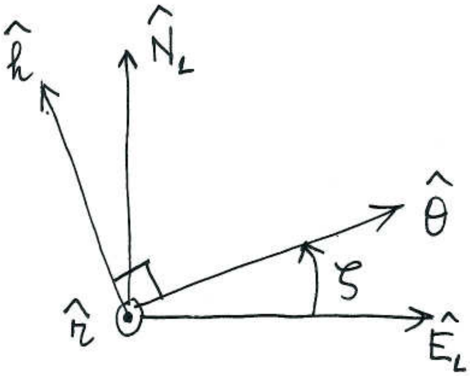


For the J_2 - perturbation

$$\frac{\partial U_{J2}}{\partial r} = \frac{3\mu_E}{r^4} R_E^2 J_2 \, \frac{3 s_\phi^2 - 1}{2}$$

$$\frac{\partial U_{J2}}{\partial \lambda_g} = 0$$

$$\frac{\partial U_{J2}}{\partial \phi} = -\frac{3\mu_E}{r^4} R_E^2 J_2 \, s_\phi c_\phi$$



ς = heading angle

Therefore, the perturbing acceleration due to J_2 is

$$\underline{f}_{J2} = \frac{3\mu_E}{r^4} R_E^2 J_2 \left[\frac{3 s_\phi^2 - 1}{2} \hat{r} - s_\phi c_\phi \hat{N}_L \right]$$

Because $\hat{N}_L = \hat{\theta} s_\zeta + \hat{h} c_\zeta$, the components of $\underline{f}_{J2}$ along $(\hat{r}, \hat{\theta}, \hat{h})$ are

$$\underline{f}_{J2} = \frac{3\mu_E}{r^4} R_E^2 J_2 \left[\frac{3 s_\phi^2 - 1}{2} \quad - s_\phi c_\phi s_\zeta \quad - s_\phi c_\phi c_\zeta \right] \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{h} \end{bmatrix}$$

In order to insert these components into the Lagrange planetary equations, it is convenient to express ϕ and ζ in terms of orbit elements. Using the following relations (derived in a previous chapter)

$$s_\phi = s_{\theta_t} s_i \qquad c_\phi s_\zeta = c_{\theta_t} s_i \qquad c_\phi c_\zeta = c_i$$

one obtains

$$\underline{f}_{J2} = \frac{3\mu_E}{r^4} R_E^2 J_2 \left[\frac{3 s_{\theta_t}^2 s_i^2 - 1}{2} \quad - s_i^2 s_{\theta_t} c_{\theta_t} \quad - s_i c_i s_{\theta_t} \right] \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{h} \end{bmatrix}$$

i.e.

$$\begin{cases} f_r^{(J2)} = \dfrac{3\mu_E}{r^4} R_E J_2 \dfrac{3 s_{\theta_t}^2 s_i^2 - 1}{2} \\[2ex] f_\theta^{(J2)} = - \dfrac{3\mu_E}{r^4} R_E^2 J_2 s_i^2 s_{\theta_t} c_{\theta_t} \\[2ex] f_h^{(J2)} = - \dfrac{3\mu_E}{r^4} R_E^2 J_2 s_i c_i s_{\theta_t} \end{cases}$$

- Averaging

The time variations of the osculating orbit elements exhibit an oscillatory behavior, in general.

Three components yield the overall time variation of an orbit element:

(a) SHORT-PERIOD oscillations (typically same scale as 1 orbit period)

(b) LONG-PERIOD oscillations

(c) SECULAR variations

Averaging is capable of obtaining more compact, useful expressions for the time derivatives of orbit elements, by "filtering" short-period oscillations, which typically are not very meaningful for applications. Instead, after averaging, the long-period time evolution (terms (b) and (c)) is described accurately.

Averaging is applied over a single orbital period, here regarded as "osculating" orbit period, with duration $2\pi\sqrt{\frac{a^3}{\mu}}$. In the right-hand sides of the Lagrange planetary equations, only the terms depending on θ_* are assumed to vary in an orbit period. Moreover, the time derivatives are converted into derivatives with respect to the true anomaly θ_*; let [ ] represent a generic orbit element,

$$\frac{d[\;]}{dt} = \frac{d[\;]}{d\theta_*}\frac{d\theta_*}{dt} \longrightarrow \frac{d[\;]}{d\theta_*} = \frac{1}{\frac{d\theta_*}{dt}}\frac{d[\;]}{dt}$$

In the previous expression $\frac{d\theta_*}{dt} \simeq \frac{h}{r^2}$ (i.e. the second term of the equation of $\dot{\theta}_*$ is here neglected, because the first one dominates)

The procedure can be applied to all orbit elements, although it is shown in detail only for Ω:

$$\dot{\Omega} = \frac{r f_h}{h s_i} s_{\theta_t} \Rightarrow \Omega' = \frac{d\Omega}{d\theta_*} = \frac{r^2}{h} \frac{r f_h}{h s_i} s_{\theta_t}$$

$$\langle \Omega' \rangle = \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2}{h} \frac{r f_h}{h s_i} s_{\theta_t} \, d\theta_* = \frac{1}{2\pi h^2 s_i} \int_0^{2\pi} r^3 s_{\theta_t} \left[-\frac{3\mu_E}{r^4} R_E^2 J_2 s_i c_i s_{\theta_t} \right] d\theta_*$$

$$= -\frac{3\mu_E R_E^2 J_2 c_i}{2\pi h^2 p} \int_0^{2\pi} (1 + e c_{\theta_*}) s^2_{\theta_t} \, d\theta_* =$$

$$= -\frac{3\mu_E R_E^2 J_2 c_i}{2\pi h^2 p} \underbrace{\int_0^{2\pi} (1 + e c_{\theta_*}) \frac{1 - \cos[2(\theta_* + \omega)]}{2} d\theta_*}_{\pi} =$$

$$= -\frac{3 R_E^2 J_2 c_i}{2 p^2}$$

As a last step, the average time derivative is obtained

$$\langle \dot{\Omega} \rangle = \langle \Omega' \dot{\theta}_* \rangle = \langle \Omega' \rangle \sqrt{\frac{\mu}{a^3}} = -\frac{3 R_E^2 J_2 \sqrt{\mu_E}}{2 a^{7/2} (1-e^2)^2} c_i$$

where in the last steps, the average motion expression, i.e. $\sqrt{\frac{\mu_E}{a^3}}$, was used. Multiplying the remaining terms of $\dot{\theta}_*$ by Ω' yields terms of order J_2^2 (neglected)

