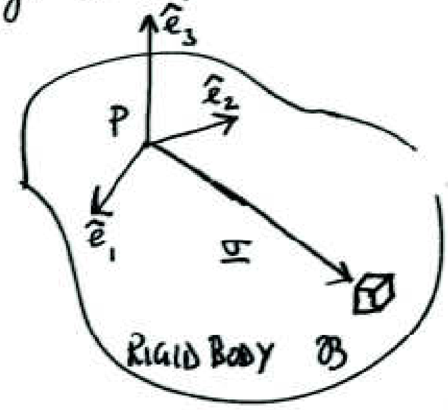
RIGID BODY KINEMATICS

RIGID-BODY-MODEL OF A SPACECRAFT

In many aerospace applications, vehicles can be modeled as rigid bodies. This means that deformations and any flexibility effect are neglected.

A rigid body has time invariant mass distribution with respect to any point that belongs to it, i.e.



$$\rho = \rho(\underline{\sigma})$$

the volumic density is function only of $\underline{\sigma}$ (not of time)

$$dm = \rho(\underline{\sigma})\,dV$$

A right-hand-sequence $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ is attached to $\mathcal{B}$ and termed BODY AXES

$\rightarrow$ they move together with the body and describe its orientation, i.e. attitude, with respect to another frame

Overall, a rigid body has

(i) 3 translational degrees of freedom (associated with position and velocity of its mass center)

(ii) 3 rotational degrees of freedom (associated with attitude and angular rate)

• ROTATION MATRIX (DIRECTION COSINE MATRIX)

First representation for attitude of $\mathcal{B}$ in an inertial frame $(\hat{E}_1, \hat{E}_2, \hat{E}_3)$ is the ROTATION MATRIX $\underset{B\leftarrow N}{R}$

$$\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}}_{\underset{B\leftarrow N}{R}} \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \end{bmatrix}$$

Its rows are the components of $\hat{e}_i$ along $(\hat{E}_1, \hat{E}_2, \hat{E}_3)$

Its columns are the components of $\hat{E}_i$ along $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$

Matrix $\underset{B\leftarrow N}{R}$ is ORTHONORMAL, i.e.

$$\underset{B\leftarrow N}{R}^{T} \equiv \underset{B\leftarrow N}{R}^{-1} \iff \underset{B\leftarrow N}{R}\ \underset{B\leftarrow N}{R}^{T} = I_{3\times3}$$

The latter relation is related to

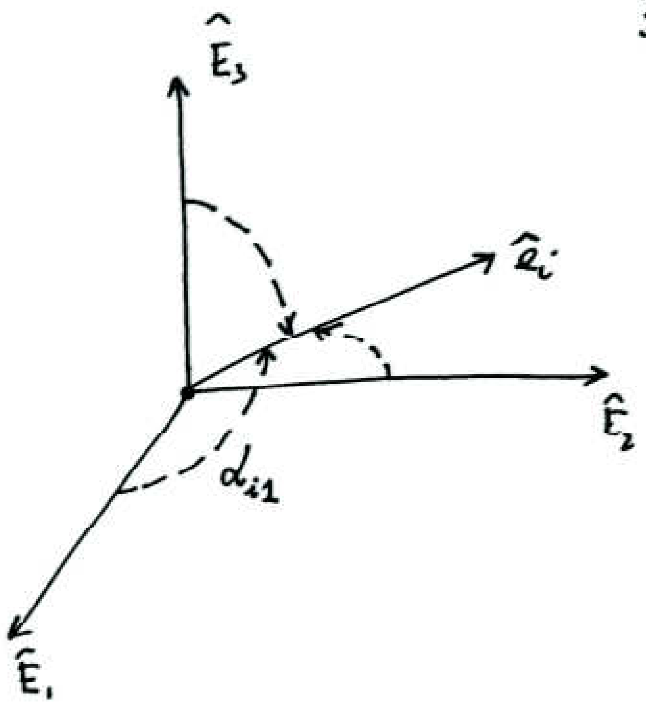
$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij} \quad \text{and} \quad \hat{E}_i \cdot \hat{E}_j = \delta_{ij}$$

$$\left(\delta_{ij} = 1 \text{ if } i = j\,;\ \delta_{ij} = 0, \text{ otherwise};\ \delta_{ij} = \text{Kronecker delta} \right)$$

This representation requires 9 elements $\{r_{ij}\}$

However, 6 relations hold, i.e. the orthonormality equations, either for the rows $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$ (6 eqs)

or for the columns $\hat{E}_i \cdot \hat{E}_j = \delta_{ij}$ (6 eqs)



This representation is also termed DIRECTION COSINE MATRIX because

$$\hat{e}_i \cdot \hat{E}_1 = r_{i1} = \cos \alpha_{i1}$$

$$\hat{e}_i \cdot \hat{E}_2 = r_{i2} = \cos \alpha_{i2}$$

$$\hat{e}_i \cdot \hat{E}_3 = r_{i3} = \cos \alpha_{i3}$$

($\alpha_{i1}, \alpha_{i2}, \alpha_{i3}$ are direction cosines of $\hat{e}_i$)

- <u>Kinematics equations</u>

If one assumes that ${}^N\underline{\omega}^B$ is Known, then the time evolution of $\underset{B \leftarrow N}{R}$ is sought

Letting ${}^N\underline{\omega}^B$ = angular velocity of B (as seen in N), for each unit vector $\hat{e}_i$, the transport theorem is applied,

$$\frac{{}^N d\hat{e}_i}{dt} = \underbrace{\frac{{}^B d\hat{e}_i}{dt}}_{=\underline{0}} + {}^N\underline{\omega}^B \times \hat{e}_i = {}^N\underline{\omega}^B \times \hat{e}_i$$

Specifically, if $(\omega_1, \omega_2, \omega_3)$ are the components of ${}^N\underline{\omega}^B$ in $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$

$$\frac{{}^N d\hat{e}_1}{dt} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ 1 & 0 & 0 \end{vmatrix} = \omega_3 \hat{e}_2 - \omega_2 \hat{e}_3$$

$$\frac{{}^N d\hat{e}_2}{dt} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ 0 & 1 & 0 \end{vmatrix} = \omega_1 \hat{e}_3 - \omega_3 \hat{e}_1$$

$$\frac{{}^N d\hat{e}_3}{dt} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ 0 & 0 & 1 \end{vmatrix} = \omega_2 \hat{e}_1 - \omega_1 \hat{e}_2$$

In compact form

$$\frac{{}^N d}{dt} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} = - \underbrace{\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}}_{} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}$$

$\tilde{\omega}$ = skew-symmetric matrix associated with ${}^N\underline{\omega}^B$

This relation, using the rotation matrix, yields

$$\frac{{}^N d}{dt} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} = - \tilde{\omega} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} = - \tilde{\omega} \underset{B \leftarrow N}{R} \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \end{bmatrix} \qquad \text{(A)}$$