The same technique can be used for the remaining orbit elements, to yield the average time derivatives reported in the following.

$$\langle \dot a \rangle = 0$$

$$\langle \dot e \rangle = 0$$

$$\langle \dot i \rangle = 0$$

No average variation of semimajor axis, eccentricity, and inclination

$$\langle \dot\Omega \rangle = -\frac{3 R_E^2 J_2 \sqrt{\mu_E}}{2 a^{7/2} (1-e^2)^2}\, c_i$$

$$\langle \dot\omega \rangle = -\frac{3 R_E^2 J_2 \sqrt{\mu_E}}{2 a^{7/2} (1-e^2)^2} \left(\frac{5}{2} s_i^2 - 2\right)$$

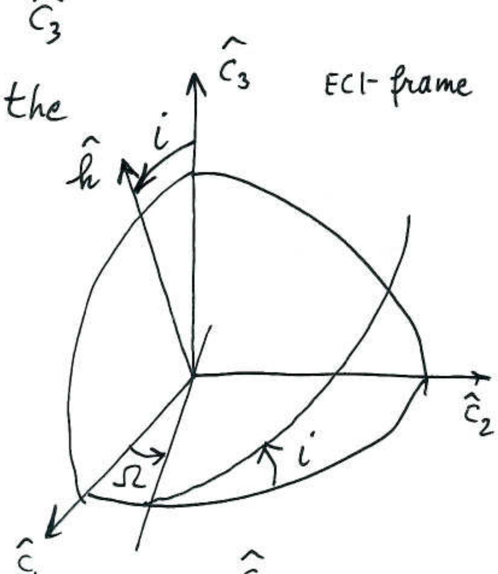
$$\langle \dot\theta_* \rangle = -\frac{3 R_E^2 J_2 \sqrt{\mu_E}}{2 a^{7/2} (1-e^2)^2} \left(1 - \frac{3}{2} s_i^2\right) + \sqrt{\frac{\mu_E}{a^3}}$$

It is worth remarking that two major effects are

(a) PRECESSION of ORBIT PLANE about axis $\hat c_3$

In fact $\langle \dot\Omega \rangle \neq 0$ (in general) while the inclination remains constant. This means that $\hat h$ rotates about the $\hat c_3$-axis with constant angle (the inclination) and period

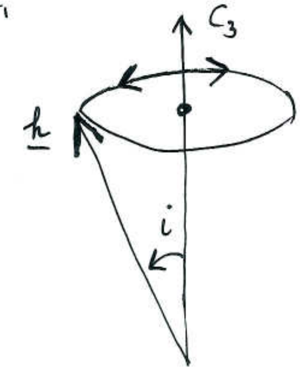
$$\frac{2\pi}{|\langle \dot\Omega \rangle|}$$



Precession occurs

(1) Clockwise if $i < \frac{\pi}{2}$ (direct orbits)

(2) Counterclockwise if $i > \frac{\pi}{2}$ (retrograde orbits)



(b) ROTATION OF APSIDAL LINE

In fact $\langle \dot{\omega} \rangle \neq 0$ (in general), therefore the periapse direction rotates.

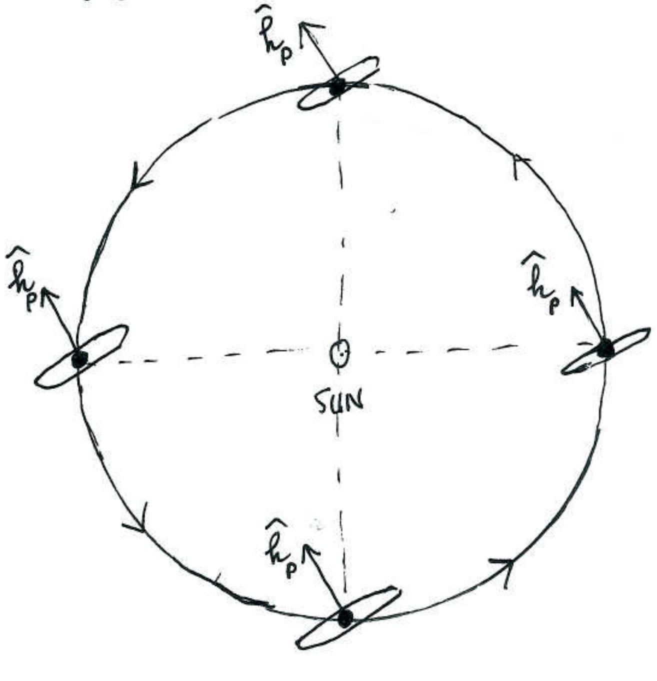
In order to obtain $\langle \dot{\omega} \rangle = 0$, one can select the so-called "critical inclinations" such that

$$5 s_i^2 - 4 = 0 \quad \text{i.e.} \quad s_i = \frac{2}{\sqrt{5}} \longrightarrow i = \begin{cases} \operatorname{asin}\left(\frac{2}{\sqrt{5}}\right) = 63.4 \text{ deg} \\ \pi - \operatorname{asin}\left(\frac{2}{\sqrt{5}}\right) = 116.6 \text{ deg} \end{cases}$$