However, the time derivative of

$\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}$ can be found also using an alternative way, i.e. writing $\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} = \underset{B \leftarrow N}{R} \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \end{bmatrix}$ in the time derivative

$$\frac{{}^N d}{dt}\begin{bmatrix}\hat{e}_1\\ \hat{e}_2\\ \hat{e}_3\end{bmatrix} = \frac{{}^N d}{dt}\left\{\underset{B\leftarrow N}{R}\begin{bmatrix}\hat{E}_1\\ \hat{E}_2\\ \hat{E}_3\end{bmatrix}\right\} = \underset{B\leftarrow N}{\dot{R}}\begin{bmatrix}\hat{E}_1\\ \hat{E}_2\\ \hat{E}_3\end{bmatrix} + \underset{B\leftarrow N}{R}\underbrace{\frac{{}^N d}{dt}\begin{bmatrix}\hat{E}_1\\ \hat{E}_2\\ \hat{E}_3\end{bmatrix}}_{\underline{0}} = \underset{B\leftarrow N}{\dot{R}}\begin{bmatrix}\hat{E}_1\\ \hat{E}_2\\ \hat{E}_3\end{bmatrix} \quad (B)$$

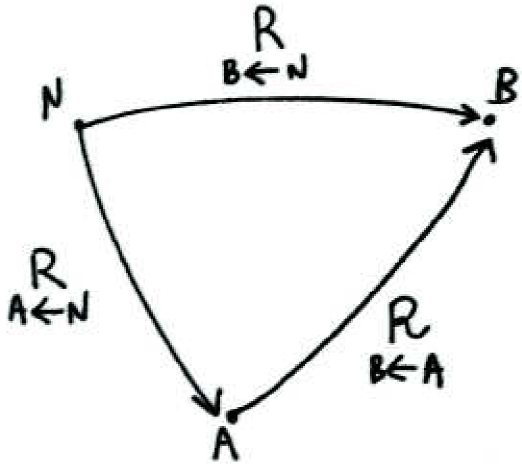
Equating the two results in (A) and (B)

$$\left(\underset{B\leftarrow N}{\dot{R}} + \tilde{\omega}\underset{B\leftarrow N}{R}\right)\begin{bmatrix}\hat{E}_1\\ \hat{E}_2\\ \hat{E}_3\end{bmatrix} = 0 \Rightarrow \underset{B\leftarrow N}{\dot{R}} = -\tilde{\omega}\underset{B\leftarrow N}{R}$$

The last matrix expression is a matrix differential equation equivalent to 9 differential equations for $\{r_{ij}\}$, i.e.

$$\begin{bmatrix}\dot{r}_{11} & \dot{r}_{12} & \dot{r}_{13}\\ \dot{r}_{21} & \dot{r}_{22} & \dot{r}_{23}\\ \dot{r}_{31} & \dot{r}_{32} & \dot{r}_{33}\end{bmatrix} = -\begin{bmatrix}0 & -w_3 & w_2\\ w_3 & 0 & -w_1\\ -w_2 & w_1 & 0\end{bmatrix}\begin{bmatrix}r_{11} & r_{12} & r_{13}\\ r_{21} & r_{22} & r_{23}\\ r_{31} & r_{32} & r_{33}\end{bmatrix}$$

- <u>Relative attitude</u>



If the attitude of B relative to A is needed, a scheme like that in figure can be used.

$$\underset{B\leftarrow A}{R} = \underset{B\leftarrow N}{R}\;\underset{N\leftarrow A}{R} = \underset{B\leftarrow N}{R}\;\underset{A\leftarrow N}{R}^{T}$$

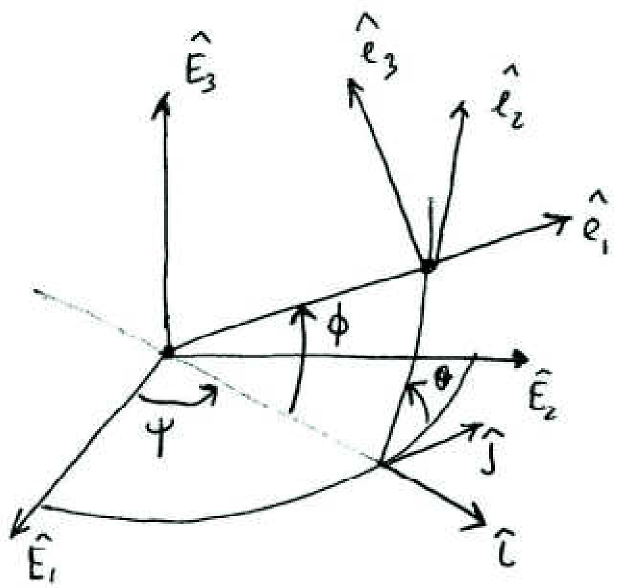
This allows writing $\underset{B\leftarrow A}{R}$ in terms of $\underset{B\leftarrow N}{R}$ and $\underset{A\leftarrow N}{R}$

SEQUENCES OF ANGLES

Only 3 out of 9 components of the direction cosine matrix are independent. This means that representations can be found that use 3 angles to define the orientation of $(\hat{e}_1, \hat{e}_2, \hat{e}_3) \leftrightarrow$ spacecraft with respect to $(\hat{E}_1, \hat{E}_2, \hat{E}_3) \leftrightarrow$ inertial frame

3 angles are associated with 3 distinct and subsequent elementary rotations. For instance

(A) EULER ANGLES (3-1-3)



$$\underset{B \leftarrow N}{R} = R_3(\phi)\, R_1(\theta)\, R_3(\psi)$$

ψ = precession angle $[-\pi, \pi[$

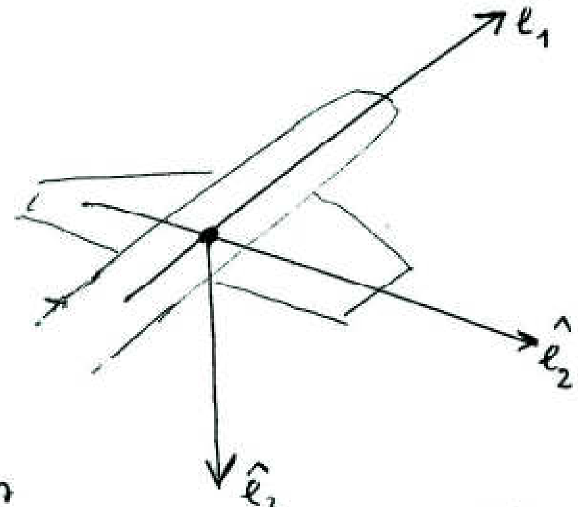
θ = nutation angle $[0, \pi]$

ϕ = spin angle $[-\pi, \pi[$

The constraints are such that any orientation corresponds to a unique sequence of values for (ψ, θ, ϕ), with the exception of the two cases $\theta = 0, \pi$, which corresponds to SINGULARITY