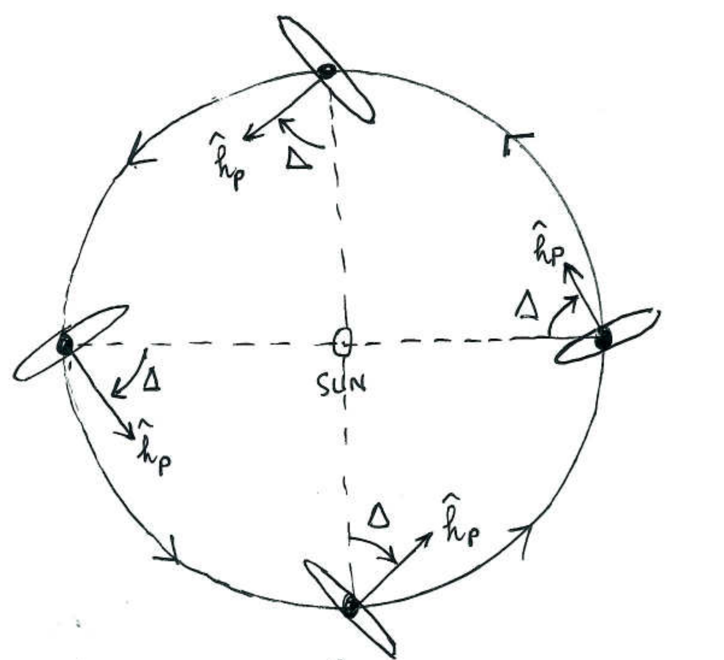
If any of these two inclinations is selected, the apse line is "frozen" in space.

- Sunsynchronous orbits

These orbits are relevant in applications, because they tend to preserve near-identical lighting conditions. This can be obtained if the orbit plane rotates with the same angular rate as that of the Earth around the Sun



Orientation of $\hat{h}_p$ in the absence of perturbations (in 1 year)



Orientation of $\hat{h}_p$ for sunsynchronous orbits (in 1 year)

In the previous two figures $\hat{h}_p$ represents the unit vector aligned with the projection of $\underline{h}$ into the ecliptic plane.

This unit vector $\hat{h}_p$ rotates with the same rate as the Earth motion around the Sun if the orbit plane of the spacecraft precedes with the same period. This phasing condition can be obtained thanks to the J_2 perturbation, because the precession period due to J_2 is

$$T_{prec} = \frac{2\pi}{|\langle\dot{\Omega}\rangle|} = \frac{2\pi \cdot 2\, a^{7/2}(1-e^2)^2}{3 R_E^2 J_2 \sqrt{\mu_E}\, |c_i|}$$

and this period must equal 1 year. Moreover, precession must occur counterclockwise (like the Earth motion around the Sun as seen from the ecliptic pole), therefore

$i > \frac{\pi}{2}$ for a sunsynchronous orbit be feasible.

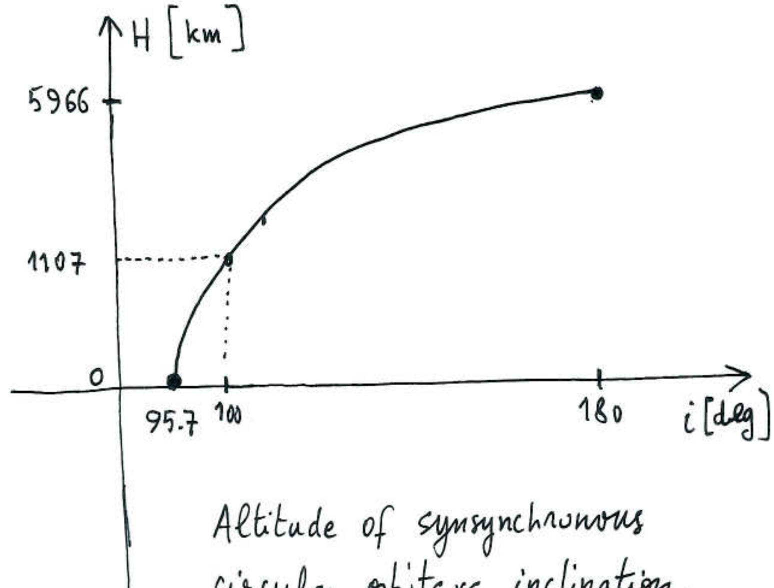
In the end, the condition for a sunsynchronous orbit is

$$\langle\dot{\Omega}\rangle = \frac{2\pi}{T_{1yr}} \quad (T_{1yr} = 1 \text{ year})$$

$$\downarrow$$

$$i = \text{acos}\left[-\frac{2\pi}{T_{1yr}} \frac{2a^{7/2}(1-e^2)^2}{3 R_E^2 J_2 \sqrt{\mu_E}}\right]$$

The right figure reports the plot (for circular orbits, $e=0$) of altitude versus inclination



Altitude of sunsynchronous circular orbits vs. inclination

Third body gravitational perturbation

In a spacecraft orbits the Earth, then the gravitational action of third bodies, such as the Moon and the Sun, can be regarded as a perturbation.

Exact derivation of the perturbing acceleration

If a spacecraft (denoted with subscript S) is subject to the simultaneous gravitational attraction of two celestial bodies (associated with subscripts 1 and 2), then

$$\frac{d^2\boldsymbol{r}_S}{dt^2} = -\frac{\mu_1}{r_{1S}^3}\boldsymbol{r}_{1S} - \frac{\mu_2}{r_{2S}^3}\boldsymbol{r}_{2S}$$

where μ_j $(j=1,2)$ is the gravitational parameter of body j, $\boldsymbol{r}_S$ is the spacecraft position vector in an inertial frame, whereas $\boldsymbol{r}_{jS} := \boldsymbol{r}_S - \boldsymbol{r}_j$ is the spacecraft position relative to attracting body j, and $r_{jS} := |\boldsymbol{r}_{jS}|$. Under the assumption that the spacecraft excerpts negligible gravitational attraction on body 1, the latter obeys the following dynamics equation:

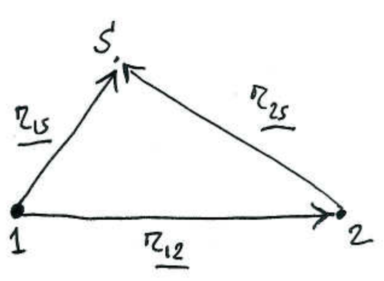
$$\frac{d^2\boldsymbol{r}_1}{dt^2} = -\frac{\mu_2}{r_{21}^3}\boldsymbol{r}_{21} \qquad \left(\boldsymbol{r}_{21} := \boldsymbol{r}_1 - \boldsymbol{r}_2 = -\boldsymbol{r}_{12};\ r_{21} = r_{12} = |\boldsymbol{r}_{21}|\right)$$