(B) BRYANT'S ANGLES (3-2-1), widely used in atmospheric flight especially for aircraft

$$\underset{B\leftarrow N}{R} = R_1(\phi)\, R_2(\theta)\, R_3(\psi)$$



ψ = yaw angle $[-\pi, \pi[$

θ = pitch angle $[-\frac{\pi}{2}, \frac{\pi}{2}]$

ϕ = roll angle $[-\pi, \pi[$

Again, any orientation corresponds to a unique set of values of (ψ, θ, ϕ), except when $\theta = \pm\frac{\pi}{2}$

In the right figure, $\hat{e}_1 \leftrightarrow$ longitudinal axis

$\hat{e}_2 \leftrightarrow$ right wing

$\hat{e}_3 \leftrightarrow$ downward direction

(in "usual" flight attitude conditions)

In this case, a possible choice for $(\hat{E}_1, \hat{E}_2, \hat{E}_3)$ is

$\hat{E}_1 = \hat{N}_L$ (local North direction at a specified time)

$\hat{E}_2 = \hat{E}_L$ (" East " " " ")

$\hat{E}_3 = -\hat{r}_L$ (local downward direction " " ")

For small angles (ψ, θ, ϕ), with this assumption

$\hat{e}_1 \sim \hat{E}_1$, $\hat{e}_2 \sim \hat{E}_2$, $\hat{e}_3 \sim \hat{E}_3$

(c) CARDAN'S ANGLES (1-2-3)

$$\underset{B\leftarrow N}{R} = R_3(\phi)\, R_2(\theta)\, R_1(\psi)$$

$$\psi \in [-\pi, \pi[\ , \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \ ; \quad \psi \in [-\pi, \pi[$$

<u>Remark</u>. Order matters! Sequence (3-2-1) yields $\underset{B\leftarrow N}{R}$ that is different from that associated with (1-2-3)

- <u>Singularities with 3 angles</u>

With 3 angles (minimal set to represent an orientation) singularities occur.

(A) EULER'S ANGLES

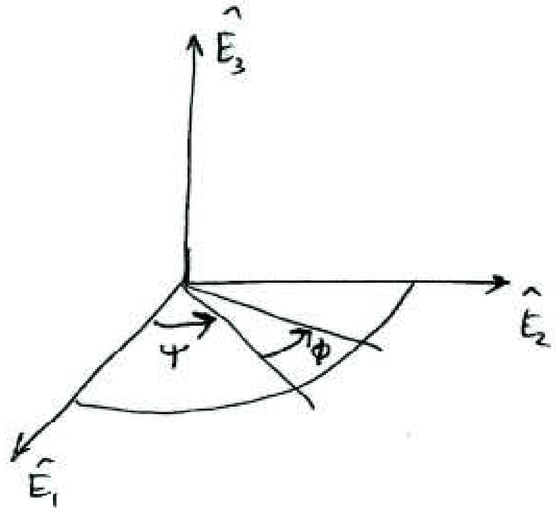
$$\underset{B\leftarrow N}{R} = \begin{bmatrix} C_\phi C_\psi - S_\phi C_\theta S_\psi & S_\phi C_\theta C_\psi + C_\phi S_\psi & S_\phi S_\theta \\ -C_\phi C_\theta S_\psi - S_\phi C_\psi & C_\phi C_\theta C_\psi - S_\phi S_\psi & C_\phi S_\theta \\ S_\theta S_\psi & -S_\theta C_\psi & C_\theta \end{bmatrix}$$

If $\theta = 0, \pi$ one has singularities, because only $(\psi + \phi)$ or $(\psi - \theta)$ is meaningful

For instance, if $\theta = 0$ one obtains

$$\underset{B\leftarrow N}{R} = \begin{bmatrix} \cos(\psi+\phi) & +\sin(\psi+\phi) & 0 \\ -\sin(\psi+\phi) & \cos(\psi+\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In fact, when $\theta = 0$, two consecutive rotations about axis 3 are equivalent to a single rotation by angle $(\psi + \phi)$



(B) BRYANT'S ANGLES

$$\underset{B \leftarrow N}{R} = \begin{bmatrix} c_\theta c_\psi & c_\theta s_\psi & -s_\theta \\ s_\phi s_\theta c_\psi - c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi c_\theta \\ c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi & c_\phi c_\theta \end{bmatrix}$$

If $\theta = \pm \frac{\pi}{2}$ one has singularities, because only $(\psi + \phi)$ or $(\psi - \phi)$ are meaningful

For instance, if $\theta = \frac{\pi}{2}$ one obtains

$$\underset{B \leftarrow N}{R} = \begin{bmatrix} 0 & 0 & -1 \\ -\sin(\psi - \phi) & \cos(\psi - \phi) & 0 \\ \cos(\psi - \phi) & \sin(\psi - \phi) & 0 \end{bmatrix}$$

- Angles from rotation matrix

(A) EULER ANGLES:

$$c_\theta = r_{33} \longrightarrow \theta = \mathrm{acos}\, r_{33} \in [0, \pi]$$

$$\left.\begin{aligned} s_\phi &= \frac{r_{13}}{s_\theta} \\ c_\phi &= \frac{r_{23}}{s_\theta} \end{aligned}\right\rangle \phi = 2\,\mathrm{atan}\frac{s_\phi}{c_\phi + 1}$$

$$\left.\begin{aligned} s_\psi &= \frac{r_{31}}{s_\theta} \\ c_\psi &= -\frac{r_{32}}{s_\theta} \end{aligned}\right\rangle \psi = 2\,\mathrm{atan}\frac{s_\psi}{c_\psi + 1}$$

(B) BRYANT'S ANGLES:

$$s_\theta = -r_{13} \longrightarrow \theta = -\mathrm{asin}\, r_{13} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

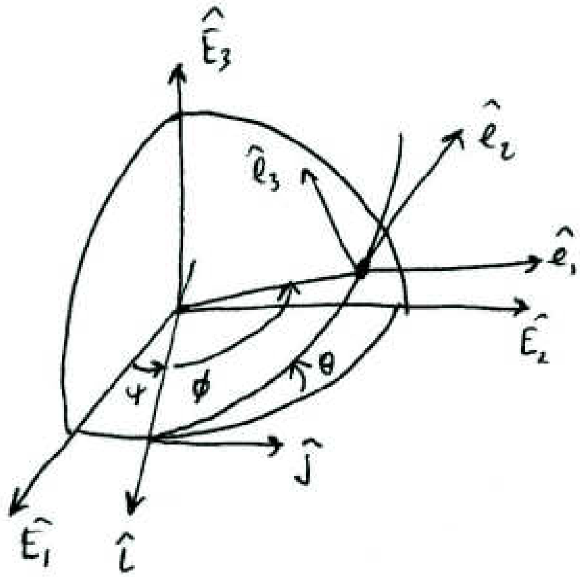
$$\left.\begin{aligned} s_\phi &= \frac{r_{23}}{c_\theta} \\ c_\phi &= \frac{r_{33}}{c_\theta} \end{aligned}\right\rangle \phi = 2\,\mathrm{atan}\frac{s_\phi}{1 + c_\phi}$$

$$\left.\begin{aligned} s_\psi &= \frac{r_{12}}{c_\theta} \\ c_\psi &= \frac{r_{11}}{c_\theta} \end{aligned}\right\rangle \psi = 2\,\mathrm{atan}\frac{s_\psi}{1 + c_\psi}$$

- **Euler angles: kinematics equations**

The angular velocity of $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ with respect to $(\hat{E}_1, \hat{E}_2, \hat{E}_3)$ can be expressed in terms of angular rates $(\dot{\psi}, \dot{\theta}, \dot{\phi})$, taking into account the sequence that is used

(A) EULER ANGLES (3-1-3)



$$\underline{w} = \dot{\psi}\,\hat{E}_3 + \dot{\theta}\,\hat{\imath} + \dot{\phi}\,\hat{e}_3$$

where

$$\hat{\imath} = c_\psi \hat{E}_1 + s_\psi \hat{E}_2$$

and $(\hat{E}_1, \hat{E}_2)$ can be written in terms of $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ using $R_{B\leftarrow N} = R_3(\phi)\, R_1(\theta) R_3(\psi)$

$(\hat{E}_1, \hat{E}_2, \hat{E}_3)$

$\updownarrow$

Inertial frame I

After some algebra,

$$\underline{w} = \begin{bmatrix} \dot{\psi} s_\phi s_\theta + \dot{\theta} c_\phi \\ \dot{\psi} c_\phi s_\theta - \dot{\theta} s_\phi \\ \dot{\psi} c_\theta + \dot{\phi} \end{bmatrix}^T \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}$$

If (w_1, w_2, w_3) denote the components of $\underline{w}$ in $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ then one can recognize that

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} s_\phi s_\theta \dot{\psi} + c_\phi \dot{\theta} \\ \dot{\psi} c_\phi s_\theta - \dot{\theta} s_\phi \\ \dot{\psi} c_\theta + \dot{\phi} \end{bmatrix} = \begin{bmatrix} s_\phi s_\theta & c_\phi & 0 \\ c_\phi s_\theta & -s_\phi & 0 \\ c_\theta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix}$$

Inversion of the previous equation yields the time evolution of (ψ, θ, ϕ) as functions of $(\omega_1, \omega_2, \omega_3)$,

$$\begin{bmatrix} \dot\psi \\ \dot\theta \\ \dot\phi \end{bmatrix} = \begin{bmatrix} \frac{s_\phi}{s_\theta} & \frac{c_\phi}{s_\theta} & 0 \\ c_\phi & -s_\phi & 0 \\ -\frac{s_\phi}{\tan\theta} & -\frac{c_\phi}{\tan\theta} & 1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

or

$$\begin{cases} \dot\psi = \frac{s_\phi}{s_\theta}\omega_1 + \frac{c_\phi}{s_\theta}\omega_2 \\ \dot\theta = c_\phi \omega_1 - s_\phi \omega_2 \\ \dot\phi = -\frac{s_\phi}{\tan\theta}\omega_1 - \frac{c_\phi}{\tan\theta}\omega_2 + \omega_3 \end{cases}$$

KINEMATICS equations

These equations are NONLINEAR and present a singularity at $\theta = 0, \pi$, the two values already found previously when dealing with the Euler angles

(B) BRYANT ANGLES (3-2-1)

$$\underline{\omega} = \dot\psi \hat E_3 + \dot\theta \hat J + \dot\phi \hat e_1 \qquad \text{where}$$

$$\hat J = -s_\psi \hat E_1 + c_\psi \hat E_2$$

and $(\hat E_1, \hat E_2, \hat E_3)$ can be written in terms of $(\hat e_1, \hat e_2, \hat e_3)$ using $\underset{B\leftarrow N}{R} = R_1(\phi) R_2(\theta) R_3(\psi)$

After some algebra,

$$\underline{\omega} = \begin{bmatrix} -S_\theta & 0 & 1 \\ S_\phi C_\theta & C_\phi & 0 \\ C_\phi C_\theta & -S_\phi & 0 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix}$$

yielding

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \frac{1}{C_\theta} \begin{bmatrix} 0 & S_\phi & C_\phi \\ 0 & C_\phi C_\theta & -S_\phi C_\theta \\ C_\theta & S_\phi S_\theta & C_\phi S_\theta \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

or

$$\begin{cases} \dot{\psi} = \dfrac{S_\phi}{C_\theta}\,\omega_2 + \dfrac{C_\phi}{C_\theta}\,\omega_3 \\ \dot{\theta} = C_\phi \omega_2 - S_\phi \omega_3 \\ \dot{\phi} = \omega_1 + S_\phi \tan\theta\, \omega_2 + C_\phi \tan\theta\, \omega_3 \end{cases}$$

These equations are again NONLINEAR and present a singularity at $\theta = \pm\frac{\pi}{2}$, the two values already found previously when dealing with the Bryant's angles.

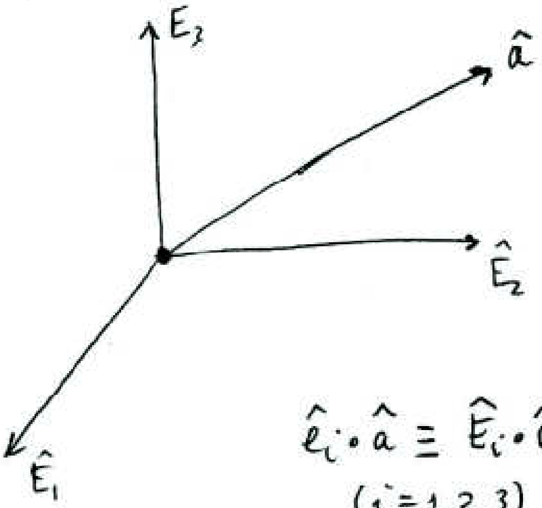
<u>Remark</u> Singularity is encountered for those values of θ that result in ψ and ϕ measured in the same plane, and regard always the second angle θ. The singular orientation is never farther than 90 deg $\Rightarrow$ Euler angles inappropriate for large rotations.