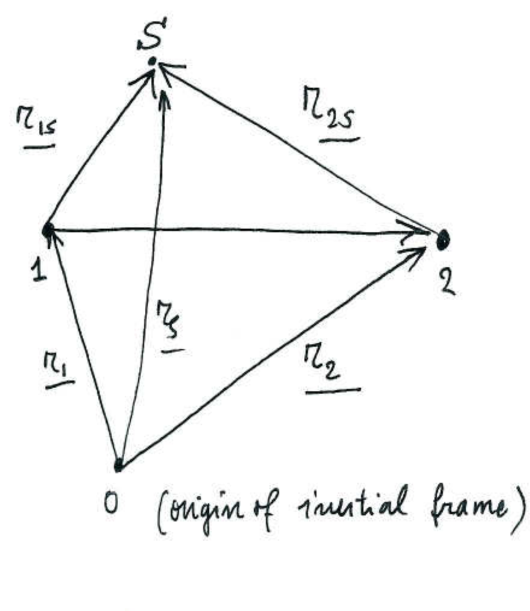
The relative position vector $\boldsymbol{r}_{1S}$ describes the spacecraft orbital motion with respect to body 1, which is assumed as the dominating celestial body. Combination of the previous equations yields

$$\frac{d^2\boldsymbol{r}_{1S}}{dt^2} = \frac{d^2(\boldsymbol{r}_S - \boldsymbol{r}_1)}{dt^2} = -\frac{\mu_1}{r_{1S}^3}\boldsymbol{r}_{1S} + \left[\frac{\mu_2}{r_{21}^3}\boldsymbol{r}_{21} - \frac{\mu_2}{r_{2S}^3}\boldsymbol{r}_{2S}\right]$$

In this equation, the term in square parentheses represents the disturbing acceleration $\boldsymbol{a}_{3B}$. Because $\boldsymbol{r}_{2S} = \boldsymbol{r}_{1S} - \boldsymbol{r}_{12}$, $\boldsymbol{a}_{3B}$ assumes the following form:

$$\boldsymbol{a}_{3B} = -\mu_2\left\{\frac{\boldsymbol{r}_{12}}{r_{12}^3} + \frac{\boldsymbol{r}_{1S} - \boldsymbol{r}_{12}}{\left[(\boldsymbol{r}_{1S} - \boldsymbol{r}_{12})\cdot(\boldsymbol{r}_{1S} - \boldsymbol{r}_{12})\right]^{\frac{3}{2}}}\right\}$$





$\underline{r}_{2S} = \underline{r}_{21} + \underline{r}_{1S} = \underline{r}_{1S} - \underline{r}_{12}$

This perturbing acceleration can be projected along $(\hat{r}, \hat{\theta}, \hat{h})$, once

$$\begin{cases} \underline{r_{12}} = \text{position of third body w.r.t. main body (i.e. Earth)} \\ \underline{r_{1S}} = \text{position of spacecraf w.r.t. main body (i.e. Earth)} \end{cases}$$

are known in $(\hat{r}, \hat{\theta}, \hat{h})$.

For the spacecraft

$$\underline{r_{1S}} = \frac{p}{1+e\,c_{\theta_*}}\,\hat{r} \qquad \left(\underline{r} := \underline{r_{1S}} \text{ in this section} \right)$$

For the third body, two cases must be considered

(1) MOON

$\underline{r_{12}}$ is the Moon position w.r.t. the Earth Center, which can be expressed in terms of Moon orbit elements

$$\underline{r_{12}} = \frac{p_M}{1+e_M c_{\theta_{*M}}} \left[c_{\theta_{tM}} c_{\Omega_M} - s_{\theta_{tM}} c_{i_M} s_{\Omega_M} \quad c_{\theta_{tM}} s_{\Omega_M} + s_{\theta_{tM}} c_{i_M} c_{\Omega_M} \quad s_{\theta_{tM}} s_{i_M} \right] \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{bmatrix}$$

where $\theta_{tM} = \omega_M + \theta_{*M}$ is the Moon argument of latitude

Then, as $\begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{h} \end{bmatrix} = R_A \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{bmatrix}$ one obtains

(see chapter on Keplerian trajectories)

$$\underline{r_{12}} = \underbrace{\frac{p_M}{1+e_M c_{\theta_{*M}}} \left[c_{\theta_{tM}} c_{\Omega_M} - s_{\theta_{tM}} c_{i_M} s_{\Omega_M} \quad c_{\theta_{tM}} s_{\Omega_M} + s_{\theta_{tM}} c_{i_M} c_{\Omega_M} \quad s_{\theta_{tM}} s_{i_M} \right] R_A^T}\begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{h} \end{bmatrix}$$

The lower bar corresponds to the three components of $\underline{r_{12}}$ along $(\hat{r}, \hat{\theta}, \hat{h})$

(2) SUN

$\underline{r}_{12}$ is the Sun position relative to the Earth. Previously, this was written in terms of ϵ (ecliptic obliquity) and θ_S (ecliptic longitude), as follows

$$\underline{r}_{12} = 1\,\mathrm{AU}\underbrace{\left[C_{\theta_S} \quad C_\epsilon S_{\theta_S} \quad S_\epsilon S_{\theta_S} \right] R_A^T}\begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{h} \end{bmatrix}$$

The lower bar corresponds to the three components of $\underline{r}_{12}$ in $(\hat{r}, \hat{\theta}, \hat{h})$

- <u>Approximate perturbing acceleration using the gravity gradient</u>

The perturbing acceleration due to a 3rd body was given by

$$\underline{f}_{3B} = -\frac{\mu_2 \underline{r}_{12}}{r_{12}^3} - \frac{\mu_2}{r_{2S}^3}\underline{r}_{2S} = \left\{ -\frac{\mu_2}{r_{2S}^3}(\underline{r}_{21} + \underline{r}_{1S}) \right\} - \left\{ -\frac{\mu_2}{r_{21}^3}\underline{r}_{21} \right\} \simeq$$

$$\simeq \underbrace{\frac{\partial}{\partial \underline{r}_{2j}}\left[-\frac{\mu_2}{r_{2j}^3}\underline{r}_{2j} \right]\Bigg|_{\underline{r}_{21}}}_{\text{GRAVITY GRADIENT DYAD}} \cdot \underline{r}_{1S}$$

assuming $|\underline{r}_{1S}| \ll |\underline{r}_{12}|$ (which is usually met)