PRINCIPAL AXIS AND ANGLE

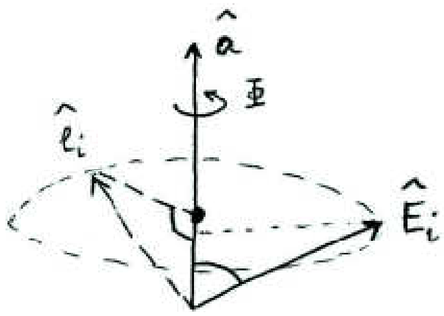
EULER'S PRINCIPAL ROTATION THEOREM :

a rigid body or coordinate frame can be brought from an arbitrary initial orientation to an arbitrary final orientation by a single rigid rotation through a principal angle Φ about the principal axis $\hat{a}$

The axis $\hat{a}$ is fixed in both the initial and the final orientation and referred to as EIGENAXIS. Its components along $B\ (\hat{e}_1, \hat{e}_2, \hat{e}_3)$ and $I\ (\hat{E}_1, \hat{E}_2, \hat{E}_3)$ are the same because the direction cosines do not change for a rotation around $\hat{a}$



$$\hat{e}_i \cdot \hat{a} \equiv \hat{E}_i \cdot \hat{a}$$
$$(i = 1, 2, 3)$$



$\hat{E}_i$ rotates along the dotted line and its component on $\hat{a}$ does not change

$$\Rightarrow \quad \hat{a} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \end{bmatrix} \equiv \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}$$

component of $\hat{E}_1$ along $\hat{a}$ — compon $\hat{E}_2$ on $\hat{a}$ — component of $\hat{E}_3$ along $\hat{a}$

However, in general $\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} = \underset{B\leftarrow N}{R} \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \end{bmatrix}$, therefore

$$\hat{a} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \underset{B\leftarrow N}{R} \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \end{bmatrix}$$

i.e. $\left(\underset{B\leftarrow N}{R}^T - I_{3\times 3} \right) \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e. (a_1, a_2, a_3) is the eigenvector of $\underset{B\leftarrow N}{R}^T$ (thus, of $\underset{B\leftarrow N}{R}$ as well) associated with eigenvalue 1

$\longrightarrow$ $\hat{a}$ is termed EIGENAXIS (or PRINCIPAL AXIS)

So far, the Euler theorem was supposed to hold. Two proofs are provided

- Proof 1 of Euler's theorem

Letting $\underline{a} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}^T$,

$\underset{B\leftarrow N}{R} \, \underline{a} = \lambda \underline{a}$ is the eigenvalue/eigenvector equation

$\Rightarrow \underline{a}^H \underset{B\leftarrow N}{R}^T = \lambda^* \underline{a}^H$, where $\underline{a}^H$ is the Hermitian of $\underline{a}$

(λ^* = complex conjugate of λ) (i.e. the components of $\underline{a}^H$ are the complex conjugate of those of $\underline{a}$ and $\underline{a}^H$ is a row vector)

Combining these two relations, one obtains

$$\underline{a}^H \underbrace{R^T_{B\leftarrow N}\ R_{B\leftarrow N}}_{I_{3\times 3}}\ \underline{a} = \lambda \lambda^* \underline{a}^H \underline{a} \rightarrow \left(|\lambda|^2 - 1 \right) = 0$$

i.e. any eigenvalue of $R_{B\leftarrow N}$ lies on the unit circle of the complex plane.

However, because $R_{B\leftarrow N}$ is a (3×3) matrix and $\det\left(R_{B\leftarrow N}\right) = 1$

and $\det\left(R_{B\leftarrow N}\right) = \lambda_1 \lambda_2 \lambda_3 = 1$

at most 2 eigenvalues are complex conjugate; the 3^{rd} eigenvalue must be real and equal to 1

- <u>Proof 2 of Euler's theorem</u>

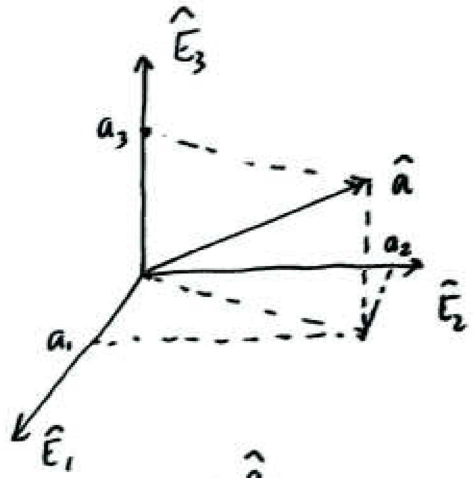
Proof 2 is a constructive proof. It assumes that $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ is obtained from $(\hat{E}_1, \hat{E}_2, \hat{E}_3)$ through a single rotation by angle Φ about axis $\hat{a}$;

the resulting rotation matrix is found in terms of (a_1, a_2, a_3) and Φ

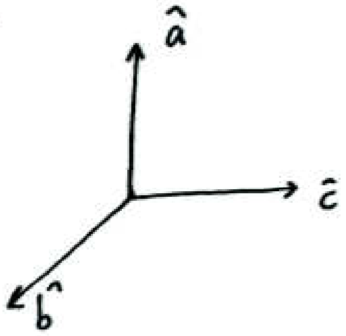
then, it is verified that any orthonormal matrix $R_{B\leftarrow N}$ can be written in the form with (a_1, a_2, a_3, Φ)

As a first step, axis $\hat{a}$ is introduced in $(\hat{E}_1, \hat{E}_2, \hat{E}_3)$, with components (a_1, a_2, a_3)

Then, two auxiliary axes are introduced, to form a right-hand sequence $(\hat{a}, \hat{b}, \hat{c})$



$$\hat{c} := \frac{\hat{a} \times \hat{E}_1}{|\hat{a} \times \hat{E}_1|} \qquad \hat{b} = \hat{c} \times \hat{a}$$



Thus

$$\hat{a} \times \hat{E}_1 = \begin{vmatrix} \hat{E}_1 & \hat{E}_2 & \hat{E}_3 \\ a_1 & a_2 & a_3 \\ 1 & 0 & 0 \end{vmatrix} = \hat{E}_2 a_3 - \hat{E}_3 a_2$$

$$|\hat{a} \times \hat{E}_1| = \sqrt{a_2^2 + a_3^2} = \sqrt{1 - a_1^2}$$

$$\hat{c} = \begin{bmatrix} 0 & \frac{a_3}{\sqrt{1-a_1^2}} & \frac{-a_2}{\sqrt{1-a_1^2}} \end{bmatrix} \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \end{bmatrix}$$

$$\hat{b} = \hat{c} \times \hat{a} = \begin{vmatrix} \hat{E}_1 & \hat{E}_2 & \hat{E}_3 \\ 0 & \frac{a_3}{\sqrt{1-a_1^2}} & \frac{-a_2}{\sqrt{1-a_1^2}} \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{bmatrix} \sqrt{1-a_1^2} & \frac{-a_1 a_2}{\sqrt{1-a_1^2}} & \frac{-a_1 a_3}{\sqrt{1-a_1^2}} \end{bmatrix} \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \end{bmatrix}$$

In compact form

$$\begin{bmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \end{bmatrix} = \underbrace{\begin{bmatrix} a_1 & a_2 & a_3 \\ \sqrt{1-a_1^2} & \frac{-a_1 a_2}{\sqrt{1-a_1^2}} & \frac{-a_1 a_3}{\sqrt{1-a_1^2}} \\ 0 & \frac{a_3}{\sqrt{1-a_1^2}} & \frac{-a_2}{\sqrt{1-a_1^2}} \end{bmatrix}}_{R_A} \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \end{bmatrix} \qquad \left(\begin{array}{l} \text{using} \\ a_1^2 + a_2^2 + a_3^2 = 1 \\ \text{repeatedly} \end{array} \right)$$