Dyads are mathematical objects such that their dot product yields a vector. In particular, the gravity gradient dyad is

$$\frac{\partial}{\partial \underline{r}_{2j}}\left[-\frac{\mu_2}{r_{2j}^3}\underline{r}_{2j} \right] = [\hat{c}_1 \quad \hat{c}_2 \quad \hat{c}_3] \begin{bmatrix} \frac{\partial}{\partial x}\left(-\frac{\mu_2 x}{r_{2j}^3}\right) & \frac{\partial}{\partial y}\left(-\frac{\mu_2 x}{r_{2j}^3}\right) & \frac{\partial}{\partial z}\left(-\frac{\mu_2 x}{r_{2j}^3}\right) \\ \frac{\partial}{\partial x}\left(-\frac{\mu_2 y}{r_{2j}^3}\right) & \frac{\partial}{\partial y}\left(-\frac{\mu_2 y}{r_{2j}^3}\right) & \frac{\partial}{\partial z}\left(-\frac{\mu_2 y}{r_{2j}^3}\right) \\ \frac{\partial}{\partial x}\left(-\frac{\mu_2 z}{r_{2j}^3}\right) & \frac{\partial}{\partial y}\left(-\frac{\mu_2 z}{r_{2j}^3}\right) & \frac{\partial}{\partial z}\left(-\frac{\mu_2 z}{r_{2j}^3}\right) \end{bmatrix} \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{bmatrix}$$

with $r_{2j} = \sqrt{x^2 + y^2 + z^2}$

In particular, the diagonal element (1,1) is

$$\frac{\partial}{\partial x}\left(-\frac{\mu_2 x}{[x^2+y^2+z^2]^{3/2}} \right) = -\frac{\mu_2}{r_{12}^3} + \frac{3\mu_2 x^2}{r_{12}^5}$$

whereas the off-diagonal element (1,2) is

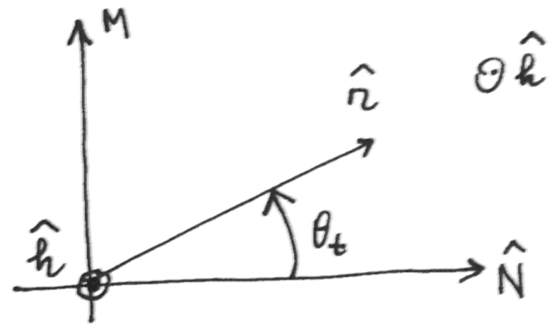
$$\frac{\partial}{\partial y}\left(-\frac{\mu_2 x}{[x^2+y^2+z^2]^{3/2}}\right) = \frac{3\mu_2 y^2}{r_{12}^5}$$

Therefore, the compact form for the inertia dyad is

$$\frac{\partial}{\partial \underline{r}_{2j}}\left[-\frac{\mu_2}{r_{2j}^3}\underline{r}_{2j}\right]\Bigg|_{\underline{r}_{21}} = \frac{\mu_2}{r_{12}^5}\left[3\,\underline{r}_{12}\,\underline{r}_{12} - r_{12}^2\,\underset{\rightarrow}{1}\right], \quad \underset{\rightarrow}{1} = \text{unit dyad}$$

i.e. $\underset{\rightarrow}{1} = \hat{c}_1\hat{c}_1 + \hat{c}_2\hat{c}_2 + \hat{c}_3\hat{c}_3$

- Average effect on circular orbits



θ_t = argument of latitude

Let the spacecraft be placed in circular orbit. This means that in the orbit plane

$$\hat{r} = \hat{N}c_{\theta_t} + \hat{M}s_{\theta_t}, \text{ where}$$

$$\dot{\theta}_t = \sqrt{\frac{\mu_1}{R^3}} \text{ is constant}$$

In general, the vector equation for $\underline{h}$ is

$$\frac{d\underline{h}}{dt} = \underline{r}\times\underline{f} \qquad (\text{with } \underline{r} = \underline{r}_{1S} \text{ in this section})$$

After inserting the expression of $\underline{f}_{3B}$ found previously,

$$\frac{d\underline{h}}{dt} = \underline{r}\times\left\{\frac{\mu_2}{r_{12}^5}\left[3\,\underline{r}_{12}\,\underline{r}_{12} - r_{12}^2\,\underset{\rightarrow}{1}\right]\cdot\underline{r}\right\} \underset{\underset{\rightarrow}{1}\cdot\underline{r}=\underline{r}}{=} \frac{3\mu_2}{r_{12}^5}(\underline{r}\times\underline{r}_{12})(\underline{r}_{12}\cdot\underline{r})$$

$$= -\frac{3\mu_2}{r_{12}^5}\,\underline{r}_{12}\times\underbrace{\underline{r}\;\underline{r}}\cdot\underline{r}_{12}, \text{ where } \underline{r}\,\underline{r} \text{ is a dyadic again}$$

(1) FIRST AVERAGING is done over an orbit period of the spacecraft. During this interval $\underline{r}_{12}$ does not change considerably, thus $\underline{r}_{12}$ is taken as constant. Only $(\underline{r}\,\underline{r})$ is to be averaged

$$\langle \underline{r}\,\underline{r}\rangle = R^2 \langle \hat{r}\hat{r}\rangle = R^2 \langle \left(\hat{N} c_{\theta_t} + \hat{M} s_{\theta_t}\right)\left(\hat{N} c_{\theta_t} + \hat{M} s_{\theta_t}\right)\rangle =$$

$$= R^2 \langle \left(\hat{N}\hat{N} c^2_{\theta_t} + \hat{M}\hat{M} s^2_{\theta_t} + \hat{N}\hat{M} c_{\theta_t} s_{\theta_t} + \hat{M}\hat{N} c_{\theta_t} s_{\theta_t}\right)\rangle =$$

$$= R^2 \left\{ \hat{N}\hat{N}\frac{1}{2\pi}\int_0^{2\pi} c^2_{\theta_t}\, d\theta_* + \hat{M}\hat{M}\frac{1}{2\pi}\int_0^{2\pi} s^2_{\theta_t}\, d\theta_* + \left(\hat{N}\hat{M} + \hat{M}\hat{N}\right)\frac{1}{2\pi}\int_0^{2\pi} s_{\theta_t} c_{\theta_t}\, d\theta_* \right\} =$$

$$= R^2\left(\frac{\hat{N}\hat{N}}{2} + \frac{\hat{M}\hat{M}}{2}\right)$$

For any orthonormal sequence, the unit dyad is defined as

$$\underset{\rightarrow}{1} = \hat{c}_1\hat{c}_1 + \hat{c}_2\hat{c}_2 + \hat{c}_3\hat{c}_3 = \hat{N}\hat{N} + \hat{M}\hat{M} + \hat{h}\hat{h}$$

This implies that $\langle \underline{r}\,\underline{r}\rangle = R^2\left[\frac{\underset{\rightarrow}{1}}{2} - \frac{\hat{h}\hat{h}}{2}\right]$. As a result

$$\langle \frac{d\underline{h}}{dt}\rangle = -\frac{3\mu_2}{2r_{12}^5}\, \underline{r}_{12} \times \left\{R^2\left[\frac{\underset{\rightarrow}{1}}{2} - \frac{\hat{h}\hat{h}}{2}\right]\right\}\cdot \underline{r}_{12} = \frac{3\mu_2 R^2}{2r_{12}^5}\left(\underline{r}_{12}\times\hat{h}\right)\left(\hat{h}\cdot\underline{r}_{12}\right)$$

(2) SECOND AVERAGING is done over an orbit period of the perturbing body, which is assumed as placed in a circular orbit about Earth; therefore

$$\langle \underline{r}_{12}\,\underline{r}_{12}\rangle = r_{12}^2\left[\frac{\underset{\rightarrow}{1}}{2} - \frac{\hat{h}_{3B}\hat{h}_{3B}}{2}\right]$$

where $\hat{h}_{3B}$ is orthogonal to the plane where the relative motion of body 3 takes place

The preceding relation is rewritten as

$$\langle \frac{d\underline{h}}{dt}\rangle = -\frac{3\mu_2 R^2}{2r_{12}^5}\, \underline{r}_{12}\cdot\hat{h}\;\hat{h}\times\underline{r}_{12} = \frac{3\mu_2 R^2}{2r_{12}^5}\,\hat{h}\cdot\underline{r}_{12}\;\underline{r}_{12}\times\hat{h}$$

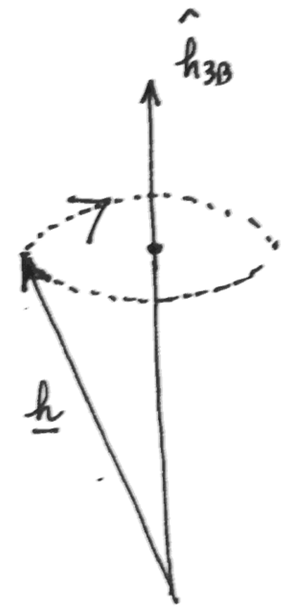
and double averaging leads to

$$\langle\langle \frac{d\underline{h}}{dt}\rangle\rangle = \frac{3\mu_2 R^2}{2r_{12}^5}\,\hat{h}\cdot\left\{r_{12}^2\left[\frac{\underset{\rightarrow}{1}}{2} - \frac{\hat{h}_{3B}\hat{h}_{3B}}{2}\right]\right\}\times\hat{h} \underset{\substack{\uparrow \\ \hat{h}\cdot\underset{\rightarrow}{1} = \hat{h}}}{=}$$

$$= \underbrace{-\frac{3}{4}\frac{\mu_2 R^2}{r_{12}^3}\left(\hat{h}\cdot\hat{h}_{3B}\right)\hat{h}_{3B}}_{h\,\underline{\omega}_h} \times \hat{h} = \underline{\omega}_h \times \underline{h}$$

One can recognize that the derivative of $\underline{h}$, after double averaging, can be written as $\underline{\omega}_h \times \underline{h}$, with

$$\underline{\omega}_h = -\frac{3}{4}\frac{\mu_2 R^2}{h\, r_{12}^3}\left(\hat{h}\cdot\hat{h}_{3B}\right)\hat{h}_{3B}$$



This means that on average $\underline{h}$ changes only due to its rotation about $\hat{h}_{3B}$, without changing its own magnitude, i.e. $\underline{h}$ (ang. momentum of the spacecraft) precedes about $\hat{h}_{3B}$

if $\hat{h}\cdot\hat{h}_{3B} > 0 \rightarrow$ clockwise precession

if $\hat{h}\cdot\hat{h}_{3B} < 0 \rightarrow$ counterclockwise precession

The precession period is $T_{prec} = \dfrac{2\pi}{|\omega_h|}$

$\dot{h} = 0$
magnitude of $\underline{h}$ does not change

- <u>Precession of Moon orbit</u>

If the previous equation is used for the Moon, regarded as a satellite of Earth, one obtains

$\hat{h}_{3B} = \hat{h}_{SUN}$ ($\hat{h}_{SUN}$ directed along the ecliptic pole, $\perp$ to ecliptic plane)

$\hat{h}_{3B}\cdot\hat{h} = \hat{h}_{SUN}\cdot\hat{h}_{MOON} = \cos\delta_M$ where $\delta_M = 5.9$ deg

and one gets

$T_{prec,moon} \cong 18$ years, period of precession of the orbit of the Moon

(the actual value is very close to 18 years)

- Appendix A: dyads

Dyads are mathematical objects from multidimensional algebra. They are denoted with $\underset{\rightarrow}{D}$ and are such that

$$(\underset{\rightarrow}{D} \cdot \underline{v}) \text{ and } (\underline{v} \cdot \underset{\rightarrow}{D}) \text{ are physical vectors}$$

In the previous expression, $\underline{v}$ itself is a physical vector, i.e. an object with components and the associated right-hand sequence of unit vectors (the "basis").