Then, both frames $(\hat{a}, \hat{b}, \hat{c})$ and $(\hat{E}_1, \hat{E}_2, \hat{E}_3)$ are rotated about axis $\hat{a}$ by angle Φ

$$
\begin{array}{ccc}
(\hat{E}_1, \hat{E}_2, \hat{E}_3) & \xrightarrow[B \leftarrow N]{R} & (\hat{e}_1, \hat{e}_2, \hat{e}_3) \\
\downarrow R_A & & \downarrow R_A \\
(\hat{a}, \hat{b}, \hat{c}) & \xrightarrow[R_1(\Phi)]{} & (\hat{a}, \hat{b}', \hat{c}')
\end{array}
$$

The same matrix R_A relates $(\hat{E}_1, \hat{E}_2, \hat{E}_3) \xrightarrow[R_A]{} (\hat{a}, \hat{b}, \hat{c})$

and $(\hat{e}_1, \hat{e}_2, \hat{e}_3) \xrightarrow[R_A]{} (\hat{a}, \hat{b}', \hat{c}')$

Looking at the preceding scheme

$$
\begin{bmatrix} \hat{a} \\ \hat{b}' \\ \hat{c}' \end{bmatrix} = R_A \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} \rightarrow \begin{bmatrix} \hat{e}_1 \\ e_2 \\ \hat{e}_3 \end{bmatrix} = R_A^T \begin{bmatrix} \hat{a} \\ \hat{b}' \\ \hat{c}' \end{bmatrix}
$$

Then, one obtains

$$
\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} = R_A^T R_1(\Phi) \begin{bmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \end{bmatrix} = \underbrace{R_A^T R_1(\Phi) R_A}_{\underset{B \leftarrow N}{R}} \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \end{bmatrix}
$$

The last step provides the rotation matrix from $(\hat{E}_1, \hat{E}_2, \hat{E}_3)$ to $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ in terms of Φ (appearing in $R_1(\Phi)$) and (a_1, a_2, a_3), which appear in R_A.

The explicit form of $\underset{B\leftarrow N}{R}$ is

$$\underset{B\leftarrow N}{R} = \begin{bmatrix} C_\Phi + a_1^2(1-C_\Phi) & a_1a_2(1-C_\Phi)+a_3S_\Phi & a_1a_3(1-C_\Phi)-a_2S_\Phi \\ a_1a_2(1-C_\Phi)-a_3S_\Phi & C_\Phi + a_2^2(1-C_\Phi) & a_2a_3(1-C_\Phi)+a_1S_\Phi \\ a_1a_3(1-C_\Phi)+a_2S_\Phi & a_2a_3(1-C_\Phi)-a_1S_\Phi & C_\Phi + a_3^2(1-C_\Phi) \end{bmatrix}$$

or, in compact form

$$\underset{B\leftarrow N}{R} = C_\Phi \, I_{3\times 3} + (1-C_\Phi)\,\underline{a}\,\underline{a}^T - S_\Phi \,\tilde{a}$$

where $\underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $\tilde{a} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$

Any orthonormal matrix $\underset{B\leftarrow N}{R}$ can be put in the preceding form, by identifying $(\Phi, \underline{a})$ from $\{r_{ij}\}$. This is being proven in next subsection

- <u>Derivation of Φ and $\underline{a}$ from $\underset{B\leftarrow N}{R}$</u>

Step 1. Sum all diagonal terms

$$r_{11} + r_{22} + r_{33} = 3C_\Phi + (1-C_\Phi)(a_1^2+a_2^2+a_3^2)$$

i.e. $$1 + 2C_\Phi = r_{11} + r_{22} + r_{33}$$

$$\rightarrow \quad \Phi = \operatorname{acos}\left\{\frac{r_{11}+r_{22}+r_{33}-1}{2}\right\}$$

The preceding relation yields an angle Φ in $[0, \pi]$

Step 2. If $\Phi \neq 0, \pi$, then consider the off-diagonal terms

$$r_{23} - r_{32} = 2 a_1 s_\Phi \longrightarrow a_1 = \frac{r_{23} - r_{32}}{2 s_\Phi}$$

$$r_{31} - r_{13} = 2 a_2 s_\Phi \longrightarrow a_2 = \frac{r_{31} - r_{13}}{2 s_\Phi}$$

$$r_{12} - r_{21} = 2 a_3 s_\Phi \longrightarrow a_3 = \frac{r_{12} - r_{21}}{2 s_\Phi}$$

Step 2a. $\Phi = 0 \iff r_{11} = r_{22} = r_{33} = 1$

and in fact $\underset{BEN}{R} = I_{3\times 3}$. This means that no rotation occurs, thus $\underline{a}$ is not defined

Step 2b. $\Phi = \pi \iff r_{11} + r_{22} + r_{33} = -1$

The greatest value among r_{11}, r_{22}, r_{33} is chosen, then if r_{jj} is this value

$$-1 + 2a_j^2 = r_{jj} \rightarrow a_j = \sqrt{\frac{r_{jj} + 1}{2}}$$

The remaining components of $\underline{a}$ are obtained from the off-diagonal terms, which are all of the form $2 a_i a_k$ $(i \neq k)$

Example: if a_1 is found first, then

$$a_2 = \frac{r_{12}}{2a_1} \qquad \text{and} \qquad a_3 = \frac{r_{13}}{2a_1}$$

The preceding steps lead to finding $(\Phi, \underline{a})$ from ${}_{B\leftarrow N}R$ in all cases, and definitely prove the Euler's theorem in a constructive way.

- <u>Remarks</u>

(a) $(\underline{a}_1, \Phi_1)$ and $(\underline{a}_2, \Phi_2)$, with $\underline{a}_2 = -\underline{a}_1$ and $\Phi_2 = -\Phi_1$ describe the same attitude (i.e. they yield the same ${}_{B\leftarrow N}R$)

(b) $(\underline{a}_1, \Phi_1)$ and $(\underline{a}_2, \Phi_2)$, with $\underline{a}_2 = -\underline{a}_1$ and $\Phi_2 = 2k\pi - \Phi_1$ $(k \in \mathbb{Z})$ describe again the same attitude, with a different rotation angle Φ_2

(c) The rotation axis is identified by $\pm\underline{a}$ and is unique, with the only exception of $\Phi = 2k\pi$ $(k \in \mathbb{Z})$. This is the case of complete rotations, ${}_{B\leftarrow N}R = I_{3\times3}$, and no eigenaxis can be identified

(d) $(\underline{a}_1, \Phi_1)$ and $(\underline{a}_2, \Phi_2)$ identify two opposite rotations (associated with matrices R_1 and R_2 such that $R_1 R_2 = I_{3\times3}$) if $(\underline{a}_2 = -\underline{a}_1$ and $\Phi_2 = 2k\pi + \Phi_1$, $k \in \mathbb{Z})$ OR $(\underline{a}_2 = \underline{a}_1$ and $\Phi_2 = 2k\pi - \Phi_1$, $k \in \mathbb{Z})$

(e) Principal axis and angle require 4 scalar quantities, i.e. (a_1, a_2, a_3, Φ). However, 1 relation holds, i.e. $a_1^2 + a_2^2 + a_3^2 - 1 = 0$ ($\underline{a}$ has norm $= 1$). This means that redundancy equals 1 (and only 2 out of 3 components of $\underline{a}$ are independent).