The dyad is a multi-dimensional vector. Also $\underline{v}\,\underline{w}$ is a dyad, because

$$\underline{z} \cdot (\underline{v}\,\underline{w}) = (\underline{z} \cdot \underline{v})\,\underline{w} \quad \text{is a vector}$$

$$(\underline{v}\,\underline{w}) \cdot \underline{z} = \underline{v}\,(\underline{w} \cdot \underline{z}) \quad \text{is a vector}$$

The preceding two relations show that in general

$$\underset{\rightarrow}{D} \cdot \underline{v} \neq \underline{v} \cdot \underset{\rightarrow}{D}$$

While a physical vector can be represented as

$$\underline{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{bmatrix}$$

a dyad has its components as elements of a square matrix D, with elements $\{d_{ij}\}$

$$\underset{\rightarrow}{D} = \begin{bmatrix} \hat{a}_1 & \hat{a}_2 & \hat{a}_3 \end{bmatrix} D \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{bmatrix} = \sum_{i,j=1}^{3,3} d_{ij}\, \hat{a}_i \hat{a}_j$$

The unit dyad $\underset{\rightarrow}{1}$ is associated with the I_{3x3} (3x3)-identity matrix in any reference frame, i.e.

$$\underset{\rightarrow}{1} = \hat{a}_1\hat{a}_1 + \hat{a}_2\hat{a}_2 + \hat{a}_3\hat{a}_3 = \hat{b}_1\hat{b}_1 + \hat{b}_2\hat{b}_2 + \hat{b}_3\hat{b}_3$$

where $(\hat{a}_1, \hat{a}_2, \hat{a}_3)$ and $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$ are arbitrary

- Appendix B: gravity gradient dyad

In the derivation of the perturbing acceleration due to a 3rd body one obtains

$$\underline{f}_{3B} = \underline{g}(\underline{r}_{21} + \underline{r}_{1S}) - \underline{g}(\underline{r}_{21}), \quad \text{with} \quad \underline{g}(\underline{r}_{2j}) = -\frac{\mu_2}{r_{2j}^3}\underline{r}_{2j}$$

Letting

$$\begin{cases} \underline{g} = g_1\hat{c}_1 + g_2\hat{c}_2 + g_3\hat{c}_3 \\ \underline{r}_{2j} = x\hat{c}_1 + y\hat{c}_2 + z\hat{c}_3 \\ \underline{r}_{1S} = \delta x\,\hat{c}_1 + \delta y\,\hat{c}_2 + \delta y\,\hat{c}_3 \end{cases}$$

the single components of $\underline{g}$ can be written as

$$\hat{c}_1)\quad g_1(\underline{r}_{21} + \underline{r}_{1S}) - g_1(\underline{r}_{21}) \cong \left.\frac{\partial g_1}{\partial x}\right|_{\underline{r}_{21}}\delta x + \left.\frac{\partial g_1}{\partial y}\right|_{\underline{r}_{21}}\delta y + \left.\frac{\partial g_1}{\partial z}\right|_{\underline{r}_{21}}\delta z$$

$$\hat{c}_2)\quad g_2(\underline{r}_{21} + \underline{r}_{1S}) - g_2(\underline{r}_{21}) \cong \left.\frac{\partial g_2}{\partial x}\right|_{\underline{r}_{21}}\delta x + \left.\frac{\partial g_2}{\partial y}\right|_{\underline{r}_{21}}\delta y + \left.\frac{\partial g_2}{\partial z}\right|_{\underline{r}_{21}}\delta z$$

$$\hat{c}_3)\quad g_3(\underline{r}_{21} + \underline{r}_{1S}) - g_3(\underline{r}_{21}) \cong \left.\frac{\partial g_3}{\partial x}\right|_{\underline{r}_{21}}\delta x + \left.\frac{\partial g_3}{\partial y}\right|_{\underline{r}_{21}}\delta y + \left.\frac{\partial g_3}{\partial z}\right|_{\underline{r}_{21}}\delta z$$

provided that $|\underline{r}_{1S}| \ll |\underline{r}_{21}|$.

$$\text{with} \quad \underline{g} = -\frac{\mu_2(x\hat{c}_1 + y\hat{c}_2 + z\hat{c}_3)}{(x^2 + y^2 + z^2)^{3/2}}$$

In vector form

$$\underline{g}\left(\underline{r}_{21}+\underline{r}_{15}\right)-\underline{g}\left(\underline{r}_{21}\right)\simeq \hat{c}_1\left[\left.\frac{\partial g_1}{\partial x}\right|_{\underline{r}_{21}}\delta x+\left.\frac{\partial g_1}{\partial y}\right|_{\underline{r}_{21}}\delta y+\left.\frac{\partial g_1}{\partial z}\right|_{\underline{r}_{21}}\delta z\right]+$$

$$+\,c_2\left[\left.\frac{\partial g_2}{\partial x}\right|_{\underline{r}_{21}}\delta x+\left.\frac{\partial g_2}{\partial y}\right|_{\underline{r}_{21}}\delta y+\left.\frac{\partial g_2}{\partial z}\right|_{\underline{r}_{21}}\delta z\right]+\hat{c}_3\left[\left.\frac{\partial g_3}{\partial x}\right|_{\underline{r}_{21}}\delta x+\left.\frac{\partial g_3}{\partial y}\right|_{\underline{r}_{21}}\delta y+\left.\frac{\partial g_3}{\partial z}\right|_{\underline{r}_{21}}\delta z\right]$$

$$=\begin{bmatrix}\hat{c}_1 & \hat{c}_2 & \hat{c}_3\end{bmatrix}\begin{bmatrix}\left.\frac{\partial g_1}{\partial x}\right|\delta x+\left.\frac{\partial g_1}{\partial y}\right|\delta y+\left.\frac{\partial g_1}{\partial z}\right|\delta z\\ \left.\frac{\partial g_2}{\partial x}\right|\delta x+\left.\frac{\partial g_2}{\partial y}\right|\delta y+\left.\frac{\partial g_2}{\partial z}\right|\delta z\\ \left.\frac{\partial g_3}{\partial x}\right|\delta x+\left.\frac{\partial g_3}{\partial y}\right|\delta y+\left.\frac{\partial g_3}{\partial z}\right|\delta z\end{bmatrix}=$$

(omitting $\underline{r}_{21}$ to the right of $\frac{\partial}{\partial}\Big|$)

$$=\begin{bmatrix}\hat{c}_1 & \hat{c}_2 & \hat{c}_3\end{bmatrix}\begin{bmatrix}\left.\frac{\partial g_1}{\partial x}\right| & \left.\frac{\partial g_1}{\partial y}\right| & \left.\frac{\partial g_1}{\partial z}\right|\\ \left.\frac{\partial g_2}{\partial x}\right| & \left.\frac{\partial g_2}{\partial y}\right| & \left.\frac{\partial g_2}{\partial z}\right|\\ \left.\frac{\partial g_3}{\partial x}\right| & \left.\frac{\partial g_3}{\partial y}\right| & \left.\frac{\partial g_3}{\partial z}\right|\end{bmatrix}\begin{bmatrix}\delta x\\ \delta y\\ \delta z\end{bmatrix}=$$

$$\begin{bmatrix}\hat{c}_1\\ \hat{c}_2\\ \hat{c}_3\end{bmatrix}\cdot\begin{bmatrix}\hat{c}_1 & \hat{c}_2 & \hat{c}_3\end{bmatrix}=I_{3\times3}$$

$$=\begin{bmatrix}\hat{c}_1 & \hat{c}_2 & \hat{c}_3\end{bmatrix}\begin{bmatrix}\left.\frac{\partial g_1}{\partial x}\right| & \left.\frac{\partial g_1}{\partial y}\right| & \left.\frac{\partial g_1}{\partial z}\right|\\ \left.\frac{\partial g_2}{\partial x}\right| & \left.\frac{\partial g_2}{\partial y}\right| & \left.\frac{\partial g_2}{\partial z}\right|\\ \left.\frac{\partial g_3}{\partial x}\right| & \left.\frac{\partial g_3}{\partial y}\right| & \left.\frac{\partial g_3}{\partial z}\right|\end{bmatrix}\overbrace{\begin{bmatrix}\hat{c}_1\\ \hat{c}_2\\ \hat{c}_3\end{bmatrix}\cdot\begin{bmatrix}\hat{c}_1 & \hat{c}_2 & \hat{c}_3\end{bmatrix}}^{I_{3\times3}}\begin{bmatrix}\delta x\\ \delta y\\ \delta z\end{bmatrix}$$

and one can recognize the dyad

$$[\hat{c}_1 \quad \hat{c}_2 \quad \hat{c}_3] \begin{bmatrix} \left.\frac{\partial g_1}{\partial x}\right| & \left.\frac{\partial g_1}{\partial y}\right| & \left.\frac{\partial g_1}{\partial z}\right| \\ \left.\frac{\partial g_2}{\partial x}\right| & \left.\frac{\partial g_2}{\partial y}\right| & \left.\frac{\partial g_2}{\partial z}\right| \\ \left.\frac{\partial g_3}{\partial x}\right| & \left.\frac{\partial g_3}{\partial y}\right| & \left.\frac{\partial g_3}{\partial z}\right| \end{bmatrix} \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{bmatrix} =: \left.\left(\frac{\partial \underline{g}}{\partial \underline{r}_{2j}}\right)\right|_{\underline{r}_{21}}$$

as well as the physical vector

$$\underline{r}_{1s} = [\hat{c}_1 \quad \hat{c}_2 \quad \hat{c}_3] \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} = \hat{c}_1 \delta x + \hat{c}_2 \delta y + \hat{c}_3 \delta z$$

Therefore, finally

$$\underline{g}(\underline{r}_{21} + \underline{r}_{1s}) - \underline{g}(\underline{r}_{21}) = \left.\left(\frac{\partial \underline{g}}{\partial \underline{r}_{2j}}\right)\right|_{\underline{r}_{21}} \cdot \underline{r}_{1s}$$

The full expression of the matrix associated with the dyad is

$$\begin{bmatrix} \frac{3\mu_2 x^2}{r_{2j}^5} - \frac{\mu_2}{r_{2j}^3} & \frac{3\mu_2 xy}{r_{2j}^5} & \frac{3\mu_2 xz}{r_{2j}^5} \\ \frac{3\mu_2 xy}{r_{2j}^5} & \frac{3\mu_2 y^2}{r_{2j}^5} - \frac{\mu_2}{r_{2j}^3} & \frac{3\mu_2 yz}{r_{2j}^5} \\ \frac{3\mu_2 xz}{r_{2j}^5} & \frac{3\mu_2 yz}{r_{2j}^5} & \frac{3\mu_2 z^2}{r_{2j}^5} - \frac{\mu_2}{r_{2j}^3} \end{bmatrix} =$$

$$= \frac{\mu_2}{r_{2j}^5} \left\{ 3 \begin{bmatrix} x \\ y \\ z \end{bmatrix} [x \quad y \quad z] - r_{2j}^2 I_{3\times3} \right\}$$ and finally one gets

$$\left. \left(\frac{\partial \underline{g}}{\partial \underline{r}_{2j}} \right) \right|_{\underline{r}_{21}} = \frac{\mu_2}{r_{21}^5} \left\{ 3 \begin{bmatrix} \hat{c}_1 & \hat{c}_2 & \hat{c}_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\underline{r}_{21}} \begin{bmatrix} x & y & z \end{bmatrix}_{\underline{r}_{21}} \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{bmatrix} - r_{21}^2 \begin{bmatrix} \hat{c}_1 & \hat{c}_2 & \hat{c}_3 \end{bmatrix} I_{3\times 3} \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{bmatrix} \right\} =$$

$$= \frac{\mu_2}{r_{21}^5} \left\{ 3\, \underline{r}_{21}\, \underline{r}_{21} - r_{21}^2\, \underset{\rightarrow}{1} \right\}$$