• EULER PARAMETERS (QUATERNIONS)

Euler parameters represent a set of 4 quantities

$q_0 = \cos\frac{\Phi}{2}$

$q_1 = a_1 \sin\frac{\Phi}{2}$

$q_2 = a_2 \sin\frac{\Phi}{2}$

$q_3 = a_3 \sin\frac{\Phi}{2}$

$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$

Redundancy = 1

(4 parameters, 1 relation)

Compact form $\{q_0, \underline{q}\}$

• Basic properties

(a) $q_0 \geq 0$ if $0 \leq \Phi \leq \pi$

(b) Same quaternion $\{q_0, \underline{q}\}$ corresponds to two choices of $(\underline{a}, \Phi)$
$(\underline{a}_1, \Phi_1)$ and $(\underline{a}_2, \Phi_2)$ such that $\underline{a}_1 = -\underline{a}_2$, $\Phi_1 = -\Phi_2$

(c) The same attitude corresponds to 2 distinct quaternions
$\{q_0, \underline{q}\} \longleftrightarrow (\underline{a}_1, \Phi_1)$ and $\{-q_0, -\underline{q}\} \longleftrightarrow (\underline{a}_2, \Phi_2)$
with $\underline{a}_1 = -\underline{a}_2$ and $\Phi_1 + \Phi_2 = 2\pi$

(d) If $\Phi = 4K\pi$ (K integer), then $q_0 = 1$ and $\underline{q} = \underline{0}$
(no rotation, thus eigenaxis not defined)

(e) If $\Phi = 2\pi + 4K\pi$ (K integer), then $q_0 = -1$ and $\underline{q} = \underline{0}$

(again, no rotation, thus eigenaxis not defined)

(f) $\{q_0, \underline{q}\}$ and $\{q_0, -\underline{q}\}$ represent two rotations, either by opposite angles around the same axis or by same angle around opposite axes; these two quaternions are termed CONJUGATE

(g) In any case, one can always identify a quaternion with $q_0 > 0$, associated with $0 \le \Phi \le \pi$

- <u>Rotation matrix and Euler parameters</u>

The rotation matrix can be found in terms of (q_0, q_1, q_2, q_3) by using the expression of $\underset{B \leftarrow N}{R}$ written in terms of (a_1, a_2, a_3, Φ). For instance, letting r_{ij} be an element of $\underset{B \leftarrow N}{R}$,

$$r_{11} = a_1^2 \left(1 - C_\Phi\right) + C_\Phi = a_1^2 \left[2 - 2\cos^2\frac{\Phi}{2}\right] + 2\cos^2\frac{\Phi}{2} - 1 =$$

$$= 2a_1^2 \sin^2\frac{\Phi}{2} + 2\cos^2\frac{\Phi}{2} - 1 = 2q_1^2 + 2q_0^2 - \left(q_0^2 + q_1^2 + q_2^2 + q_3^2\right) =$$

$$= q_0^2 + q_1^2 - q_2^2 - q_3^2$$

The same steps are repeated for the remaining elements, and yield $\underset{B \leftarrow N}{R}$:

$$\underset{B \leftarrow N}{R} = \begin{bmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1 q_2 + q_0 q_3) & 2(q_1 q_3 - q_0 q_2) \\ 2(q_1 q_2 - q_0 q_3) & q_0^2+q_2^2-q_1^2-q_3^2 & 2(q_2 q_3 + q_0 q_1) \\ 2(q_1 q_3 + q_0 q_2) & 2(q_2 q_3 - q_0 q_1) & q_0^2+q_3^2-q_1^2-q_2^2 \end{bmatrix}$$

Letting $\underline{q} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}^T$ one can recognize that

$$\underset{B \leftarrow N}{R} = \left(q_0^2 - \underline{q}^T \underline{q}\right) I_{3\times3} + 2\,\underline{q}\,\underline{q}^T - 2 q_0 \tilde{q}$$

where $\tilde{q} = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix}$

By inspection of $\underset{B \leftarrow N}{R}$ one can find the inverse transformation

(1) $r_{11} + r_{22} + r_{33} = 3q_0^2 - q_1^2 - q_2^2 - q_3^2 = 3q_0^2 - (1 - q_0^2) = 4q_0^2 - 1$

$$\Rightarrow \quad q_0 = \pm \frac{1}{2}\sqrt{r_{11} + r_{22} + r_{33} + 1}$$

(2) $r_{23} - r_{32} = 4 q_0 q_1 \Rightarrow \quad q_1 = \dfrac{r_{23} - r_{32}}{4 q_0}$

(3) $r_{31} - r_{13} = 4 q_0 q_2 \Rightarrow \quad q_2 = \dfrac{r_{31} - r_{13}}{4 q_0}$

(4) $r_{12} - r_{21} = 4 q_0 q_3 \Rightarrow \quad q_3 = \dfrac{r_{12} - r_{21}}{4 q_0}$

If $q_0 = 0$, then a different approach is used. In fact, one can solve for the diagonal terms, together with the normalization relation

$$r_{11} = q_0^2 + q_1^2 - q_2^2 - q_3^2$$

$$r_{22} = q_0^2 + q_2^2 - q_1^2 - q_3^2$$

$$r_{33} = q_0^2 + q_3^2 - q_1^2 - q_2^2$$

$$1 = q_0^2 + q_1^2 + q_2^2 + q_3^2$$

i.e.
$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} q_0^2 \\ q_1^2 \\ q_2^2 \\ q_3^2 \end{bmatrix} = \begin{bmatrix} r_{11} \\ r_{22} \\ r_{33} \\ 1 \end{bmatrix}$$

Inversion of the preceding system yields $\{q_0^2, q_1^2, q_2^2, q_3^2\}$

Then, one selects $q_i^2 = \max\{q_0^2, q_1^2, q_2^2, q_3^2\}$ and $q_i = +\sqrt{q_i^2}$

The remaining q_j are found from the off-diagonal terms.

Example if q_2^2 is max, then $q_2 = +\sqrt{q_2^2}$ and

$$q_1 = \frac{r_{12} + r_{21}}{4 q_2} \quad \text{and} \quad q_3 = \frac{r_{23} + r_{32}}{4 q_2}$$

- Euler parameters: Kinematics equations

These equations govern the time evolution of q_0, q_1, q_2, q_3 and are derived from inspection of $\underset{B\leftarrow N}{R}$ (cf. page 23) provided that $q_0 \neq 0$

$$\begin{cases} q_0 = \pm\frac{1}{2}\sqrt{r_{11}+r_{22}+r_{33}+1} \\ q_1 = \dfrac{r_{23}-r_{32}}{4q_0} \\ q_2 = \dfrac{r_{31}-r_{13}}{4q_0} \\ q_3 = \dfrac{r_{12}-r_{21}}{4q_0} \end{cases}$$

For instance, time differentiation of the first one yields

$$\dot{q}_0 = \frac{\dot{r}_{11}+\dot{r}_{22}+\dot{r}_{33}}{8q_0}$$

where the terms $\dot{r}_{ij}$ come from the Kinematics equations for the direction cosine matrix (cf. page 4)

$$\begin{cases} \dot{r}_{11} = \omega_3 r_{21} - \omega_2 r_{31} \\ \dot{r}_{22} = -\omega_3 r_{12} + \omega_1 r_{32} \\ \dot{r}_{33} = \omega_2 r_{13} - \omega_1 r_{23} \end{cases}$$

Insertion of these equations yields

$$\dot{q}_0 = \frac{1}{2}\left\{-\frac{r_{23}-r_{32}}{4q_0}\omega_1 - \frac{r_{31}-r_{13}}{4q_0}\omega_2 - \frac{r_{12}-r_{21}}{4q_0}\omega_3\right\}$$

In the right-hand side one can recognize q_1, q_2, q_3.

Therefore

$$\dot{q}_0 = -\frac{1}{2}\left[\omega_1 q_1 + \omega_2 q_2 + \omega_3 q_3\right]$$

For the other components one obtains

$$\dot{q}_1 = \frac{1}{2}\left[\omega_1 q_0 + \omega_3 q_2 - \omega_2 q_3\right]$$

$$\dot{q}_2 = \frac{1}{2}\left[\omega_2 q_0 - \omega_3 q_1 + \omega_1 q_3\right]$$

$$\dot{q}_3 = \frac{1}{2}\left[\omega_3 q_0 + \omega_2 q_1 - \omega_1 q_2\right]$$

or, in compact form

$$\begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix}\begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

and, in matrix form

$$\dot{q}_0 = -\frac{1}{2}\underline{\omega}^T \underline{q}$$

$$\dot{\underline{q}} = \frac{1}{2}\left[q_0 \underline{\omega} - \underline{\tilde{\omega}}\, \underline{q}\right]$$

All of the forms that relate $(\dot{q}_0, \dot{\underline{q}})$ to $(q_0, \underline{q})$ and $\underline{\omega}$ are BILINEAR

- **Relative quaternion using quaternion algebra**

A given quaternion is associated with a "hypercomplex" quantity $\underset{N\to A}{Q} = q_0^{(A)} + q_1^{(A)} i + q_2^{(A)} j + q_3^{(A)} k$

where (i, j, k) are to be regarded as 3 imaginary axes

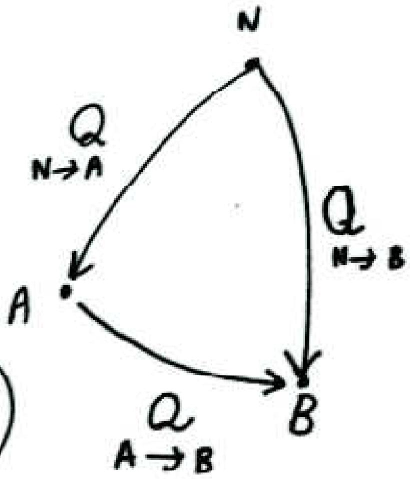
Imaginary units (i, j, k) satisfy

$$ii = jj = kk = ijk = -1 \quad \text{leading to}$$

$$ij = -ji = k \qquad ik = -ki = -j \qquad jk = -kj = i$$

Two subsequent rotations can be written in terms of Q

$$\underset{N\to A}{Q}\ \underset{A\to B}{Q} = \underset{N\to B}{Q} \quad \text{leading to}$$



$$\overbrace{\underset{N\to A}{Q^*}\ \underset{N\to A}{Q}}^{1}\ \underset{A\to B}{Q} = \underset{N\to A}{Q^*}\ \underset{N\to B}{Q}$$

$$\left(\text{where } \underset{N\to A}{Q^*} = q_0^{(A)} - q_1^{(A)} i - q_2^{(A)} j - q_3^{(A)} k\right)$$

i.e., after several steps (here omitted)

$$\begin{aligned}\underset{A\to B}{Q} = \underset{N\to A}{Q^*}\ \underset{N\to B}{Q} = & \left[q_0^{(A)} q_0^{(B)} + \underline{q}^{(A)T} \underline{q}^{(B)}\right] + \\ & + i\left[q_0^{(A)} q_1^{(B)} - q_0^{(B)} q_1^{(A)} - q_2^{(A)} q_3^{(B)} + q_3^{(A)} q_2^{(B)}\right] \\ & + j\left[q_0^{(A)} q_2^{(B)} - q_0^{(B)} q_2^{(A)} - q_3^{(A)} q_1^{(B)} + q_1^{(A)} q_3^{(B)}\right] \\ & + k\left[q_0^{(A)} q_3^{(B)} - q_0^{(B)} q_3^{(A)} - q_1^{(A)} q_2^{(B)} + q_2^{(A)} q_1^{(B)}\right]\end{aligned}$$

This leads to identifying the relative quaternion associated with $\underset{A \to B}{Q}$ (i.e. $\underset{B \leftarrow A}{R}$) and denoted

$$q_{0,e} = q_0^{(A)} q_0^{(B)} + \underline{q}^{(A)T} \underline{q}^{(B)}$$

$$\underline{q_e} = q_0^{(A)} \underline{q}^{(B)} - q_0^{(B)} \underline{q}^{(A)} + \tilde{q}^{(B)} \underline{q}^{(A)}$$

where $$\tilde{q}^{(B)} = \begin{bmatrix} 0 & -q_3^{(B)} & q_2^{(B)} \\ q_3^{(B)} & 0 & -q_1^{(B)} \\ -q_2^{(B)} & q_1^{(B)} & 0 \end{bmatrix}$$

• COMPARISON SEQUENCES OF ANGLES VS. QUATERNIONS

SEQUENCES OF ANGLES	QUATERNIONS (Euler parameters)
3 quantities (redundancy = 0)	4 quantities (redundancy = 1)
intuitive	less intuitive
singularity for 2 values of the intermediate angle	singularity-free
Kinematics eqs exhibit singularities (same values of preceding point)	no singularity in Kinematics eqs
Kinematics eqs are nonlinear → computationally expensive numerical integration	Kinematics eqs are bilinear (in ω and $(q_0, \underline{q})$) → less computationally expensive
Misalignment not clearly measurable	Misalignment can be expressed in a relatively simple way (cf. page 29)